On the Chromatic Number of Set Systems

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ABSTRACT: An (r, l)-system is an r-uniform hypergraph in which every set of l vertices lies in at most one edge. Let $m_k(r, l)$ be the minimum number of edges in an (r, l)-system that is not k-colorable. Using probabilistic techniques, we prove that

$$a_{r,l}(k^{r-1} \ln k)^{l/(l-1)} \leq m_k(r, l) \leq b_{r,l}(k^{r-1} \ln k)^{l/(l-1)},$$

where $b_{r,l}$ is explicitly defined and $a_{r,l}$ is sufficiently small. We also give a different argument proving (for even k)

$$m_k(r, l) \geq a'_{r,l} k^{(r-1)l/(l-1)},$$

where $a'_{r,l} = (r - l + 1)/r(2^{r-1}re)^{-l/(l-1)}$.

Our results complement earlier results of Erdős and Lovász [10] who mainly focused on the case $l = 2$, k fixed, and r large. Random Struct. Alg., 19, 87–98, 2001

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1. INTRODUCTION

A hypergraph H is *k-colorable* if its vertex set can be partitioned into k color classes, such that no edge is monochromatic. The chromatic number $\chi(H)$ of H is the minimum k such that H is k-colorable. A classical extremal problem is to determine the minimum number of edges in an r-uniform hypergraph (r-graph for short) that is not k-colorable. This minimum has been denoted $m_k(r)$ (see [2, 3, 6, 8, 9, 12, 14] for the results in the case $k = 2$ and [1] for large k). If we restrict to the class of *simple* hypergraphs, i.e., those where every two distinct vertices lie in at most one edge, then the corresponding parameter is denoted by $m_k^*(r)$. This parameter was first studied by Erdős and Lovász [10]. They proved the bounds

$$\frac{k^{2(r-2)}}{16r(r-1)^2} \leq m_k^*(r) \leq 1600r^4k^{2(r+1)}, \tag{1}$$

which imply that

$$\lim_{r\to\infty} m_k^*(r)^{1/r} = k^2.$$

We consider a larger class of hypergraphs. A partial (r, l)-system [henceforth, (r, l)-system], is an r-uniform hypergraph in which every set of l vertices lies in at most one edge. Let $m_k(r, l)$ be the minimum number of edges in an (r, l)-system that is not k-colorable; thus $m_k^*(r) = m_k(r, 2)$.

The works [13, 15, 11] on Steiner systems with small independence number yield results for (r, l)-systems, and imply upper bounds on $m_k^*(r)$ which improve (1) for k very large in comparison with r. In particular, Grable, Phelps, and Rödl [11] for every r and infinitely many k constructed simple hypergraphs (in fact, Steiner systems) with chromatic number at least $k+1$ and at most $c4^r r^2 k^{2r-2} \ln^2 k$ edges. Thus, for such r and k,

$$m_k^*(r) \leq c4^r r^2 k^{2r-2} \ln^2 k. \tag{2}$$

Our first result improves the upper bound in (1) in the range $k^4 > 0.01r^6(\ln^2 ek)$. It has the advantage over (2) that it applies for every $l \geq 2$. We write $(r)_l$ for $r(r-1)\cdots(r-l+1)$.

Theorem 1. *Let $r \geq 3, l \geq 2$. Then*

$$m_k(r, l) \leq 2\frac{(c_{r,l})^l}{(r)_l}\left(k^{r-1} \ln ek\right)^{l/(l-1)},$$

where $c_{r,l} = (2r^{3l})^{1/(l-1)}$.

We also improve the lower bound in (1) for $r \geq 3$ and large k.

Theorem 2. *Let $r \geq 3$. If k is even, then*

$$m_k(r, l) \geq d_{r,l}k^{(r-1)l/(l-1)},$$

where

$$d_{r,l} = \left[\frac{1}{(2^{r-1}re)^l}\prod_{i=1}^{l-1}\left(1-\frac{i}{r}\right)\right]^{1/(l-1)}.$$

It is easy to see that this implies the result stated in the abstract. In the case where r is fixed, we match the order of magnitude of the upper bound of Theorem 1.

