*Coloring Uniform Hypergraphs with Few Colors**

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ABSTRACT: Let $m(r, k)$ denote the minimum number of edges in an r-uniform hypergraph that is not k-colorable. We give a new lower bound on $m(r, k)$ for fixed k and large r. Namely, we prove that if $k \geq 2^n$, then $m(r, k) \geq \epsilon(k)k^r(r/\ln r)^{n/(n+1)}$. © 2003 Wiley Periodicals, Inc. Random Struct. Alg., 24: 1–10, 2004

1. INTRODUCTION

Let $m(r, k)$ denote the minimum number of edges in an r-uniform hypergraph that is not k-colorable. The most studied case is the case $k = 2$. Erdős [3, 4] proved that $2^{r-1} \leq m(r, 2) \leq r^2 2^r$. Then Beck [2] improved the lower bound to $2^r r^{1/3-\epsilon}$, and Spencer [7] presented a simpler proof of Beck's bound based on random recoloring. Radhakrishnan and Srinivasan [6] proved

Theorem 1 [6]. *For every $c < 1/\sqrt{2}$, there exists an $r_0 = r_0(c)$ such that for every $r > r_0$,*

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$$m(r, 2) \geq c2^r \sqrt{r/\ln r}.$$

Erdős [4] and Erdős and Lovász [5] said that "perhaps, the order of magnitude of $m(r, 2)$ is $r2^r$." Repeating the argument of Erdős [3, 4], one can see that for every $k \geq 2$, there exists $C = C(k)$ such that $k^{r-1} \leq m(r, k) \leq Cr^2k^r$.

In this paper we consider large but fixed k and huge r. Our aim is to prove

Theorem 2. *For every positive integer k, let $\epsilon = \epsilon(k) = \exp\{-4k^2\}$ and $n = n(k) = \lfloor \log_2 k \rfloor$. Then for every $r > \exp\{2\epsilon^{-2}\}$,*

$$m(r, k) \geq \epsilon k^r \left(\frac{r}{\ln r}\right)^{n/(n+1)}.$$

This result supports the insight of Erdős. The proof extends the Spencer–Radhakrishnan–Srinivasan idea of semirandom local recoloring. Note that for $k \gg r$, the local recoloring does not seem promising. Best known bounds on $m(r, k)$ for $k \gg r$ are due to Alon [1], who used a global deterministic recoloring.

One can also consider $m_l(r, k)$—the minimum number of edges in an r-uniform hypergraph with list chromatic number $k + 1$. In fact, the proof of Radhakrishnan and Srinivasan [6] works for list coloring as well, and gives the same bound for $m_l(r, 2)$. On the other hand, the proof of Theorem 2 and Alon's proof [1] for $k \gg r$ do not work for list coloring.

The structure of the paper is as follows. In the next section a semi-random Procedure EVOLUTION is described and some of its simple properties are derived. In Section 3 we concentrate on studying typical ("smooth") outcomes of EVOLUTION. In the next section we give the outline of the proof of Theorem 2. In the final Section 5 we describe some variations of EVOLUTION allowing easier estimation of important probabilities needed for the proof.

2. A COLORING PROCEDURE AND ITS PROPERTIES

Let $r \geq \exp\{2\epsilon^{-2}\}$. Fix some $0 < p < 2^{-k^r r}$. Then there is the unique positive integer s such that $sp \leq (\ln r)/(n + 1)r < (s + 1)p$. Let $G = (V, E)$ be an r-uniform hypergraph with $|E| = m \leq \epsilon k^r(r/(\ln r))^{n/(n+1)}$.

The coloring procedure EVOLUTION described below consists of $n + 1$ stages and every stage apart from Stage 0 consists of s steps.

STAGE 0. Color every vertex randomly and independently, with a color in $\{0, 1, 2, \ldots, k - 1\}$, chosen uniformly in this set.

STAGE l, $l = 1, \ldots, n$.

STEP $i + s(l - 1)$, $1 \leq i \leq s$. For every $v \in V(G)$ (in some order), do the following. If (i) v never changed its color before this step, and (ii) v belongs to an edge which was monochromatic, say, of color α, before this stage and no vertex of the edge changed its color during the stage (including the current step), then with probability p we recolor v with color $\alpha + 2^{l-1}$ (modulo k). Otherwise, do nothing with v.

