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Decomposing Graphs into Long Paths

Dedicated to the memory of Ivan Rival

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Abstract. It is known that the edge set of a 2-edge-connected 3-regular graph can be decomposed into paths of length 3. W. Li asked whether the edge set of every 2-edge-connected graph can be decomposed into paths of length at least 3. The graphs C_3, C_4, C_5, and $K_4 - e$ have no such decompositions. We construct an infinite sequence $\{F_i\}_{i=0}^{\infty}$ of nondecomposable graphs. On the other hand, we prove that every other 2-edge-connected graph has a desired decomposition.

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1. Introduction

A *k-path* in a graph is a path with k edges. A path in a graph is *long* if its length is at least three. A *long path decomposition* (*an lp-decomposition*, for short) of a graph G is a decomposition of the edges of G into long paths.

On the meeting 'Graphs and Order' in 1984 organized by Ivan Rival, Li [1] asked whether the edge set of every 2-edge-connected graph can be decomposed into long paths. The first author of the present paper learned about this question from Ivan. There was some joint work with Ivan on the problem, but the project was not finished.

The following folklore fact (see, e.g., [3]) together with the Petersen's theorem on perfect matchings in 3-regular graphs [2] implies that the answer to the question of Li is 'yes' for 3-regular graphs.

LEMMA 0. *A* 3*-regular graph* $H = (V, E)$ *decomposes into* 3*-paths if and only if it has a perfect matching.*

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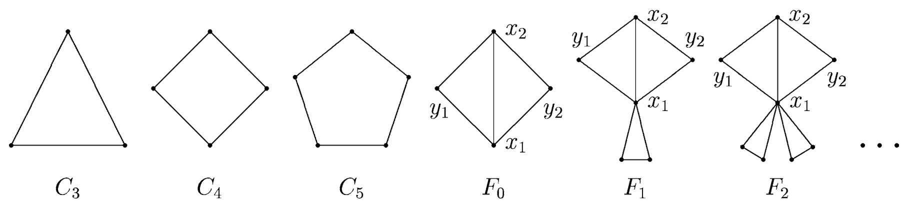


Figure 1. Bad graphs.

Thus, the question attempts to generalize a property of 3-regular graphs to general graphs. Formally, the answer to the question is 'no': the cycles C_3, C_4, and C_5 have too few edges for two long paths. Furthermore, the graphs F_0, F_1, $F_2, \ldots$ depicted on Figure 1 also do not admit any *lp*-decomposition. Indeed, for every i, the degree of x_1 in F_i is $3 + 2i$. Therefore, any *lp*-decomposition of F_i would require at least $i+2$ paths. But F_i has only $3i+5$ edges. This shows that the extension is not immediate. On the other hand, we prove that the examples above are the only exceptions from the rule. The main result of the paper is

THEOREM 1. *Let G be a 2-edge-connected graph distinct from C_3, C_4, C_5 and from every F_i, $i = 0, 1, \ldots$. Then G has an lp-decomposition.*

In the rest of the paper we prove Theorem 1. In the next section we discuss transformations of graphs preserving 2-edge-connectivity. In Section 3 we reduce the problem of finding an *lp*-decomposition to the easier problem of finding so-called *cp*-decompositions. Then we study properties of a hypothetical minimum counterexample G_0: the block structure in Section 4, neighborhoods of vertices of degree 2 in Section 5, and the subgraphs isomorphic to F_0 (i.e., $K_4 - e$) in Section 6. Using this knowledge, we prove in Section 7 that the maximum degree of every block in G_0 is at most 3, and in Section 8 we finish the proof of Theorem 1.

2. Graph Transformations Preserving 2-Connectivity

An edge e of a 2-edge-connected graph G is *critical*, if $G - e$ is not 2-edge-connected. A well known observation is that deleting a chord of a cycle in a 2-connected graph leaves the graph 2-connected and that deleting a chord of an Eulerian subgraph in a 2-edge-connected graph leaves the graph 2-edge-connected. This can be stated as follows.

LEMMA 1. *Deleting any number of chords of a given cycle in a 2-connected graph leaves the graph 2-connected. Similarly, deleting any number of chords of a given Eulerian subgraph in a 2-edge-connected graph leaves the graph 2-edge-connected. In particular, no chord of a Eulerian subgraph in a 2-edge-connected graph is a critical edge.*

The next fact is a variation of an old lemma by Mader. We give a proof of it for completeness.

LEMMA 2. *Let v be a vertex in a 2-edge-connected graph H with degree at least 4 and such that the set $N_H(v)$ of neighbors of v does not induce a complete graph. If v is not a cut vertex in H, then there are nonadjacent neighbors u_1 and u_2 of v such that $H - \{vu_1, vu_2\} + u_1u_2$ is 2-edge-connected.*

Proof. Consider arbitrary nonadjacent neighbors u_1 and u_2 of v. Let $H(u_1, u_2) = H - \{vu_1, vu_2\} + u_1u_2$. It is connected, since v is not a cut vertex. Assume that it has a cut edge e_1, and let $H_1 = H(u_1, u_2) - e_1$. Again, since v is not a cut vertex, none of u_1u_2 and the edges incident with v is e_1. It follows that u_1 and u_2 are in the same component of H_1, and all the vertices in $N_H(v) - u_1 - u_2$ are in the other. In other words, e_1 separates in $H - v$ vertices u_1 and u_2 from all the other vertices of $N_H(v)$. In particular, some vertex $u_3 \in N_H(v) - u_1 - u_2$ is adjacent to neither of u_1 and u_2.

Consider $H(u_1, u_3) = H - \{vu_1, vu_3\} + u_1u_3$. As above, there should be an edge e_2 in $H - v$ separating vertices u_1 and u_3 from all the other vertices of $N_H(v)$. Let u_4 be a neighbor of v different from u_1, u_2, u_3 and let P be a u_1, u_4-path in $H - v$. Then P contains the cut edges e_1 and e_2 of $H - v$. If e_1 is closer to u_1 on P than e_2, then e_2 does not separate u_1 from u_2, otherwise, e_1 does not separate u_1 from u_3. This contradiction proves the lemma. □

3. Reduction to *cp*-Decompositions

A *k-lollipop* is a graph consisting of a k-cycle and a path sharing exactly one vertex, and this vertex is an end vertex of the path (see Figure 2(a)). It is easy to check that no 3-lollipop has an *lp*-decomposition.

