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PRECOLORING EXTENSIONS OF BROOKS' THEOREM*

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Abstract. Let G be a connected graph with maximum degree k (other than a complete graph or odd cycle), let W be a precolored set of vertices in G inducing a subgraph F, and let D be the minimum distance in G between components of F. If the components of F are complete graphs and $D \geq 8$ (for $k \geq 4$) or $D \geq 10$ (for $k = 3$), then every proper k-coloring of F extends to a proper k-coloring of G. If the components of F are single vertices and $D \geq 8$, and the vertices outside W are assigned color lists of size k, then every k-coloring of F extends to a proper coloring of G with the color on each vertex chosen from its list. These results are sharp.

Key words. coloring extension, list coloring, Brooks' Theorem

AMS subject classification. 05C15

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1. Introduction.

For $k \geq 3$, the famous theorem of Brooks [6] states that a graph with maximum degree k is k-colorable if it does not have K_{k+1} as a component. Our general aim in this paper is to strengthen this result by allowing some vertices to have arbitrarily specified colors.

Albertson [1] proved that if a set W of vertices in an r-colorable graph is separated pairwise by distance at least 4, then every coloring of W from a set of $r + 1$ colors extends to a proper $(r+1)$-coloring of W. The result was generalized by letting $G[W]$ be a disjoint union of complete graphs with at most j vertices, where $G[W]$ denotes the subgraph of G induced by W. If the components of $G[W]$ are far enough apart, then every proper $(r + 1)$-coloring of $G[W]$ extends to a proper $(r + 1)$-coloring of G. Albertson showed that distance $6j - 2$ is enough, Kostochka (see [2]) lowered the threshold to $4j$, and Albertson and Moore [2] showed that distance $3j$ suffices when $j = r$.

These results require an extra color; generally a partial coloring of an r-chromatic graph may not extend to a proper r-coloring, regardless of the distance between the precolored vertices. Can the extension conclusions be strengthened when more colors are allowed? With $\Delta(G) + 1$ colors allowed, every partial proper coloring extends, even in a list coloring sense. That is, if each uncolored vertex has a list of $\Delta(G) + 1$ available colors, then we can extend a partial coloring in an arbitrary vertex order; when we reach a vertex, there is always a color in its list that has not already been used on any neighbor. What happens when only $\Delta(G)$ colors are allowed?

THEOREM 1.1. *Let W be a set of vertices in a graph G with $\Delta(G) \geq 4$ and $K_{\Delta(G)+1} \not\subseteq G$. If the components of $G[W]$ are complete graphs and the distance between any two such components is at least 8, then every proper $\Delta(G)$-coloring of*

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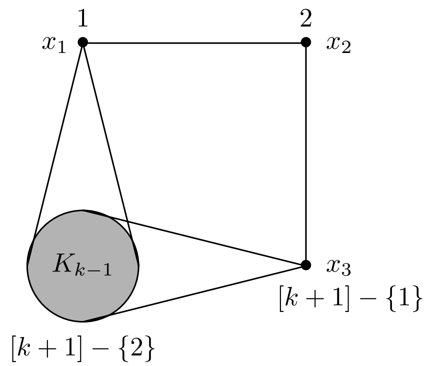

FIG. 1. *Failure of list coloring extension.*

$G[W]$ *extends to a proper* $\Delta(G)$*-coloring of* G*. The same statement holds for* $\Delta(G) = 3$ *with* 10 *in place of* 8*.*

We will shortly present examples showing that Theorem 1.1 is sharp, except that distance 8 suffices when $\Delta(G) = 3$ if the components of $G[W]$ are isolated vertices (as guaranteed by Theorem 1.2 below).

In the special case when G has chromatic number $\Delta(G)$, Theorem 1.1 provides an extension theorem using no "extra" colors. As discussed in [3], such results are rare. Also, in comparison to the earlier result of Kostochka, Theorem 1.1 shows that with $\Delta(G)$ colors, the sizes of the components of $G[W]$ are irrelevant, and there is a constant distance that suffices.

Our second result, proved together with the first, is a list version of the theorem when W is an independent set. This result was also proved independently by Axenovich [4].

THEOREM 1.2. *Let* W *be a set of vertices in a graph* G *with* $\Delta(G) \geq 3$*. Let* L *be a function that assigns to each vertex a list of* $\Delta(G)$ *available colors. If the distance between any two vertices of* W *is at least* 8*, then every coloring of* W *chosen from the lists extends to a proper coloring* f *of* G *such that* $f(v) \in L(v)$ *for all* $v \in V(G)$*.*

Using the word "list" for the set of colors available for a vertex is standard in this setting. A function L assigning a list to each vertex is a *list assignment* for a graph G, and a proper coloring f such that $f(v) \in L(v)$ for all $v \in V(G)$ is an L*-coloring*. Since L can assign the same list of $\Delta(G)$ colors at each vertex, Theorem 1.2 strengthens the special case of Theorem 1.1 where the components of $G[W]$ are single vertices. Since the claim is made for each choice of colors on W, we may view the precoloring on W as lists of size 1. In discussing lists, it is helpful to use the notation $[k]$ for the set $\{1, \ldots, k\}$.