Remark 1. Note that any vertex can be recolored at most once.

Remark 2. In fact, every step consists of $|V(G)|$ smaller steps (one per vertex).

Lemma 1. *For every $w \geq 1$ and every set $W \subseteq V$ with $|W| = w$, the probability that all vertices of W will be recolored during some fixed q steps is at most $(qp)^w$.*

Proof. For every vertex v_j and every $i \geq 1$, under any conditions (!), the probability that v_j will be recolored at Step i is at most p. Thus, under any conditions the probability that v_j will be recolored during some given q steps is at most qp. This yields the lemma. ■

Throughout the paper we will use the following notation: $c = -\ln \epsilon = 4k^2$, $z = \lfloor cr/\ln r \rfloor$.

Lemma 2. *For an edge $e \in E$, let $X(e)$ be the event that more than z vertices of e are recolored during Steps $1, 2, \ldots, ns$. Then, for every $e \in E$,*

$$\mathbf{P}\{X(e)\} \leq \epsilon^{0.5r}.$$

Proof. By Lemma 1, this probability is at most

$$\binom{r}{z+1}(nsp)^{z+1} \leq \left(\frac{er}{z+1}\right)^{z+1}\left(\frac{n \ln r}{(n+1)r}\right)^{z+1} \leq \left(\frac{e \ln r}{z+1}\right)^{z+1}$$
$$\leq \left(\frac{e(\ln r)^2}{cr}\right)^{cr/\ln r} < e^{-0.5cr} = \epsilon^{0.5r}. \quad ■$$

Lemma 3. *If all vertices of an edge $e \in E$ are of color α at the end of Stage l, $l \geq 1$, then at the end of Stage 0 the vertices of e can be colored only with colors α, $\alpha - 2^0$, $\alpha - 2^1, \ldots, \alpha - 2^{l-1}$ (modulo k).*

Proof. By Remark 1, every vertex can be recolored at most once and by the definition, a vertex of color β can be recolored during Stage j only with color $\beta + 2^{j-1}$ (modulo k). ■

If an edge e_0 becomes monochromatic of color α during Stage l, then it must contain at the end of Stage 0 a vertex of color $\alpha - 2^{l-1}$. Suppose that, at the end of Stage 0, it contained vertices of colors $\alpha - 2^{i_1-1}, \ldots, \alpha - 2^{i_h-1}$, where $i_h = l$ and $i_1 < i_2 < \cdots < i_h$. Then, for every $1 \leq j \leq h$, there exists an edge e_j and a vertex $v_j \in e_0 \cap e_j$ such that

(a) e_j was monochromatic of color $\alpha - 2^{i_j-1}$ at the end of Stage $i_j - 1$;
(b) v_j was recolored with α during Stage i_j and it was the last vertex of this color in e_0 recolored with α.

In this case we say that e_0 *i_j-blames* e_j. We also say that $e_1, e_2, \ldots, e_h$ are the *edges of level 1* of the *cause tree with root* e_0.

Remark 3. Since on every step of Procedure EVOLUTION the vertices are considered consecutively, every edge e can blame only an edge sharing exactly one vertex with e.

If e_0 i_j-blames e_j, then e_j can i-blame some other edge for some $i < i_j$ or be monochromatic at the end of Stage 0. Let us call the edges blamed by the edges of level 1 the *edges of level 2* of the *cause tree with root* e_0, and so on. Thus, if an edge e of a cause tree has exactly t distinct colors in it at the end of Stage 0, then e blames either $t - 1$ or t other edges.

Let us call an outcome of Procedure EVOLUTION a *smooth outcome,* if in no edge more than z vertices were recolored. By Lemma 2, the probability of a smooth outcome is at least $1 - m\epsilon^{0.5r} = 1 - o(1)$. Also, if the outcome is smooth and an edge e of a cause tree has exactly t distinct colors in it at the end of Stage 0, then e blames exactly $t - 1$ other edges.