LEMMA 3. *Let $H = (V, E)$ be a connected graph that can be decomposed into a cycle and a path. If $|E| \geqslant 6$ and H is not a 3-lollipop, then H has an lp-decomposition.*

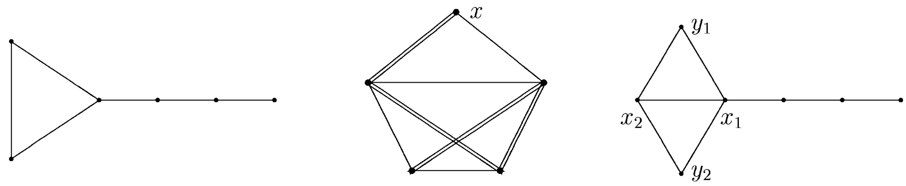
Proof. Assume that H is a counterexample to the lemma. Among edge decompositions of H into a path and a cycle, choose a decomposition into a path $P = u_0u_1 \dots u_t$ and a cycle $C = (v_1, \dots, v_r)$ with the maximum possible r. If $t = 0$, then H is a cycle and the lemma is evident. Let $t > 0$. Let $u_{i_1} = v_{j_1}, \dots, u_{i_s} = v_{j_s}$ be the common vertices of P and C with $i_1 < i_2 < \cdots < i_s$. We may also assume that

$$i_1 \geqslant t - i_s. \tag{1}$$

We will derive properties of H in a series of claims.

CLAIM 1. *For every $1 \leqslant l \leqslant s - 1$, vertices u_{i_l} and $u_{i_{l+1}}$ are not neighbors in C.*

Proof. If, say, $u_{i_l} = v_k$ and $u_{i_{l+1}} = v_{k+1}$, then replacing the edge v_kv_{k+1} in C by the path $P(u_{i_l}, u_{i_{l+1}})$ we obtain a longer cycle C_1 such that $E(H) - E(C_1)$ spans a path. This contradicts the choice of C and P. □



(a) A 3-lollipop. (b) Graph H for Lemma 3. (c) A graph F_0^+.

Figure 2.

Claim 1 immediately implies that

$$\text{if} \quad s \geqslant 2, \quad \text{then} \quad r \geqslant 4. \tag{2}$$

CLAIM 2. $s \geqslant 4$.

Proof. Let $s \leqslant 3$. Since $u_{i_s-1} \notin \{v_{j_s-1}, v_{j_s+1}\}$ by Claim 1, at least one of v_{j_s-1} and v_{j_s+1} is not in $V(P)$. We may assume that $v_{j_s+1} \notin V(P)$. If $i_s \geqslant 2$, let the path P_1 be obtained by adding to $P(u_0, u_{i_s})$ the edge $v_{j_s}v_{j_s+1}$. Then $E(H) - E(P_1)$ also spans a path P_2. Moreover, by construction, $|E(P_1)| \geqslant 3$, and the only possibility that $|E(P_2)| \leqslant 2$ is that $r = 3$ and $t = i_s$. But then by (2), $s = 1$ and thus H is a 3-lollipop.

If $i_s \leqslant 1$, then by (1), $i_s = 1$ and $t \leqslant 2$. If $t = 2$, then $s = 1$ and the path $P_3 = (u_0, u_1, v_{j_1+1}, v_{j_1+2})$ possesses the property that $E(H) - E(P_3)$ spans a long path. Finally, if $t = 1$, then $r \geqslant 5$. In this case we may assume that $v_{j_1+2} \neq u_0$; and again P_3 is such that $E(H) - E(P_3)$ forms a long path. □

By Claims 1 and 2, v_{j_2} is not adjacent in C with v_{j_1} and v_{j_3}. Therefore, $r \geqslant 5$.

CLAIM 3. $r = 5$ *and* $i_s = t$.

Proof. Note that by Claim 2, $i_s \geqslant 3$ and $P(u_0, u_{i_s})$ is a long path. If $r \geqslant 6$ or $t > i_s$, then $C \cup P(u_{i_s}, u_t)$ is a k-lollipop with at least 6 edges and $k \geqslant 5$ which easily decomposes into two long paths. □

CLAIM 4. $t = 3$.

Proof. If $t > 3$, then $P(u_0, u_{t-1})$ is a long path and C together with the incident edge $u_{t-1}u_t$ easily decomposes into two long paths. □

The only possible graph H such that $r = 5$, $s = 4$, and $t = 3$ is depicted on Figure 2(b). Its edge set easily decomposes into two Hamiltonian paths starting at x, as shown on Figure 2(b). □

LEMMA 4. *Let $H = (V, E)$ be a connected graph whose edge set can be decomposed into two cycles, C_1 and C_2. Then H has an lp-decomposition.*

Proof. Let $|E(C_1)| \geqslant |E(C_2)|$ and x be a common vertex in C_1 and C_2. If $|E(G)| \geqslant 9$, then $|E(C_1)| \geqslant 5$ and after deleting some three consecutive edges of C_1 not incident with x the remaining graph satisfies the conditions of Lemma 3. Thus in this case the lemma follows.

Let $|E(G)| \leqslant 8$. Then G has a vertex v of degree two. Let G_1 be obtained from G by splitting v into two vertices v' and v'' of degree one. The graph G_1 satisfies the conditions of Lemma 3. Therefore, G_1 has an *lp*-decomposition. It consists of two paths, since $|E(G)| \leqslant 8$. If v' and v'' are the ends of different paths, say, P_1 and P_2, in this decomposition, then we are done. If both v' and v'' are the ends of the same path, then this decomposition gives a decomposition of $E(G)$ into a path and a cycle of total length at least 6. Since G is Eulerian, it is not a lollipop. Thus applying Lemma 3 again, we are done. □

By F_0^+ we will denote a graph consisting of F_0 and a nontrivial path P whose end vertex is a vertex of degree 3 in F_0 (see Figure 2(c)). Most often, we will use F_0^+ with P consisting of just one edge. Clearly, any F_0^+ decomposes into two long paths. A *cp-decomposition* of a graph G is a decomposition of G into cycles, long paths and copies of F_0^+. Every *lp*-decomposition is a *cp*-decomposition. In general, a graph can have a *cp*-decomposition, but not an *lp*-decomposition (an example is C_4). But for large 2-edge-connected graphs this cannot happen.