Theorem 3. *Let $r > l \geq 2$ be fixed. Then there exists c depending only on r and l such that for sufficiently large k we have $m_k(r, l) \geq c(k^{r-1} \ln k)^{l/(l-1)}$.*

We prove Theorem 1 in Section 2 and Theorem 2 in Section 3. In Section 4 we generalize a result from [13] about the chromatic number of hypergraphs with large independent sets; this result is used in Section 5 in the proof of Theorem 3.

2. THE UPPER BOUND

The bounds of the kind (2) in [13, 15, 11] hold for all r and k, but apply only to large k as written. Our construction also works for every $r > 2$ and $k \geq 2$. It is an example of a random greedy algorithm.

Proof of Theorem 1. Consider the following procedure:

1. Order all r-element subsets of the set $\{1, 2, \ldots, n\}$ at random: $R_1, \ldots, R_{\binom{n}{r}}$;
2. Construct the family $G_0, \ldots, G_{\binom{n}{r}}$ of hypergraphs with the vertex set $V = \{1, \ldots, n\}$ as follows: G_0 has no edges and for $j = 1, \ldots, \binom{n}{r}$ if $G_{j-1} + R_j$ is an (r, l)-system, then we let $G_j = G_{j-1} + R_j$, otherwise, $G_j = G_{j-1}$;
3. Let $G(n, r) = G_{\binom{n}{r}}$.

Clearly, Part 2 is a deterministic procedure once the ordering is defined. Our aim is to prove that if $n = [c_{r,l}(k^{r-1} \ln ek)^{1/(l-1)}]$, where $c_{r,l} = (2r^{3l})^{1/(l-1)}$, then with positive probability $G(n, r)$ has no independent set of size $\lceil n/k \rceil$. Thus such a hypergraph has no k-colorings. Since $G(n, r)$ is an (r, l)-system by construction, this will give us (for $r \geq 3$) an example of an (r, l)-system with chromatic number at least $k + 1$ and the number of edges at most

$$\frac{\binom{n}{l}}{\binom{r}{l}} \leq \frac{n(n-1)^{l-1}}{(r)_l} \leq 2\frac{c_{r,l}^l}{(r)_l}\left(k^{r-1} \ln ek\right)^{l/(l-1)}.$$

The proof follows from the following claim.

Claim. *For an arbitrary set X of vertices in $G(n, r)$ of cardinality $x = [n/k]$, the probability that X induces no edges in $G(n, r)$ is less than $\binom{n}{x}^{-1}$.*

Proof. Fix an X of size $x = [n/k]$. Let B_X be the event that X induces no edges in $G(n, r)$. Observe that B_X implies that every r-set $T \subseteq X$ must be preceded (in the random ordering) by some r-set R not in X such that $R \in G(n, r)$ and $|R \cap T| \geq l$.

Consequently, $l \le |R \cap X| \le r-1$. Let us call such an R a *witness* for $T \in [X]^r$ not being included in $G(n,r)$. The point is that if B_X happens, then we must have a large number of witnesses in $G(n,r)$, and the probability of the latter is small. Indeed, each $R \in G(n,r)$ can be a witness for at most $\binom{r-1}{l}\binom{x-l}{r-l}$ r-sets $T \subset X$. This means that in order to prevent all $\binom{x}{r}$ r-sets T of X to appear in $G(n,r)$, the number of witnesses has to be at least

$$m = \left\lceil \frac{\binom{x}{r}}{\binom{r-1}{l}\binom{x-l}{r-l}} \right\rceil = \left\lceil \frac{(x)_l}{(r)_l\binom{r-1}{l}} \right\rceil.$$

For $j \ge 1$, let $A_j = A_{X,j}$ denote the event that the first j edges $R_{l_1}, R_{l_2}, \dots, R_{l_j}$ in $G(n,r)$ such that $|R_{l_i} \cap X| \ge l$ are not contained in X, i.e., $l \le |R_{l_i} \cap X| \le r-1$. The previous paragraph yields that if B_X occurs, then A_m also occurs.