3. SMOOTH OUTCOMES

Consider a smooth outcome of Procedure EVOLUTION. If an edge e was monochromatic of some color α on some step, then it contains at least $r - z$ vertices of this color on every step. In this case, we say that α is the *main color,* $\mu(e)$, of e and that e is an *unlucky edge.*

Lemma 4. *In a smooth outcome, for every unlucky edge e_0, the main colors of all the edges of the cause tree with root e_0 are distinct.*

Proof. If e and e' are edges of the cause tree with root e_0, then there exist two sequences $e_0, e_1, \dots, e_l = e$ and $e'_0 = e_0, e'_1, \dots, e'_{l'} = e'$ such that e_j i_j-blames e_{j+1} for $j = 0, 1, \dots, l - 1$ and e'_j i'_j-blames e'_{j+1} for $j = 0, 1, \dots, l' - 1$. Furthermore, $i_0 > i_1 > \cdots > i_{l-1}$, $i'_0 > i'_1 > \cdots > i'_{l-1}$, and the sequences $i_0, i_1, \dots, i_l$ and $i'_0, i'_1, \dots, i'_l$ are not identical. Thus, the numbers $2^{i_0-1} + 2^{i_1-1} + \cdots + 2^{i_{l-1}-1}$ and $2^{i'_0-1} + 2^{i'_1-1} + \cdots + 2^{i'_{l'-1}-1}$ are distinct and differ by less than k. On the other hand, by the definition, the main color of e is $\alpha - 2^{i_0-1} - 2^{i_1-1} - \cdots - 2^{i_{l-1}-1}$ and the main color of e' is $\alpha - 2^{i'_0-1} - 2^{i'_1-1} - \cdots - 2^{i'_{l'-1}-1}$. This proves the lemma. ∎

Lemma 5. *In a smooth outcome, if e and e' are edges of the cause tree with root e_0 and neither of them blames the other, then e and e' are disjoint.*

Proof. Assume that e and e' have a common vertex v and both belong to a cause tree with the root e_0. Then there exist two sequences $e_0, e_1, \dots, e_l = e$ and $e'_0 = e_0, e'_1, \dots, e'_{l'} = e'$ such that e_j i_j-blames e_{j+1} for $j = 0, 1, \dots, l - 1$ and e'_j i'_j-blames e'_{j+1} for $j = 0, 1, \dots, l' - 1$. Furthermore, $i_0 > i_1 > \cdots > i_{l-1}$, $i'_0 > i'_1 > \cdots > i'_{l-1}$.

Claim 1. $i_{l-1} \neq i'_{l'-1}$.

Proof of Claim. If $i_{l-1} = i'_{l'-1}$, then e and e' both were monochromatic at the end of Stage $i_{l-1} - 1$. But, by Lemma 4, their main colors differ. This proves the claim.

Thus below we can assume that $i_{l-1} < i'_{l'-1}$. It follows that e ceased to be monochromatic before e' did. In particular, v was recolored from $\mu(e)$ to $\mu(e')$. This yields the following claim.

Claim 2. $\mu(e') - \mu(e)$ *(modulo k) is a power of* 2.

We can say more.

Claim 3. $\mu(e') - \mu(e) = 2^{i_{l-1}-1}$.

Proof of Claim. Recall that

$$\mu(e') - \mu(e) = (\alpha - 2^{i'_0-1} - 2^{i'_1-1} - \cdots - 2^{i'_{l'-1}-1}) - (\alpha - 2^{i_0-1} - 2^{i_1-1} - \cdots - 2^{i_{l-1}-1}).$$

In this expression, α cancels out and every other summand apart from $2^{i_{l-1}-1}$ is divisible by $2^{i_{l-1}}$. Together with Claim 2, this yields the claim.

Claim 3 implies that v was recolored during Stage i_{l-1} and thus $\mu(e') = \mu(e_{l-1})$. This contradicts Lemma 4. ■

Lemma 5 implies that every cause tree in every smooth outcome is an r-uniform hypergraph tree in the usual sense rooted at an edge. A cause tree with root e_0 in a smooth outcome S will be called the (S, e_0)-tree.