LEMMA 5. *Let $H = (V, E)$ be a 2-edge-connected graph with $|E| \geqslant 6$ that admits a cp-decomposition. Then H has an lp-decomposition.*

Proof. Consider a *cp*-decomposition $\mathcal{D}$ of G with fewest parts that are not paths. Since F_0^+ easily decomposes into two long paths, $\mathcal{D}$ has no parts isomorphic to F_0^+. Suppose that a cycle C is a part of $\mathcal{D}$. If $C = G$, then $|E(C)| \geqslant 6$ and we are done. Otherwise, there is another part C' of $\mathcal{D}$ sharing a vertex with C. By Lemma 4, C' is a path, and then by Lemma 3, C is a 3-cycle sharing with C' only one vertex that is an end vertex of C'. Since G is 2-edge-connected, there exists one more path C'' sharing with C exactly one vertex that is an end vertex of C''. Then we can add one edge of C to C' and two edges of C to C'' so that C' and C'' with added edges still form paths. That would give a *cp*-decomposition of G with fewer than in $\mathcal{D}$ parts that are not paths. This proves the lemma. □

In view of Lemma 5 we often will look for just *cp*-decompositions instead of *lp*-decompositions.

4. Block Structure

Let $G_0 = (V, E)$ be a counterexample to the theorem with fewest edges, and among counterexamples with fewest edges, let G_0 have fewest cut vertices.

LEMMA 6. *None of pendant blocks of G_0 is a cycle.*

Proof. Assume that the lemma does not hold. Choose a shortest cycle C that is a pendant block in G_0. Let v be the cut vertex in C and $G_1 = G_0 - (V(C) - v)$. By construction, G_1 is 2-edge-connected. If G_1 is not forbidden then by the minimality of G_0 it has an *lp*-decomposition, and thus G_0 has a *cp*-decomposition. Hence G_1 is a forbidden graph. If G_1 is a cycle, then we have a *cp*-decomposition of G_0.

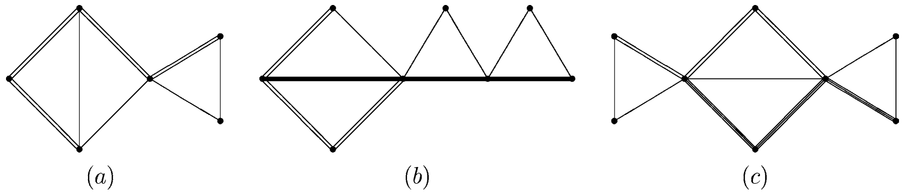


Figure 3. Some *lp*-decompositions for Lemma 6.

Similarly, if $G_1 = F_i$ and $|V(C)| \geqslant 4$, then G_0 decomposes into $C - e$ (where e is an edge incident to v), $F_0 + e$, and 3-cycles.

Finally, let $G_1 = F_i$ and $|V(C)| = 3$. If $i = 0$ and $G_0 \neq F_1$, then G_0 is the graph on Figure 3(a) (with its decomposition shown). This contradicts the choice of G_0. Let $i \geqslant 1$. If after deleting any other end block that is a 3-cycle from G_0, we again get the F_i, then either $G_0 = F_{i+1}$ or $i = 1$ and G_0 is one of the two graphs on Figure 3(b) and Figure 3(c). The *cp*-decompositions of these graphs shown on the picture finish the proof of the lemma. □

LEMMA 7. *Let B_1 be an end block isomorphic to F_0 in G_0. Let v be the cut vertex of G_0 in $V(B_1)$ and $G_2 = G_0 - (V(B_1) - v)$. Then*

(a) *the degree of v in B_1 is 3;*
(b) *the degree of v in G_2 is even.*

Proof. If G_2 is forbidden, then by Lemma 6, G_2 is F_0 or F_1. It follows that G_0 is one of the graphs on Figure 4 (depicted together with their *cp*-decompositions). So, assume that G_2 is not forbidden and hence has an *lp*-decomposition $\mathcal{D}$. If v is an end vertex of some path P in $\mathcal{D}$, then regardless of what degree in B_1 has v, adding to P a 2-path P_0 containing the chord of F_0 and the vertex v yields an *lp*-decomposition of G_0. In particular, this proves (b). If v has degree 2 in B_1 and v divides a path P in $\mathcal{D}$ into two nontrivial paths P_1 and P_2, then we add P_0 to P_1 and the remaining edges of G_1 to P_2. This proves (a). □

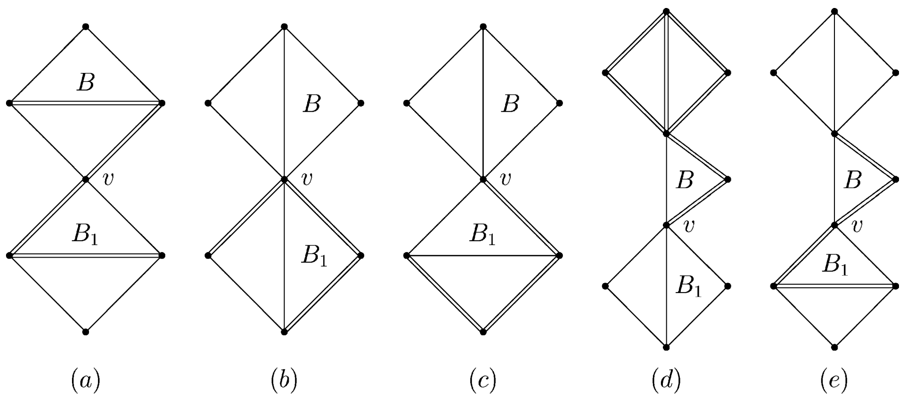


Figure 4. Some *cp*-decompositions for Lemma 7.

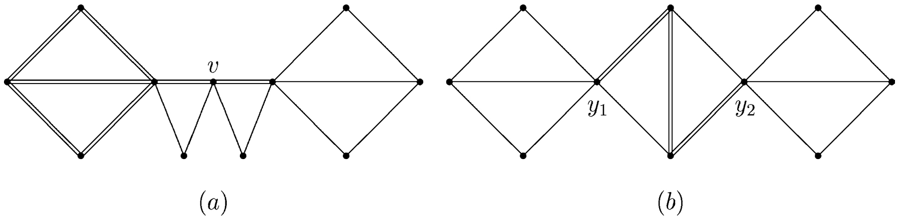


Figure 5. Some *cp*-decompositions for Lemmas 8 and 9.