The rest of the proof consists of bounding the probability of A_m from above by $\binom{n}{x}^{-1}$.

For this calculation, we further assume that R_{l_1} is the witness that appears first in the ordering, and that for each $1 < j \le m$, R_{l_j} is the first witness which comes after $R_{l_{j-1}}$. Let $G^j = G_{l_j - 1}$ be the family of all r-sets included in $G(n,r)$ before the jth witness R_{l_j} is chosen. For $1 \le j \le m$, let $\mathcal{S}_j$ be the collection of all r-sets S, such that $|X \cap S| \ge l$ and $|R \cap S| < l$ for all $R \in G^j$.

Since $A_m \subset A_{m-1} \subset \cdots \subset A_2 \subset A_1$, we have

$$\mathbf{P}\{A_m\} = \mathbf{P}\{A_1\} \cdot \mathbf{P}\{A_2 \mid A_1\} \cdot \cdots \cdot \mathbf{P}\{A_m \mid A_{m-1}\}.$$

To estimate these probabilities we first note that each of the events A_1 and $A_{j+1} \mid A_j$, $j = 1, \dots, m-1$ corresponds to a random choice from the set $\mathcal{S}_j$ with the result that the chosen set belongs to $\mathcal{S}_j - [X]^r$.

Since $|\mathcal{S}_1| \le \binom{x}{l}\binom{n}{r-l}$ we have

$$\mathbf{P}\{A_1\} = \frac{|\mathcal{S}_1| - \binom{x}{r}}{|\mathcal{S}_1|} \le 1 - \frac{\binom{x}{r}}{\binom{x}{l}\binom{n}{r-l}}.$$

Furthermore, suppose that $j > 1$, and let $j \le m_0 = \lceil m/2 \rceil$. Assume now that the event A_j occurred. Since

$$\mathbf{P}\{A_j \mid A_{j-1}\} = \frac{|\mathcal{S}_j - [X]^r|}{|\mathcal{S}_j|} = 1 - \frac{|\mathcal{S}_j \cap [X]^r|}{|\mathcal{S}_j|},$$

we need to estimate the cardinality of the set $\mathcal{S}_j \cap [X]^r$.

The hypergraph G^j contains precisely $j-1$ r-sets R with $l \le |R \cap X| \le r-1$. Each of these is a witness for at most

$$\binom{|X \cap R|}{l}\binom{x-l}{r-l} \le \binom{r-1}{l}\binom{x-l}{r-l}$$

r-sets. Consequently, the number of r-sets T in X with no witness at this stage is

$$\begin{aligned} |\mathcal{S}_j \cap [X]^r| &\ge \binom{x}{r} - (j-1)\binom{r-1}{l}\binom{x-l}{r-l} \\ &\ge \binom{x}{r} - (m_0-1)\binom{r-1}{l}\binom{x-l}{r-l} \ge \frac{1}{2}\binom{x}{r}, \end{aligned}$$

where the last inequality follows from the choice of m_0. Summarizing, we infer that

$$\mathbf{P}\{A_j \mid A_{j-1}\} \le 1 - \frac{\frac{1}{2}\binom{x}{r}}{\binom{n}{r-l}\binom{x}{l}}.$$