Lemma 6. *Let $\lambda(l)$ denote the maximal possible number of edges in a cause tree T under the condition that $\mu(e_1) - \mu(e_2) \in \{1, 2, \ldots, 2^{l-1}\}$ (modulo k) for every pair of edges (e_1, e_2) such that e_1 blames e_2. Then for every $l \geq 0$, $\lambda(l) \leq 2^l$.*

Proof. If e_1 i_1-blames e_2 and e_2 i_2-blames e_3, then $i_2 < i_1$. Thus, under conditions of the lemma, for the root e_0 and an arbitrary edge e of the tree, we have

$$\mu(e_0) - \mu(e) \in \{1, 2, \ldots, 2^{l-1} + 2^{l-2} + \cdots + 1\} = \{1, 2, \ldots, 2^l - 1\}.$$

Now, Lemma 4 implies that T has at most $1 + (2^l - 1)$ vertices. ■

4. THE OUTLINE OF A PROOF OF THE MAIN RESULT

Let $G = (V, E)$ be an r-uniform hypergraph with $|E| = m \leq \epsilon k^r (r/(\ln r))^{n/(n+1)}$. We want to prove that with a positive probability EVOLUTION produces a proper k-coloring of G. To do this, we will split the event of a "bad" outcome into smaller events and estimate the probabilities of these smaller events. For an edge $e \in E$ and a color α, let $W(e, \alpha)$ be the event that the outcome is smooth and e is monochromatic of color α at the end of Stage n. The goal is to prove that the probability $\mathbf{Pr}(e, \alpha)$ of $W(e, \alpha)$ is less than $1/2km$ for every $e \in E$ and every $\alpha \in \{0, 1, \ldots, k - 1\}$. Since the probability that the outcome is not smooth is at most $me^{-0.5cr} = o(1)$, this would immediately yield the theorem.

Consider a smooth outcome S such that $W(e_0, \alpha)$ happens. Then e_0 is the root of a cause tree (the (S, e_0)-tree) T, and every edge e of T has the main color $\mu_S(e)$. An (S, e_0)-tree in which every edge e is assigned the color $\mu_S(e)$ will be called *the colored* (S, e_0)*-tree*. Note that there are at most m^{b-1} possible edge sets of cause trees with root e_0 and b edges. Each such set can generate at most k^{b-1} different colored cause trees. Thus, in order to prove that $\mathbf{Pr}(e_0, \alpha) < 1/2km$, it is enough to show that the probability $\mathbf{Pr}(e_0, \alpha, T)$ of the event $W(e_0, \alpha, T)$ that $W(e_0, \alpha)$ happens and the colored cause tree is a given T (i.e., we fix the set of edges of T and the main colors of the edges) is less than $m^{-b}k^{-b+1}(1/2k) = m^{-b}k^{-b}/2$. In the next section, we prove that

$$\mathbf{Pr}(e, \alpha, T) \le \epsilon m^{-b}. \tag{1}$$

Since $b \le k$ by Lemma 6 and thus $k^{-b}/2 \ge e^{-k(\ln k)-1} > \epsilon$, the inequality (1) is sufficient to prove the theorem.

5. AUXILIARY PROCEDURES

In order to prove (1), we will introduce auxiliary coloring procedures on small subgraphs of G, then prove that the probabilities of appropriate events in the new probability spaces dominate the values we want to estimate, and finally estimate the probabilities of these new events.

For every smooth outcome S and a colored (S, e_0)-tree T, we will consider the following modification EVOLUTION(T) of Procedure EVOLUTION, restricted to the vertices of the edges in T.

STAGE 0. Color every vertex randomly and independently, with a color in $\{0, 1, \ldots, k-1\}$, chosen uniformly in this set.

STAGE l, $l = 1, \ldots, n$.

STEP $i + s(l-1)$, $1 \le i \le s$. Let v be a vertex of T. If v has once changed its color, it never changes the color again. If it hasn't ever changed its color, then we consider two cases.

CASE 1. v belongs to exactly one edge $e(v)$. Let $\gamma(v)$ denote the current color of v. With probability p we recolor v with color $\gamma(v) + 2^{l-1}$ (modulo k) if either $e(v)$ is monochromatic of color $\mu(e(v))$ or if $\mu(e(v)) - \gamma(v) = 2^{l-1}$. Otherwise, do nothing with v.