LEMMA 8. *Let v be a cut vertex in G_0, and G_1 and G_2 be two connected subgraphs of G_0 such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = \{v\}$. Then either G_1 or G_2 is an F_0.*

Proof. If none of G_1 and G_2 is forbidden, then by the minimality of G_0, each of them has an *lp*-decomposition, and thus so does G_0. Hence, we may assume that G_1 is forbidden. Then by Lemma 6, G_1 must be either F_0 or F_1. If $G_1 = F_0$, then we are done. Assume that $G_1 = F_1$ and let v_1 be the cut vertex of G_1. Then the triangle block (call it C_1) of F_1 is not a pendant block of G_0. If G_2 is forbidden then similarly $G_2 = F_1$ with the triangle block C_2 being not a pendant block of G_0. Then (taking into account Lemma 7) the only possible G_0 is depicted (together with an *lp*-decompositions) on Figure 5(a), a contradiction. Thus G_2 is not forbidden and has an *lp*-decomposition $\mathcal{D}$. If v is an end vertex of a path P in $\mathcal{D}$, then adding to P the two edges of a 2-path P_0 from v to v_1, we get a *cp*-decomposition of G_0 (paths plus a copy of F_0^+). If v divides a path P in $\mathcal{D}$ into two nontrivial paths P_1 and P_2, then we add P_0 to P_1 and two edges ($v_1 v$ and the chord of the F_0) to P_2. This gives a *cp*-decomposition of G_0 (paths plus a C_4). □

Remark 1. Lemma 8 yields that G_0 has the *main block B* such that every other block is isomorphic to F_0 and is an end block. Moreover, every cut vertex belongs to exactly two blocks.

LEMMA 9. *The main block B of G_0*

(a) *is not a cycle,*
(b) *is not an F_0.*

Proof. Assume that B is a cycle. Let $B_1, \ldots, B_k$ be the other blocks of G_0, and $v_1, \ldots, v_k$ be the corresponding cut vertices in cyclic order on B. By Lemma 6, $k \geqslant 2$. Then G_0 decomposes into F_0^+-s, each consisting of $E(B_i)$ and the edges of the v_i, v_{i+1}-path in B. By Lemma 5, this contradicts the choice of G_0.

Now assume that B is an F_0 with vertices x_1 and x_2 of degree 3 and vertices y_1 and y_2 of degree 2. By Lemma 7, only y_1 and y_2 can be cut vertices. Since $G_0 \neq F_0$, at least one of them is a cut vertex. If only one of y_1 and y_2 is a cut vertex, then we have the situation on Figure 4(c), and if both are cut vertices, then we have the situation on Figure 5(b). □

5. On Vertices of Degree 2

LEMMA 10. *If a vertex x of the main block B has exactly two neighbors u_1 and u_2 in G_0, then*

(a) *u_1 and u_2 are adjacent;*
(b) *neither of u_1 and u_2 is a cut vertex;*
(c) *the degree of each of u_1 and u_2 is at least* 4.

Proof. If $u_1u_2 \notin E(G_0)$, then consider G_1 obtained from G_0 by deleting x and adding the edge u_1u_2. This graph is 2-edge-connected. If G_1 is a forbidden graph, then by Lemmas 9 and 6, G_1 is F_0. But subdivision of any edge makes from F_0 an *lp*-decomposable graph. Therefore, G_1 is not forbidden, and has an *lp*-decomposition $\mathcal{D}$. If u_1u_2 belongs to a path P in $\mathcal{D}$, then replacing u_1u_2 with the path u_1xu_2 in P creates an *lp*-decomposition of G_0. This proves (a).

Suppose that u_1 is the common vertex of B and an end block B_1 (isomorphic to F_0). By Lemma 9, $V(B) \neq \{x, u_1, u_2\}$. Hence $B - x$ contains a u_1, u_2-path P not using the edge u_1u_2. This and Lemma 1 imply that u_1u_2 is not critical, and that the graph G' obtained from G by deleting $V(B_1) - x$ and the edge u_1u_2 is 2-edge-connected. If G' has a *cp*-decomposition, then this decomposition together with u_1u_2 and $E(B_1)$ gives a *cp*-decomposition of G_0. Thus G' is a forbidden graph that is not a cycle (a cycle has the trivial *cp*-decomposition). But the nonadjacent in G' vertices u_1 and u_2 are in the same block of G' containing x (of degree 2) which does not happen in any F_i. This proves (b).

Suppose that $\deg(u_1) \leqslant 3$. If $\deg(u_1) = 2$, then by (a), the 3-cycle (x, u_1, u_2) is a pendant block in G_0, a contradiction to Lemma 6. Thus, we may assume that $N_{G_0}(u_1) = \{x, u_2, u_3\}$.

Case 1. $u_2u_3 \in E(G_0)$. If also $\deg(u_2) = 3$, then $V(B) = \{x, u_1, u_2, u_3\}$. This contradicts Lemma 9(b). Hence u_2 has a neighbor $u_4 \notin \{x, u_1, u_3\}$. Then $B - u_2$ contains a u_4, u_3-path which cannot pass through x or u_1. Hence the graph $G_1 = G_0 - x - u_2u_3$ is 2-edge-connected. It cannot be an F_i, since the neighbors u_2 and u_3 of the vertex u_1 of degree 2 in G_1 belong to the same block and are nonadjacent. Hence, G_1 has a *cp*-decomposition $\mathcal{D}$. Adding to $\mathcal{D}$ the path u_1, x, u_2, u_3, we get a *cp*-decomposition of G_0.

Case 2. $u_2u_3 \notin E(G_0)$. Since u_1u_3 is not a cut edge, $B - u_1$ contains a u_2, u_3-path. Hence the graph $G_2 = G_0 - x - u_1 + u_2u_3$ is 2-edge-connected. If G_2 is forbidden, then by Lemma 6, G_2 is either a cycle, or F_0, or F_1. If G_2 is a cycle, then the long path $E(G_2) - u_2u_3 + u_3u_1$ and the 3-cycle (x, u_1, u_2, x) decompose G_0. If $G_2 = F_1$, then the 3-cycle block in F_1 is the remainder of B with vertices, say, u_2, u_3, and u_4. But in this case, either u_3 or u_4 is a vertex of degree 2 in G_0 adjacent to a cut vertex, a contradiction to (b). If $G_2 = F_0$, then G_0 is one of the three graphs on Figure 6. They easily decompose into two 4-paths (as shown). Thus, G_2 is not forbidden and has an *lp*-decomposition $\mathcal{D}$. Let $P = v_1, \ldots, v_k$ be the path in $\mathcal{D}$ containing the edge u_2u_3 with $u_2 = v_i$ and $u_3 = v_{i+1}$. If $i \geqslant 3$, then replacing P with $P_1 = v_1, \ldots, v_i, u_1$ and $P_2 = v_k, \ldots, v_{i+1}, u_1, x, u_2$ gives an

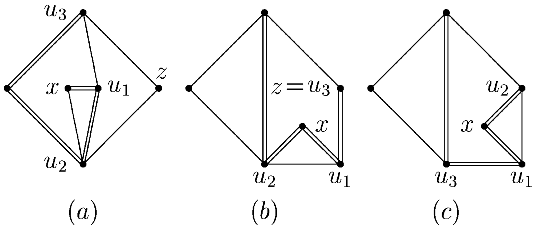


Figure 6.