This yields

$$\begin{aligned}\mathbf{P}\{A_m\} \le \mathbf{P}\{A_{m_0}\} &= \mathbf{P}\{A_1\}\cdot\mathbf{P}\{A_2 \mid A_1\}\cdot\cdots\cdot\mathbf{P}\{A_{m_0} \mid A_{m_0-1}\}\\ &\le \left(1 - \frac{\frac{1}{2}\binom{x}{r}}{\binom{n}{r-l}\binom{x}{l}}\right)^{m_0} \le \left(1 - \frac{\frac{1}{2}\binom{x}{r}}{\binom{n}{r-l}\binom{x}{l}}\right)^{(x)_l/2(r)_l\binom{r-1}{l}}\\ &\le \exp\left\{-\frac{(x)_r}{4n^{r-l}\binom{r}{l}(r)_l\binom{r-1}{l}}\right\}.\end{aligned}$$

In order to prove the claim we will show that the last expression is less than $\binom{n}{x}^{-1}$. Since $\binom{n}{x} < (ne/x)^x = \exp\{x\ln(en/x)\}$, and $x = \lceil n/k\rceil$, we have

$$\mathbf{P}\{A_m\}\cdot\binom{n}{x} < \exp\left\{x\left(\ln\frac{en}{x} - \frac{(x-1)_{r-1}}{4n^{r-l}\binom{r}{l}(r)_l\binom{r-1}{l}}\right)\right\}.$$

By the choice of n, it is easy to observe that $x \ge n/k \ge (r-1)$.[3] Thus for $r \ge 3$,

$$(x-1)_{r-1} \ge \left(1 - \frac{r-1}{x}\right)^{r-1} x^{r-1} \ge \left(1 - \frac{1}{(r-1)^2}\right)^{r-1} x^{r-1} \ge \frac{1}{2}x^{r-1} \ge \frac{1}{2}\left(\frac{n}{k}\right)^{r-1}.$$

Consequently, $\mathbf{P}\{A_m\}\cdot\binom{n}{x}$ is strictly less than

$$\exp\left\{x\left(\ln ek - \frac{\frac{1}{2}\left(\frac{n}{k}\right)^{r-1}}{4n^{r-l}\binom{r}{l}(r)_l\binom{r-1}{l}}\right)\right\} \le 1,$$

where the last inequality follows since

$$n \ge (2r^{3l})^{1/(l-1)}(k^{r-1}\ln ek)^{1/(l-1)} \ge \left[8\binom{r}{l}(r)_l\binom{r-1}{l}\right]^{1/(l-1)}(k^{r-1}\ln ek)^{1/(l-1)}. \quad \blacksquare$$

3. LOWER BOUNDS FROM THE LOVÁSZ LOCAL LEMMA

In this section, we prove Theorem 2. Our main tool is the symmetric version of the Lovász local lemma which we state below (see [5] for a proof).

Lemma 4 (Local lemma). *Let $A_1, \ldots, A_n$ be events in a probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that* $\text{Prob}[A_i] \le p$ *for all i. If $ep(d+1) \le 1$, then* $\text{Prob}[\wedge\overline{A_i}] > 0$.

We use the following lemma from [10], whose proof we supply for completeness.

Lemma 5. *Let H be an r-graph. If every vertex in H has degree at most k^{r-1}/er, then $\chi(H) \le k$.*

Proof. We color the vertices of H with k colors, with each color being assigned to each vertex independently with equal probability. The probability that a given edge is monochromatic is $1/k^{r-1}$. The event A_F that edge F is monochromatic is independent of all events $A_{F'}$ with $F \cap F' = \emptyset$. The number of F' with $F \cap F' \neq \emptyset$ is at most $r(k^{r-1}/(er) - 1) \le k^{r-1}/e - 1$. The local lemma (Lemma 4) therefore implies that there is a k-coloring with no monochromatic edge. ■

Proof of Theorem 2. Let H be an (r, l)-system with at most $z = c_{r,l} k^{(r-1)l/(l-1)}$ edges, where

$$c_{r,l} = \left[\frac{1}{(2^{r-1}re)^l} \prod_{i=1}^{l-1} \left(1 - \frac{i}{r}\right)\right]^{1/(l-1)}.$$

Let

$$A = \left\{v \in V(H)\colon \deg(v) > \frac{k^{r-1}}{er2^{r-1}}\right\}.$$

Let $B = V(H) - A$, and let H_A and H_B be the subhypergraphs induced by A and B, respectively. By Lemma 5, there is a proper $k/2$-coloring of H_B.