CASE 2. v belongs to exactly two edges $e(v)$ and $e'(v)$, and the level of $e(v)$ in T is less than the level of $e'(v)$. If $e'(v)$ is monochromatic of color $\mu(e'(v))$, then with probability p we recolor v with color $\gamma(v) + 2^{l-1}$ (modulo k). Otherwise, do nothing with v.

Remark 4. The reader can check again the proofs of Lemmas 1, 2, and 3 and see that their statements hold also for EVOLUTION(T). And the notions of *blaming* and *cause trees* introduced immediately after Lemma 3 make complete sense for EVOLUTION(T), as well.

Thus, we can define the event $W_T(e_0, \alpha, T)$ that the edge e_0 is monochromatic of color α at the end of Procedure EVOLUTION(T) and the colored cause tree is the T itself. Of course, this might happen only if $\alpha = \mu(e_0)$. Let $\mathbf{Pr}_T(e_0, \alpha, T)$ denote the probability of $W_T(e_0, \alpha, T)$.

Lemma 7. $\mathbf{Pr}_T(e_0, \alpha, T) \geq \mathbf{Pr}(e_0, \alpha, T)$.

Proof. Consider what should happen in EVOLUTION (respectively, EVOLUTION(T)) if $W(e_0, \alpha, T)$ (respectively, $W_T(e_0, \alpha, T)$) takes place. Note that Stage 0 for vertices in T is the same in both procedures, EVOLUTION and EVOLUTION(T). For both events $W(e_0, \alpha, T)$ and $W_T(e_0, \alpha, T)$, by the definitions of the cause trees for e_0, every vertex v belonging to two edges $e(v)$ and $e'(v)$ of T and such that the level of $e(v)$ in T is less than the level of $e'(v)$ must be recolored from $\mu(e'(v))$ to $\mu(e(v))$ during Stage $\log_2(\mu(e(v)) - \mu(e'(v)))$ and $e'(v)$ must be monochromatic [of color $\mu(e'(v))$] at this moment. By Lemma 1, for every vertex v in T belonging to exactly one edge $e(v)$, the probability that v becomes eventually of the color $\mu(e(v))$ in EVOLUTION(T) is at least as big as in EVOLUTION (since in EVOLUTION v might be recolored into some other color). And the stage when the vertices of an edge e can be recolored from $\mu(e)$ to other colors is defined already by Stage 0, which is the same in both procedures. Also, in EVOLUTION some vertices of color $\mu(e)$ in an edge $e \in E(T)$ might be recolored from $\mu(e)$ because of some other edges, while in EVOLUTION(T) the only reason for it is that e is monochromatic. This proves the lemma. ■

The reason to introduce EVOLUTION(T) is that some auxiliary events in this model are independent and hence some probabilities are easier to estimate. For every edge e of T, let T_e be the subtree of T with the root e induced by e and all its descendants in T, where the main colors of all edges of T_e are the same as in T. For every e in T and every $0 \leq l \leq n$, let $W_T(e, \mu(e), T_e, l)$ denote the event that at the end of Stage l of EVOLUTION(T) the edge e will be monochromatic of color $\mu(e)$ and the cause tree for e will be T_e. Let $\mathbf{Pr}_T(e, \mu(e), T_e, l)$ denote the probability of $W_T(e, \mu(e), T_e, l)$

Lemma 8. *Let e_1 and e_2 be two edges of a colored cause tree T with root e_0 such that neither of them is an ancestor of the other. Then the events $W_T(e_1, \mu(e_1), T_{e_1}, l_1)$ and $W_T(e_2, \mu(e_2), Te_2, l_2)$ are independent for every l_1 and l_2.*

Proof. By Lemma 5, the vertex sets of T_{e_1} and T_{e_2} are disjoint. According to the definition of EVOLUTION(T), $W_T(e_1, \mu(e_1), T_{e_1}, l_1)$ does not depend on the colors of vertices not belonging to T_{e_1}. ■

Lemma 8 allows us to estimate $\mathbf{Pr}_T(e_0, \mu(e_0), T)$. The following lemma is crucial.