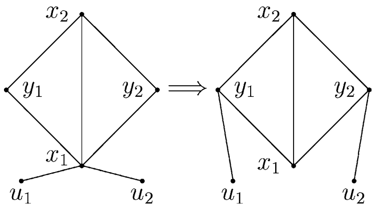


Figure 7.

lp-decomposition of G_0. Similarly, if $i = 2$, then we replace P with $P_1' = v_1, v_2, x, u_1$ and $P_2' = v_k, \ldots, v_3, u_1, u_2$. If $i = 1$, then we add to $P - v_1$ the edge u_3u_1, and the remaining edges of G_0 form the cycle u_2, u_1, x, u_2. This proves the lemma. □

LEMMA 11. *No cut vertex of* G_0 *has degree* 2 *in* B.

Proof. Suppose that B_1 is a block of G_0 distinct from B, and the vertex $x \in V(B) \cap V(B_1)$ has exactly two neighbors u_1 and u_2 in B. Let $V(B_1) = \{x_1, x_2, y_1, y_2\}$, where $\deg_{G_0}(y_1) = \deg_{G_0}(y_2) = 2$. Consider the graph G_1 obtained from G_0 by deleting edges x_1u_1 and x_1u_2 and adding edges y_1u_1 and y_2u_2 (see Figure 7). Graph G_1 is 2-edge-connected and differs from any forbidden graph (the block containing $V(B_1)$ has many vertices). It has the same number of edges as G_0 but fewer cut vertices. Thus by the choice of G_0, G_1 has an *lp*-decomposition $\mathcal{D}$. Since the degree of each of x_1, x_2, y_1, y_2 is 3, each of them is an end vertex of a path in $\mathcal{D}$. Since only two edges connect $V(B_1)$ with the rest of G_1, all edges of some path, say, P_0, in $\mathcal{D}$ are in $E(B_1)$. Since $V|(B_1)| = 4$, we have $|E(P_0)| = 3$. For $i = 1, 2$, let P_i be the path in $\mathcal{D}$ containing the edge u_iy_i, and $P_i' = P_i - V(B_1)$ (possibly, $P_2 = P_1$). By the above, P_1 and P_2 together have exactly two edges in $E(B_1)$ and hence

$$|E(P_1')| + |E(P_2')| \geqslant 2. \tag{3}$$

Because of (3) and of the symmetry of the configuration, it is enough to consider the following three cases.

Case 1. $P_2 \neq P_1$, and $|E(P_1')| \geqslant 2$. Let P_1'' be obtained from P_1' by adding the edge u_1x_1 and P_2'' be obtained from P_2' by adding the path u_2, x_1, x_2, y_2 (it is possible that P_2' had only one vertex, namely, u_2). Then replacing in $\mathcal{D}$ the paths P_1,

P_2, and P_0 with P_1'', P_2'', and the path y_2, x_1, y_1, x_2, we obtain an *lp*-decomposition of G_0.

Case 2. $P_2 \neq P_1$, and $|E(P_1')| = |E(P_2')| = 1$. Let P_1'' be obtained from P_1' by adding the path u_1, x_1, x_2 and P_2'' be obtained from P_2' by adding the path u_2, x_1, y_1. Then replacing in $\mathcal{D}$ the paths P_1, P_2, and P_0 with the paths P_1'', P_2'', and y_1, x_2, y_2, x_1, we obtain an *lp*-decomposition of G_0.

Case 3. $P_2 = P_1$. If $|E(P_1)| \geqslant 6$, then we can split it into two long paths and get one of the previous cases. Since P_1 has two edges in $E(B_1)$, the only possibility for $|E(P_1)| \leqslant 5$ is that $u_1u_2 \in E(G_0)$ and $E(P_1') = \{u_1u_2\}$. It follows that $\mathcal{D}_1 = \mathcal{D} - \{P_0, P_1\}$ is an *lp*-decomposition of $G_0 - V(B_1) - u_1u_2$. Let P_3 be a path in $\mathcal{D}_1$ containing u_1 (by Lemma 10, $\deg_{G_0}(u_1) \geqslant 4$). Suppose that u_1 divides P_3 into two paths P_3' and P_3'' such that $u_2 \notin V(P_3'')$ and that P_3' has at least one edge. Let P_4' be obtained from P_3' by adding the path u_1, x_1, x_2, and P_4'' be obtained from P_3'' by adding the path u_1, u_2, x_1, y_1. Then replacing in $\mathcal{D}'$ the path P_3 with the two paths P_4' and P_4'' and adding the path y_1, x_2, y_2, x_1, we obtain an *lp*-decomposition of G_0. □

Remark 2. We conclude that every vertex of degree 2 in B has degree 2 in G_0.

6. Subgraphs of B Isomorphic to F_0

LEMMA 12. *B contains no K_4.*

Proof. Assume that the subgraph H of B induced by $\{x, y, z, u\}$ is K_4. Consider $G_1 = G_0 - E(H)$.

Case 1. G_1 has an x, y-path P not containing z and u. Then P together with the edges xz, zu, and uy forms a cycle, whose chords are ux, xy, and yz. By Lemma 1, the graph $G_2 = G_0 - ux - xy - yz$ is 2-edge-connected and all x, y, z and u are in he same block of G_2. Since this block has at least 5 vertices, G_2 is not forbidden and has an *lp*-decomposition, unless $G_2 = G_5$. If $G_2 = G_5$, then G_2 has the trivial *cp*-decomposition. In both cases, adding path u, x, y, z to the decomposition creates a *cp*-decomposition of G_0.

Because of the symmetry between x, y, z, and u, we have to consider only the following case.

Case 2. $V(B) = \{x, y, z, u\}$. By Lemma 7(b), none of x, y, z, and u is a cut vertex. Then $G_0 = B = H = K_4$. This proves the lemma. □

In this section, we consider a subgraph H of B isomorphic to F_0, and will assume that $V(H) = \{x_1, x_2, y_1, y_2\}$ and $y_1y_2 \notin E(G_0)$. Also, throughout this section, $G_2 = G_0 - E(H)$.