Now color H_A randomly using a new set of $k/2$ colors, where each color appears on each vertex independently with equal probability. Since H is an (r, l)-system, every vertex in H_A has degree (in H_A) at most

$$\Delta = \frac{(a-1)(a-2)\cdots(a-l+1)}{(r-1)(r-2)\cdots(r-l+1)},$$

where $a = |A|$. Consequently, each edge E in H_A is incident with at most $d = \Delta r - r$ other edges. Moreover, since H has at most z edges and $zr > ak^{r-1}/(er2^{r-1})$, we infer that $a < zer^2 2^{r-1}/k^{r-1}$. Consider the space of colorings with each vertex being colored randomly and independently of others. For each edge E in H_A, let M_E be the event that E is monochromatic. Since $p = \mathbf{P}\{M_E\} = (2/k)^{r-1}$ and

$$\begin{aligned} ep(d+1) &\le e\left(\frac{2}{k}\right)^{r-1}\left(\frac{(a-1)(a-2)\cdots(a-l+1)}{(r-1)(r-2)\cdots(r-l+1)}r - r + 1\right) \\ &\le e\left(\frac{2}{k}\right)^{r-1} \frac{a^{l-1}r}{(r-1)(r-2)\cdots(r-l+1)} \\ &\le e^l \frac{r^{2l-1}}{(r-1)\cdots(r-l+1)}\left(\frac{2}{k}\right)^{(r-1)l} z^{l-1} < 1, \end{aligned}$$

the local lemma implies that there is a proper $k/2$-coloring of H_A. These two colorings together yield a proper k-coloring of H. ■

4. FROM INDEPENDENT SETS TO PROPER COLORINGS

In this section, we prove a preliminary Lemma 8 to our main lower bound, Theorem 3, which might be interesting of its own. A special case appears in [13]. The following fact was kindly pointed out to us by a referee.

Lemma 6. *Let $f(m)$ be a monotonically nondecreasing function, $f(1) = 1$, and $f(m) \leq m$ for every m. Let G be a graph on n vertices. Let $I_1, \ldots, I_t$ be a family of disjoint independent sets in G with $i_l = |I_l|$ for $l = 1, \ldots, t$. Let $x_0 = 0$ and $x_l = \sum_{j=1}^{l} i_j$. If $i_j \geq f(n - x_{j-1})$ for every $j = 1, \ldots, t$, then $t \leq \sum_{l=n-x_t+1}^{n} (1/f(l))$.*

Proof. Since $f(m)$ is monotonically nondecreasing and $i_j \geq f(n - x_{j-1})$, we have

$$\sum_{l=n-x_t+1}^{n} \frac{1}{f(l)} = \sum_{j=1}^{t} \sum_{l=n-x_j+1}^{n-x_{j-1}} \frac{1}{f(l)} \geq \sum_{j=1}^{t} \sum_{l=n-x_j+1}^{n-x_{j-1}} \frac{1}{f(n - x_{j-1})}$$
$$= \sum_{j=1}^{t} \frac{i_j}{f(n - x_{j-1})} \geq t. \qquad \blacksquare$$

This lemma (due to a referee) directly implies the following nice corollary.