Lemma 9. *Let T be a colored cause tree with root e_0. Let e be an edge of T and $0 \leq l \leq n$. If T_e has b edges, then*

$$\mathbf{Pr}_T(e, \mu(e), T_e, l) \leq \epsilon m^{-b}\left(\frac{r}{\ln r}\right)^{(n-l)/(n+1)}.$$

Proof. We use induction on l. Consider first $l = 0$. In this case, T_e should be a single edge e. By the definition of Stage 0 (recall that $m \leq \epsilon k^r (r/(\ln r))^{n/(n+1)}$),

$$\mathbf{Pr}_T(e, \mu(e), T_e, 0) = (k)^{-r} \leq \frac{\epsilon}{m}\left(\frac{r}{\ln r}\right)^{n/(n+1)}.$$

This proves the case $l = 0$.

Now, suppose that the lemma holds for every $l' < l$. Consider a colored cause subtree T_e of T with root e and b edges.

CASE 1. e was monochromatic of color $\mu(e)$ already at the end of Stage $l - 1$ of EVOLUTION(T). By the induction assumption, the probability of this event is at most

$$\epsilon m^{-b}\left(\frac{r}{\ln r}\right)^{(n-l+1)/(n+1)}.$$

As in the proof of Lemma 2, the probability that more than z vertices of e had changed their colors is less than $\epsilon^{0.5r}$. If at most z vertices of e had changed their colors, then, on each of the s steps of Stage l, each of the remaining at least $r - z$ vertices of e with probability p will change its color. The probability that none of them succeeds is at most

$$((1 - p)^{r-z})^s \leq \exp\left\{-psr\left(1 - \frac{c}{\ln r}\right)\right\}.$$

Since $p < 2^{-k'r}$ and $ps \geq (\ln r)/(n + 1)r - p$ by the definition of p and s, the last expression is at most

$$\exp\left\{-\left(\frac{\ln r}{n+1} - pr\right)\left(1 - \frac{c}{\ln r}\right)\right\} \leq \exp\left\{-\frac{\ln r}{n+1} + \frac{c}{n+1} + pr\right\}$$
$$\leq r^{-1/(n+1)}e^c = \frac{1}{\epsilon}r^{-1/(n+1)}.$$

Thus,

$$\mathbf{Pr}_T(e, \mu(e), T_e, l) \leq \epsilon^{0.5r} + \epsilon m^{-b}\left(\frac{r}{\ln r}\right)^{(n-l+1)/(n+1)}\epsilon^{-1}r^{-1/(n+1)},$$

and the last expression is at most what we need when $\ln r \geq 2/\epsilon$.

CASE 2. e becomes monochromatic of color $\mu(e)$ at Stage l of EVOLUTION(T). Let $e_1, \ldots, e_q$ be the edges of level 1 in T_e and b_j be the number of edges in T_{e_j}, $j = 1, \ldots, q$. Suppose also that $\mu(e) - \mu(e_j) = 2^{i_j - 1}$ and $i_1 > i_2 > \cdots > i_q$. By the definition of our case, $i_1 = l$. We estimate the probability of the event $W_T(e, \mu(e), T_e, l)$ under the condition that the set of vertices of color $\mu(e_j)$ at the end of Stage 0 in e apart from v_j is A_j, and $|A_j| = a_j$ for $j = 1, \ldots, q$. Consider an arbitrary e_j.

In order to recolor the common vertex v_j of e and e_j with $\mu(e)$, the edge e_j must be monochromatic of color $\mu(e_j)$ at the end of Stage $i_j - 1$. By the induction assumption, the probability of it is at most

$$\epsilon m^{-b_j}\left(\frac{r}{\ln r}\right)^{(n-i_j+1)/(n+1)}.$$

Assume that v_j gets color $\mu(e)$ at Step h_j of Stage $i_j - 1$. If at most z vertices of e_j had changed their colors, then the probability that none of the remaining at least $r - z$ vertices of e_j changes its color on first $h_j - 1$ steps of Stage $i_j - 1$ is at most $((1-p)^{r-z})^{h_j-1}$. The probability that v_j is recolored at Step h_j is at most p. By Lemma 1, the probability that all vertices in A_j are recolored with $\mu(e)$ at the end of Step h_j is at most $(h_j p)^{a_j}$.