LEMMA 13. *G_2 contains no x_1, y_1-path that is vertex disjoint from any x_2, y_2-path. Similarly, it contains no x_1, y_2-path that is vertex disjoint from any x_2, y_1-path.*

Proof. By symmetry, it is enough to prove the first statement. Assume that G_2 has an x_1, y_1-path P_1 vertex disjoint from an x_2, y_2-path P_2. Then the edges of the path $P_0 = y_1, x_1, x_2, y_2$ are chords of the cycle $C = x_1 P_1 y_1, x_2 P_2 y_2, x_1$. Let $G_3 = G_0 - E(P_0)$. By Lemma 1, G_3 is 2-edge-connected. Since G_3 contains the cycle C with $|V(C)| \geqslant 6$, it is not forbidden. Hence G_3 has an *lp*-decomposition $\mathcal{D}$, and $\mathcal{D}$ together with $E(P_0)$ forms an *lp*-decomposition of G_0. □

LEMMA 14. *G_2 does contain any x_1, x_2-path vertex disjoint from any y_1, y_2-path.*

Proof. Assume that G_2 has an x_1, x_2-path P_1 vertex disjoint from an y_1, y_2-path P_2. Note that by Lemma 12, P_2 has at least two edges. The rest of the proof repeats that of Lemma 13, replacing C with $C_1 = x_1 P_1 x_2, y_1 P_2 y_2, x_1$. □

LEMMA 15. *G_2 is not connected.*

Proof. Assume that G_2 is connected. Consider a subtree T of G_2 with end vertices x_1, x_2, y_1, y_2. By Lemmas 13 and 14, T consists of a 'central' vertex v joined with each of x_1, x_2, y_1, y_2 by internally disjoint paths. Then $S = T + y_1x_2 + y_2x_1$ is an Eulerian subgraph of G_0, and the edges of $P_0 = y_1, x_1, x_2, y_2$ are chords of S. Again, by Lemma 1, the graph $G_3 = G_0 - E(P_0)$ is 2-edge-connected. If G_3 has an *lp*-decomposition $\mathcal{D}$, then $\mathcal{D}$ together with $E(P_0)$ yields an *lp*-decomposition of G_0. Thus, G_3 is forbidden. Since $|E(G_3)| \geqslant 6$, G_3 is an F_i with $i \geqslant 1$. Since $\deg_{G_3}(v) \geqslant 4$, it is the cut vertex of G_3. The cut vertex of any F_i is all-adjacent. It follows that G_0 contains a K_4, a contradiction to Lemma 12. □

LEMMA 16. *The vertices x_1 and x_2 are in different components of G_2.*

Proof. Suppose that G_2 contains an x_1, x_2-path P. Then by Lemmas 15 and 14, y_1 and y_2 are in different components of G_2. By symmetry, we may assume that y_1 is the only vertex of the set $\{x_1, x_2, y_1, y_2\}$ in a component Q of G_2. This means that $\deg_B(y_1) = 2$ and hence by Lemma 11, $\deg_{G_0}(y_1) = 2$.

Case 1. $y_2 \in V(P)$. Then the edges x_1x_2, x_1y_2, and x_2y_2 are chords of the cycle $y_1x_1Px_2y_1$ of length at least 6 and deleting them from G_0 leads to a 2-edge-connected nonforbidden graph G'. Any *cp*-decomposition of G' together with the 3-cycle $x_1y_2x_2x_1$ forms a *cp*-decomposition of G_0.

Case 2. P does not contain y_2. Consider $G_3 = G_0 - y_1 - x_1x_2$. Because of cycle $C = x_1Px_2y_2x_1$, graph G_3 is 2-edge-connected. If G_3 has a *cp*-decomposition $\mathcal{D}$, then $\mathcal{D}$ together with the 3-cycle x_1, y_1, x_2, x_1 yields a *cp*-decomposition of G_0. Thus, G_3 is forbidden and is not a cycle. Since C is not a 3-cycle, it must be a 4-cycle, say, x_2, y_2, x_1, z, x_2 in a subgraph of G_3 isomorphic to F_0. Furthermore, $zy_2 \in E(G_3)$, because $x_1x_2 \notin E(G_3)$. Then $\{x_1, x_2, y_2, z\}$ induces a K_4 in G_0, a contradiction. □

LEMMA 17. *Neither of y_1 and y_2 has degree 2 in G_0.*

Proof. Assume that $\deg_{G_0}(y_1) = 2$. This means that y_1 is an isolated vertex in G_2. Now, Lemma 16 implies that G_2 has at least three components. Hence, one of

x_1 and x_2, say, x_1 has degree 3 in B. Then by Lemma 7(b), $\deg_{G_0}(x_1) = 3$. This contradicts Lemma 10(c). □

LEMMA 18. *Each of* x_1 *and* x_2 *has degree* 3 *in* G_0.

Proof. Suppose that $\deg_{G_0}(x_1) \geqslant 4$. Then by Lemma 7(b), x_1 is connected in G_2 with some of y_1, y_2, and x_2, but by Lemma 16, not with x_2. By symmetry, we may assume that G_2 contains an x_1, y_1-path P_1 not passing through y_2. By Lemmas 17 and 11, y_2 is also connected with some of y_1, x_1, and x_2. By Lemmas 13 and 16, y_2 is not connected with x_2 in G_2. It follows that some component Q_1 of G_2 contains x_1, y_1, and y_2, but not x_2. In particular, Q_1 contains a path P_2 connecting y_2 with P_1. Now Lemma 7(b) yields that $\deg_{G_0}(x_2) = 3$. Observe that x_1y_1 and x_1x_2 are chords of the cycle $C = y_1P_1x_1, y_2, x_2, y_1$ in G_0. Hence the graph G_4 obtained from G_0 by deleting x_1y_1 and x_1x_2 and replacing the path y_1, x_2, y_2 with the edge y_1y_2 is 2-edge-connected. Assume that G_4 is forbidden. Then the block of G_4 containing P_1 and y_2 must be an F_0. It follows that P_1 has only one internal vertex, say, w. Furthermore, the path P_2 has no internal vertices at all. Then $y_2w \in E(G_0)$, and G_0 is the graph on 5 vertices obtained from K_5 by deleting two disjoint edges. Such graph decomposes into two 4-paths. Therefore, G_4 is not forbidden and has an *lp*-decomposition $\mathcal{D}$. Consider the path P in D containing the edge y_1y_2.