Lemma 7. *Let $f(m)$ be a monotonically nondecreasing function, $f(1) = 1$, and $f(m) \leq m$ for every m. Let G be a graph on n vertices. If for every $2 \leq m \leq n$, the independence number of every m-vertex subgraph of G is at least $f(m)$, then $\chi(G) \leq \sum_{j=1}^{n} (1/f(j))$.*

Lemma 8. *Let $0 \leq \alpha < 1$ and $\beta < 1 - \alpha$. Let H be a hypergraph with n vertices. Suppose that every subhypergraph P of H (including H itself) with $m \geq 2$ vertices has an independent set of size at least $f(m) = cm^{\alpha}(\ln em)^{\beta}$ for some constant $c > 0$. Then there is a $d = d(c, \alpha, \beta) > 0$ such that $\chi(H) \leq dn^{1-\alpha}(\ln en)^{-\beta}$.*

Proof. Define $f(1) = 1$. Then by Lemma 7,

$$\chi(H) \leq \sum_{j=1}^{n} \frac{1}{f(j)} \leq 1 + \int_1^n \frac{1}{c} x^{-\alpha} (\ln ex)^{-\beta}\, dx$$
$$\leq 1 + \frac{1}{c(1 - \alpha - \beta)} \int_1^n x^{-\alpha} (\ln ex)^{-\beta} \left(1 - \alpha - \frac{\beta}{\ln ex}\right) dx$$
$$= 1 + \frac{1}{c(1 - \alpha - \beta)} \Big|_1^n x^{1-\alpha} (\ln ex)^{-\beta} = 1 + \frac{1}{c(1 - \alpha - \beta)} \big(n^{1-\alpha} (\ln en)^{-\beta} - 1\big).$$

This proves the lemma. $\blacksquare$

We use Lemma 8 to prove that (r, l)-systems with not too many vertices are k-colorable. The following result of Rödl and Šinajová guarantees large independent sets in such r-graphs.

Theorem 9 (Rödl–Šinajová [15]). *Let H be an (r, l)-system on n vertices. Then H has an independent set of size at least $cn^{(r-l)/(r-1)}(\ln n)^{1/(r-1)}$, where c is a positive constant depending only on r and l.*

Theorem 9 together with Lemma 8 implies the following theorem.

Theorem 10. *Let H be an (r, l)-system on n vertices. Then $\chi(H) \leq c(n^{l-1}/\ln n)^{1/(r-1)}$ for some constant c depending only on r and l. Moreover, there is another constant c' (also depending only on r and l) such that, if $n \leq c'(k^{r-1}\ln k)^{1/(l-1)}$, then $\chi(H) \leq k$.*

5. THE MAIN LOWER BOUND

In this section we prove Theorem 3. The main idea to properly k-color the (r, l)-system is to greedily take maximal independent sets. We therefore need a lower bound on the size of a maximal independent set in an r-graph. Such a bound is provided for a fairly restricted class by a result of Ajtai et al. [4].

Theorem 11 [4]. *Let G be an r-uniform hypergraph without 2-, 3-, and 4-cycles. If $|E(G)|/|V(G)|$ is very large in comparison with r, then*

$$\alpha(G) \geq c\frac{|V(G)|^{r/(r-1)}}{|E(G)|^{1/(r-1)}}\left(\ln\frac{|E(G)|}{|V(G)|}\right)^{1/(r-1)}, \tag{3}$$

where c depends only on r.

We remark that the condition $|E(G)|/|V(G)|$ being large can be removed by changing the constant c. Duke, Lefmann, and Rödl [7] extended this bound (with a different constant) to the class of simple hypergraphs. We need the following generalization of [7] for (r, l)-systems; the proof follows from the idea in [15].

Theorem 12. *Let r, l be integers with $r > l > 1$, and let $\delta = (r-l)/(8r-10)$. Suppose that F is an (r, l)-system with $|V(F)| = n$ and $|E(F)| \geq n^{l-\delta}$. Then*

$$\alpha(F) \geq c_1\, n\left(\frac{\ln w}{w}\right)^{1/(r-1)}, \tag{4}$$

where $w = |E(F)|/n$ and c_1 depends only on r and l.