By Lemma 8, with fixed A_j and h_j the corresponding events are mutually independent for different j. Therefore,

$$\begin{aligned}\mathbf{Pr}_T(e, \mu(e), T_e, l) &\le (b\epsilon^{0.5r}) + k^{-r} \\ &\times \sum_{a_1=0}^{r-q}\sum_{a_2=0}^{r-q-a_1}\cdots\sum_{a_q=0}^{r-q-a_1-\cdots-a_{q-1}}\binom{r-q}{a_1}\binom{r-q-a_1}{a_2}\cdots\binom{r-q-a_1-\cdots-a_{q-1}}{a_q} \\ &\times \prod_{j=1}^{q}\epsilon m^{-b_j}\left(\frac{r}{\ln r}\right)^{(n-i_j+1)/(n+1)} p\sum_{h_j=1}^{s}((1-p)^{r-z})^{h_j-1}(h_j p)^{a_j}.\end{aligned}$$

Here the first summand in the first line stands for the probability that the outcome is not smooth. For every j, we can estimate

$$\begin{aligned}p\sum_{a_j=0}^{r-q-a_1-\cdots-a_{j-1}}&\binom{r-q-a_1-\cdots-a_{j-1}}{a_j}\sum_{h_j=1}^{s}((1-p)^{r-z})^{h_j-1}(h_j p)^{a_j} \\ &\le p(1-p)^{-zs-r}\sum_{h_j=1}^{s}((1-p)^{r})^{h_j}\sum_{a_j=0}^{r}\binom{r}{a_j}(h_j p)^{a_j} \\ &\le pe^{p/(1-p)(zs+r)}\sum_{h_j=1}^{s}(1-p)^{rh_j}(1+h_j p)^{r} \\ &\le pe^{(pr)/(1-p)+(z\ln r)/[(1-p)(n+1)r]}\sum_{h_j=1}^{s}(1-p)^{rh_j}(1+p)^{rh_j} \le (ps)e^{(pr)/(1-p)+c/[(1-p)(n+1)]} \\ &\le \frac{\ln r}{(n+1)r}e^{c} = \frac{\ln r}{(n+1)r\epsilon}.\end{aligned}$$

Thus,

$$\mathbf{Pr}_T(e, \mu(e), T_e, l) \le (b\epsilon^{0.5r}) + k^{-r}m^{-b_1-\cdots-b_q}\prod_{j=1}^{q}\epsilon\left(\frac{r}{\ln r}\right)^{(n-i_j+1)/(n+1)}\frac{\ln r}{(n+1)r\epsilon}.$$

Note that

$$k^{-r}m^{-b_1-\cdots-b_q} = k^{-r}m^{-b+1} \le m^{-b}\epsilon\left(\frac{r}{\ln r}\right)^{n/(n+1)}$$

and that, for every j,

$$\epsilon\left(\frac{r}{\ln r}\right)^{(n-i_j+1)/(n+1)} \frac{\ln r}{(n+1)r\epsilon} \le \frac{1}{n+1} < 1.$$

Recall that $i_1 = l$. Hence,

$$\begin{aligned}\mathbf{Pr}_T(e, \mu(e), T_e, l) &\le (b\epsilon^{0.5r}) + m^{-b}\epsilon\left(\frac{r}{\ln r}\right)^{n/(n+1)} \epsilon\left(\frac{r}{\ln r}\right)^{(n-i_1+1)/(n+1)} \frac{\ln r}{(n+1)r\epsilon} \\ &\le (b\epsilon^{0.5r}) + 0.5\epsilon m^{-b}\left(\frac{r}{\ln r}\right)^{(n-l)/(n+1)} \le \epsilon m^{-b}\left(\frac{r}{\ln r}\right)^{(n-l)/(n+1)}. \quad \blacksquare\end{aligned}$$

Applying Lemma 9 to e_0 for $l = n$, we get

$$\mathbf{Pr}_T(e_0, \mu(e_0), T_{e0}) \le \epsilon m^{-b}.$$

But $\mathbf{Pr}_T(e, \mu(e_0), T_{e_0}, n) = \mathbf{Pr}_T(e, \mu(e_0), T)$. Thus, the above inequality together with Lemma 7 yields (1) and therefore the theorem.

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