If y_1 is an end vertex of P, then replace y_1y_2 in P by y_2x_2; the unused edges of G_0 form the 3-cycle x_1, x_2, y_1, x_1. This gives a *cp*-decomposition of G_0.

If y_2 is an end vertex of P, then replace y_1y_2 in P by y_1x_2; the unused edges of G_0 form the long path y_1, x_1, x_2, y_2. This gives an *lp*-decomposition of G_0.

If each of the connected components P' (containing y_1) and P'' (containing y_2) of $P - y_1y_2$ has at least one edge and P' does not contain x_1, then let $P_1 = P' \cup \{y_1x_1, x_1x_2\}$ and $P_2 = P'' \cup \{y_2x_2, x_2y_1\}$. Replacing P in $\mathcal{D}$ with P_1 and P$_2$ yields an *lp*-decomposition of G_0.

Finally, if each of P' and P'' has at least one edge and P' contains x_1, then P' has at least 2 edges. In this case, let $P_1 = P' \cup \{y_1x_2\}$ and $P_2 = P'' \cup \{y_2x_2, x_2x_1, x_1y_1\}$. Again, replacing P in $\mathcal{D}$ with P_1 and P_2 yields an *lp*-decomposition of G_0. □

7. Main Lemma

LEMMA 19. *The degree in* B *of every vertex* $v \in V(B)$ *is at most* 3.

Proof. Suppose that the lemma does not hold. Among vertices having degree at least 4 in B, choose a vertex v of minimum degree (in B). By Lemma 12, the neighborhood of v is not a complete graph. Hence by Lemma 2, there exist nonadjacent neighbors u_1 and u_2 of v such that $G_1 = G_0 - \{vu_1, vu_2\} + u_1u_2$ is 2-edge-connected. Since only the degree of v in G_1 differs from that in G_0 and no vertices of degree 2 are adjacent in G_0 (by Lemma 10), the graph G_1 is not a cycle. It cannot be F_0, since B had at least 5 vertices, and it cannot be an F_i for $i \geqslant 3$ by Lemma 6. If $G_1 = F_1$, then since G_0 has no adjacent vertices of degree two, either

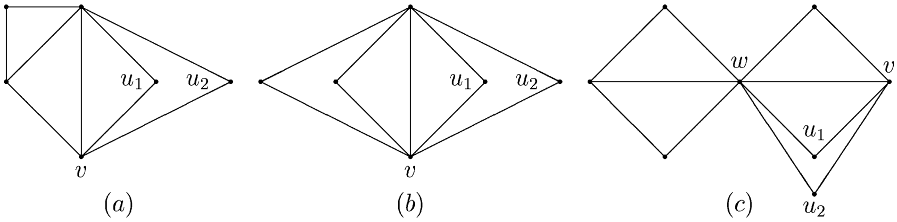


Figure 8.

u_1 and u_2 are the vertices of degree two in the 3-cycle block of G_1, or this block contains v. In this case, G_0 is one of the two graphs on Figure 8(a) and Figure 8(b), but each of these graphs contradicts Lemma 18. Let $G_1 = F_2$. By Lemma 6, one of the triangle blocks of G_1 contains u_1u_2 and the other contains v which is not adjacent to u_1 and u_2. Hence the cut vertex w of F_2 is a common neighbor of v, u_1, and u_2 (see Figure 8(c)). But then the subgraph of G_0 induced by $\{v, w, u_1, u_2\}$ is F_0, a contradiction to Lemma 18.

Therefore, G_1 has an *lp*-decomposition $\mathcal{D}$. We may assume that the length of every path in $\mathcal{D}$ is at most 5 (otherwise, we cut each too long path into two or more). Let P be the path in $\mathcal{D}$ containing the edge u_1u_2. If P does not contain v, then replacing u_1u_2 in P with the path u_1, v, u_2 produces an *lp*-decomposition of G_0.

Suppose $v \in V(P)$. Then the set $E(P) - u_1u_2 + u_1v + u_2v$ forms in G_0 a union P' of a cycle and a path. By Lemmas 3 and 5, $|E(P')| \leqslant 5$, and since P is a long path, $|E(P')| \geqslant 4$. The four possible forms of P' are depicted on Figure 9. By construction, in every form

$$\{u_1, u_2, v, z\} \subseteq V(B) \tag{4}$$

(vertex w might be not in B).

Note that in P' of Form 3, $vx \notin E(G_0)$ by Lemma 18, and hence by Lemma 10(a), the degrees of u_1 and z are at least 3. Because of the symmetry between u_1 and z for every form, we may assume that $\deg_B(u_1) \geqslant 3$ in all other forms, as well. Then there exists a path $Q = y_1, \ldots, y_q$ in $\mathcal{D}$ containing u_1. As remarked above, $q \leqslant 6$. We may assume that $u_1 = y_t$ for some $t \leqslant (q+1)/2$. The vertex u_1 partitions Q into two subpaths $Q_1 = y_1, \ldots, y_t$ and $Q_2 = y_t, \ldots, y_q$.

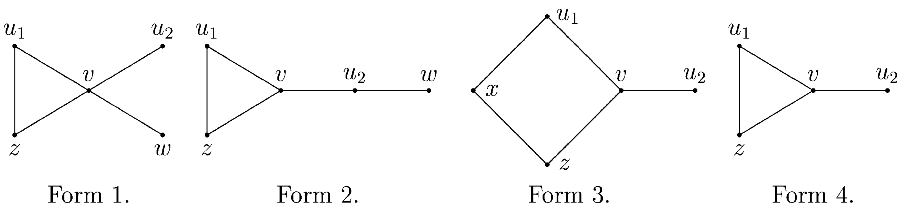


Figure 9. Different forms of P'.

We will say that a P'-configuration of the Form i *is better than* that of the Form j if $i < j$. In the rest of the proof, we will decompose $E(P') \cup E(Q)$ into either long paths and cycles or into long paths, cycles, and a better P'. That would prove the lemma.

Case 1. P' is of the Form 1. If y_{t-1} or y_{t+1} coincides with w or u_2, then G_0 contains an F_0 in which the degree of v is 3. But the degree of v in the whole G_0 is at least 4, a contradiction to Lemma 18. Similarly, v and z cannot coincide with y_{t-2} or y_{t+2} (in case of $z \in \{y_{t-2}, y_{t+2}\}$ either u_1 or z would have degree at least 4). Hence

$$\{y_{t-1}, y_{t+1}\} \cap V(P') = \emptyset, \quad v, z \notin \{y_{t-2}, \ldots, y_{t+2}\}. \tag{5}$$

Subcase 1.1. $q \geqslant 5$ and $1 \leqslant t \leqslant 2$. A possible decomposition is $\{E(Q) - y_1y_2, \{y_1y_2, u_1v, vu_2\}, \{u_1z, zv, vw\}\}$.