Proof (Sketch). Let $\epsilon_0 = (4l-5)/(4r-5)$ and $\epsilon_1 = (l-1-\delta)/(r-1) < (l-1)/(r-1)$. Set $\epsilon = (\epsilon_0+\epsilon_1)/2$. Consider a random induced subsystem H of F, where every vertex in H is included with probability $p = n^{-\epsilon}$ independently of all other vertices. The expected number of vertices in H is pn and the expected number of edges in H is $p^r|E(F)|$. In [15] it is proven that with positive probability, we can delete at most half of the vertices of H to obtain a subsystem G of F with

1. no cycles of length less than five,
2. $|V(G) = pn/2$, and
3. $|E(G)| \leq 2p^r|E(F)|$.

We apply Theorem 11 to G. With given $|V(G)|$, the bound we seek for $\alpha(G)$ decreases when $|E(G)|$ grows. Therefore, letting $z = |E(F)|$, we obtain that $\alpha(F)$ is at least

$$\begin{aligned}\alpha(G) &\geq c\frac{(pn/2)^{r/(r-1)}}{(2p^r z)^{1/(r-1)}}\left(\ln\frac{4p^r z}{pn}\right)^{1/(r-1)}\\ &= c\frac{(n/2)^{r/(r-1)}}{(2z)^{1/(r-1)}}\left(\ln\frac{4z}{n^{\epsilon(r-1)+1}}\right)^{1/r-1}\\ &\geq c_1\frac{n^{r/(r-1)}}{z^{1/(r-1)}}\left(\ln\frac{z}{n}\right)^{1/(r-1)}.\end{aligned}$$

The last inequality follows by replacing the exponent inside the logarithm by a factor outside of the logarithm. ■

Proof of Theorem 3. Let H be an (r, l)-system with at most $c_2(k^{r-1}\ln k)^{l/(l-1)}$ edges. Set $c_3 = 1/(er3^{r-1})$. Partition $V(H)$ into two parts:

V_0—vertices of degree at most c_3k^{r-1}, and

V_1—vertices of degree greater than c_3k^{r-1}.

By Lemma 5 there is a proper coloring of the vertices in V_0 with $k/3$ colors. It remains to properly color the vertices of V_1 with at most $2k/3$ colors.

Let $H_1 = H(V_1)$ and $n_1 = |V_1|$. Let c_4 be chosen so that by Theorem 10, every (r, l)-system with at most

$$n_0 = c_4\left(k^{r-1}\ln k\right)^{1/(l-1)} \tag{5}$$

vertices is $\frac{k}{3}$-colorable. Because

$$n_1c_3k^{r-1} \leq \sum_{v\in V_1}\deg(v) \leq r|E(H)|,$$

we obtain

$$n_1 \leq \frac{rc_2}{c_3}\left(k^{r-1}\ln^l k\right)^{1/(l-1)} = \frac{rc_2}{c_3c_4}n_0 \ln k. \tag{6}$$

Let $a_1 = n_1/n_0$. If $a_1 \leq 1$, then we are done, and by (6), in any case,

$$a_1 \leq \frac{rc_2}{c_3c_4}\ln k. \tag{7}$$

Let $i \geq 1$ and consider the following procedure.

Step i.

Case a. If $a_i > 1$, then we distinguish between two cases depending on whether $|E(H_i)|$ is large.

- **(I)** If $|E(H_i)| \geq n_i^{l-\delta}$, where δ is as in Theorem 12, then we can apply Theorem 12. Choose in H_i a maximum independent set I_i, let $H_{i+1} = H_i - I_i, n_{i+1} = |V(H_{i+1})| = n_i - |I_i|$ and $a_{i+1} = n_{i+1}/n_0$. Now go to Step $i+1$.
- **(II)** If $|E(H_i)| < n_i^{l-\delta}$, then partition $V(H_i)$ into two sets X and Y, where X consists of all vertices of H_i with degree less than dk^{r-1}, with $d = 1/(er6^{r-1})$.