Subcase 1.2. $q = 5$ and $t = 3$. If view of (5), the sets $Q_2 \cup \{u_1z, zv, vu_2\}$ and $Q_1 \cup \{u_1v, vw\}$ are paths or cycles.

Subcase 1.3. $q = 6$ and $t = 3$. We decompose $E(P') \cup E(Q)$ into Q_2, $\{u_1z, zv, vu_2\}$, and $Q_1 \cup \{u_1v, vw\}$.

Subcase 1.4. $q = 4$ and $t = 2$. Similarly to Subcase 1.2, we decompose $E(P') \cup E(Q)$ into $Q_2 \cup \{u_1z, zv, vu_2\}$ and $Q_1 \cup \{u_1v, vw\}$.

Subcase 1.5. $q = 4$ and $t = 1$. Assume first that $y_4 \neq v$. By the symmetry between u_2 and w, we can assume that $y_3 \neq w$. Then by (5), $Q + u_1v + vw$ is a long path or a cycle, and $P' - u_1v - vw$ is a long path, contrary to our assumption. It follows that every path in $\mathcal{D}$ containing u_1 is a u_1, v-path of length 3. Hence $\deg_{G_0}(u_1) < \deg_B(v)$, since $u_2 \in V(B) \cap N_{G_0}(v)$. By the choice of v, we have $\deg_B(u_1) = 3$. Then by Lemma 7(b), $\deg_{G_0}(u_1) = 3$ and by Lemma 10, $\deg_B(z) > 2$. Repeating our argument for z in place of u_1, we conclude that $\deg_{G_0}(z) = 3$ and $\mathcal{D}$ contains a z, v-path $Q' = z, z_2, z_3, v$ with $\{z_2, z_3\} \cap V(P') = \emptyset$ (see Figure 10). In this case, we decompose $E(P') \cup E(Q)$ into $Q + vw$, $Q' + vu_1$, and the path u_2, v, z, u_1.

Case 2. P' is of the Form 2 or Form 3.

Subcase 2.1. $q \geqslant 5$. If $1 \leqslant t \leqslant 2$, then a possible decomposition for Form 2 is $\{E(Q) - y_1y_2, \{y_1y_2, u_1z, zv\}, \{u_1v, vu_2, u_2w\}\}$ and a possible decomposition for the Form 3 is $\{E(Q) - y_1y_2, \{y_1y_2, u_1v, vu_2\}, \{u_1x, xz, zv\}\}$.

Let $t = 3$, $q = 6$. Then for both forms our parts will be Q_2, $Q_1 + u_1v$, and $P' - vu_1$. Finally, let $t = 3$, $q = 5$. In this case, the first part for both forms will be $Q_1 + u_1v$; the second part for Form 2 will be $Q_2 + u_1z$ and for Form 3 will be $Q_2 + u_1x$. For both forms, the remaining edges of P' form a long path.

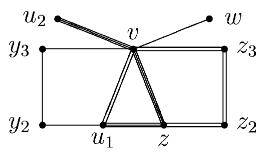


Figure 10. A configuration for Subcase 1.5.

Subcase 2.2. $q = 4$, $t = 2$. If $y_1 = u_2$ in Form 2 or $y_1 = z$ in Form 3, then G_0 contains an F_0 in which the degree of u_1 is 3. But the degree of u_1 in the whole G_0 is at least 4, a contradiction to Lemma 18. It follows that the decomposition of $E(P') \cup E(Q)$ into $Q_2 + u_1v$ and $P' - u_1v + y_1y_2$ is what we need.

Subcase 2.3. $q = 4$, $t = 1$. If $y_3 \neq v$, then we decompose $E(P') \cup E(Q)$ into $Q + u_1v$ and $P' - u_1v$. Suppose that $y_3 = v$. This is impossible for Form 2 by Lemma 18. In the case of Form 3, we can decompose $E(P') \cup E(Q)$ into the path u_1, x, z, v and a new P' of Form 1 with the same central vertex v.

Case 3. P' is of the Form 4. Similarly to Case 1, (5) holds.

Subcase 3.1. $t = 1$. Consider the decomposition of $E(P') \cup E(Q)$ into $Q + u_1v$ and $P' - u_1v$. The latter is a long path; the former is a path or a cycle unless $q = 6$ and $v \in \{y_4, y_5\}$ or $q = 5$ and $y_4 = v$. If $q = 6$, then by Lemma 3, $Q + u_1v$ decomposes into two long paths. If $q = 5$ and $y_4 = v$, then $Q - y_4y_5$ is a long path and $P' + y_4y_5$ makes a configuration of Form 1.

Subcase 3.2. $t = \lfloor q/2 \rfloor$ or $t = (q + 1)/2$. By (5), the decomposition of $E(P') \cup E(Q)$ into $Q_2 + u_1v$ and $Q_1 \cup P' - u_1v$ is what we need.

Subcase 3.3. $q = 6$ and $t = 2$. We decompose $E(P') \cup E(Q)$ into $\{y_3y_4, y_4y_5, y_5y_6\}$, $\{y_1y_2, u_1v, vu_2\}$, and $\{y_3y_2, u_1z, zv\}$. This completes the proof of the lemma. □

8. Completion of the Proof

Lemma 19 implies that the degree structure of G_0 is simple. Indeed, since B has no vertices of degree 4 or more, by Lemmas 7(b) and 11, it has no cut vertices of G_0. Therefore, $\Delta(G_0) \leqslant 3$, and by Lemma 10, G_0 has no vertices of degree 2, i.e., G_0 is 3-regular. By Petersen's theorem [2], G_0 has a perfect matching. Applying Lemma 0 finishes the proof. To be on the safe side, we present a proof of the part of Lemma 0 that we use.

Partial proof of Lemma 0. Let $H = (V, E)$ be a 3-regular graph on $2k$ vertices. Assume that H has a perfect matching $M = \{e_1, \ldots, e_k\}$. Then $T = E - M$ forms a 2-factor in H. Orient the edges of T so that every cycle of T is a directed cycle. For $i = 1, \ldots, k$, let E_i consist of e_i and the two arcs in T entering the ends of e_i. Since the tails of all arcs in T are distinct, every E_i forms a 3-path. □

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