By the choice of d, Lemma 5 implies that the hypergraph induced by X can be properly $k/6$-colored. Let d' be chosen so that, by Theorem 10, every (r, l)-system on at most $d'(k^{r-1} \ln k)^{1/(l-1)}$ vertices is properly $k/6$-colorable. Because

$$|Y|dk^{r-1} \leq \sum_{v \in Y} \deg(v) \leq r|E(H_i)| \leq rn_i^{l-\delta} < rn_1^{l-\delta},$$

we conclude that since k is sufficiently large

$$|Y| \leq \frac{rn_1^{l-\delta}}{dk^{r-1}} \leq \frac{r}{dk^{r-1}}\left[\frac{rc_2}{c_3} \ln k(k^{r-1} \ln k)^{(1/l-1)}\right]^{l-\delta} \leq d'(k^{r-1} \ln k)^{(1/l-1)}. \qquad (8)$$

Consequently, the subhypergraph of H_i induced by Y can be properly $k/6$-colored. These two colorings together yield a proper $k/3$-coloring of H_i. Color H_i properly with $k/3$ colors. Since all vertices of H are now colored, we stop the procedure.

Case b. if $a_i \leq 1$, then the number of vertices in the uncolored hypergraph is at most n_0. We apply Theorem 10 to color these vertices with $k/3$ colors. Now we stop the procedure.

Suppose that the procedure stops on Step $t+1$. That means that on Steps $i = 1, \ldots, t$, we were in Case (a), part (I). We will prove that $t \leq k/3$. Observe that this implies that H is k-colorable.

- **(α)** We used $k/3$ colors to color $H(V_0)$,
- **(β)** We used t colors for $I_1, \ldots, I_t$,
- **(γ)** Regardless of whether we stopped the procedure due to Case (a), part (II), or Case (b), in each situation we used $k/3$ new colors. This yields the required k-coloring of H.

In order to complete the argument, we will show that in (β) we have $t \leq k/3$.
By the definition of a_i, we have

$$\begin{aligned}|E(H_i)|/|V(H_i)| &\leq \frac{c_2(k^{r-1} \ln k)^{l/(l-1)}}{a_i c_4(k^{r-1} \ln k)^{1/(l-1)}} \\ &\leq \frac{c_2(k^{r-1} \ln k)^{l/(l-1)}}{a_i c_4(k^{r-1} \ln k)^{1/(l-1)}} = \frac{c_2 k^{r-1} \ln k}{a_i c_4}\end{aligned}$$

and hence by Theorem 12 for large k,

$$|I_i| \geq c_1 n_i \left(\frac{a_i c_4}{c_2 k^{r-1} \ln k}\right)^{1/(r-1)} \left(\ln \frac{c_2 k^{r-1} \ln k}{a_i c_4}\right)^{1/(r-1)}$$
$$\geq \left(\frac{a_i c_4}{c_2}\right)^{1/(r-1)} \frac{c_1 n_i}{k} = \left(\frac{c_4}{c_2 n_0}\right)^{1/(r-1)} \frac{c_1 n_i^{r/(r-1)}}{k}. \quad (9)$$

Let $c_5 = c_1(\frac{c_4}{c_2})^{1/(r-1)}$. Then by (9), the conditions of Lemma 6 are satisfied with $f(m) = \frac{c_5 n_i^{r/(r-1)}}{k n_0^{1/(r-1)}}$. Hence by Lemma 6,

$$t \leq 1 + \sum_{l=n_0+1}^{n} \frac{1}{f(l)} = 1 + \frac{k n_0^{1/(r-1)}}{c_5} \sum_{l=n_0+1}^{n} l^{-(r/r-1)} \leq 1 + \frac{k n_0^{1/(r-1)}}{c_5} \int_{n_0}^{n} x^{-r/(r-1)}\, dx$$
$$\leq 1 + \frac{k(r-1) n_0^{1/(r-1)}}{c_5} \left(n_0^{-1/(r-1)} - n^{-1/(r-1)}\right) \leq 1 + \frac{k(r-1)}{c_5}.$$

Thus if we choose c_2 small enough to make $c_5 > 6(r-1)$, then for large k we will have $t < k/3$. This proves the bound. ∎

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