When $G[W]$ is not an independent set, no list extension theorem is possible. For $k \geq 2$, consider the graph shown in Figure 1. It consists of a path with vertices x_1, x_2, x_3 in order and a copy of K_{k-1} whose vertices are adjacent to x_1 and x_3. All vertices have degree k, except that x_2 has degree 2. The colors on x_1 and x_2 are specified as 1 and 2, respectively. Let $L(x_3) = [k+1] - \{1\}$, and let $L(v) = [k+1] - \{2\}$ for each v outside $\{x_1, x_2, x_3\}$. In a proper extension of the coloring on $\{x_1, x_2\}$, some color j outside $\{1, 2\}$ must be used on x_3. Since the remaining vertices have list $[k+1] - \{2\}$, no color in $\{1, 2, j\}$ can be used on these vertices, which leaves only $k - 2$ available colors in $[k+1]$ for the copy of K_{k-1}.

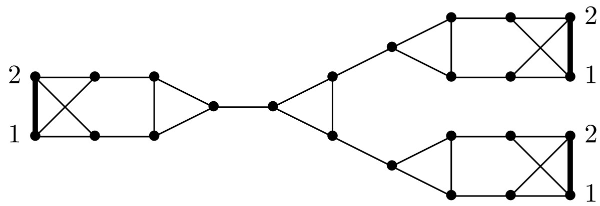

FIG. 2. *Failure of* Δ*-extension when* $\Delta(G) = 3$ *and distance is* 9.

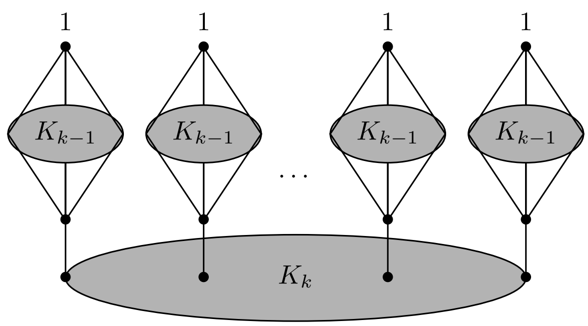

FIG. 3. *Failure of extension for distance* 7.

For $\Delta(G) = 2$, there is no extension theorem with $\Delta(G)$ colors, since extendibility of a coloring of two points on a long path depends on the parity of the distance between them.

For $\Delta(G) = 3$, the graph in Figure 2 shows that the distance threshold of 10 for the extension theorem with nontrivial cliques is sharp. Here the precolored set W consists of the three peripheral 2-cliques shown in bold. A proper 3-coloring that extends this must have the third color on each vertex neighboring the central triangle, but then the coloring cannot be extended to the center. The distance between two precolored cliques is 9.

For $\Delta(G) \geq 4$ in Theorem 1.1 and $\Delta(G) \geq 3$ in Theorem 1.2, the graph in Figure 3 shows that the distance threshold of 8 is sharp. In the list case, we use the same list $[k]$ on all vertices of $G - W$. To construct G, first let H consist of K_{k+1} with one edge deleted and instead a pendant edge attached to one of the deficient vertices. Let G consist of k disjoint copies of H plus edges making the pendant vertices in the copies of H into a clique. Let W consist of the vertices of degree $k - 1$, all given the same color. Although G has maximum degree k, and the distance between any two vertices of W is 7, this coloring of W does not extend to a proper k-coloring of G.

The proofs of the upper bounds use many common ideas, so we develop them together. The general approach is to derive contradictory properties for a minimal counterexample.

2. Background and preliminaries. A precoloring extension problem can be modeled as a list coloring problem. The colors on the precolored vertices are removed from the lists of colors available at their neighbors. If the number of colors available at a vertex is at least its degree, then this remains true after the precolored vertices are deleted, because at most one color is lost for each neighbor deleted.

We use $d_G(v)$ or simply $d(v)$ to denote the degree of a vertex v in a graph G (all our graphs are simple). Also $d_G(u, v)$ or $d(u, v)$ denotes the distance between vertices u and v in G. We extend this notation to vertex subsets: $d_G(A, B)$ is the minimum of the distances in G between a vertex of A and a vertex of B.

A graph G is *degree-choosable* if it has an L-coloring whenever L is a list assignment with $|L(v)| \geq |d(v)|$ for all $v \in V(G)$. We say that such a list assignment is *supervalent*.

Given a supervalent list assignment L for G, Vizing [9] showed that if the size of some list exceeds the degree of the vertex, or if G is 2-connected and the lists are not identical, then G has an L-coloring. This and its consequence that a connected graph having a degree-choosable induced subgraph is also degree-choosable are easy to prove.

These observations lead to the characterization of degree-choosable graphs by Erdős, Rubin, and Taylor [7]: A connected graph fails to be degree-choosable if and only if it is a *Gallai tree*, which is a connected graph in which every block is a complete graph or an odd cycle. Furthermore, the lists in a supervalent list assignment not permitting a proper coloring have a restricted form.

THEOREM 2.1 (Borodin [5], Erdős–Rubin–Taylor [7]). *If L is a supervalent list assignment for a connected graph G and there is no L-coloring of G, then*

(a) $|L(v)| = d(v)$ *for every* $v \in V(G)$.

(b) *G is a Gallai tree.*

(c) $L(v) = \cup_{B \in \mathcal{B}(v)} L_B$ *for all* $v \in V(G)$*, where* $\mathcal{B}(v)$ *is the set of blocks containing* v*, and for each block* B*,* L_B *is a set of* $\chi(B) - 1$ *colors.*

A short proof of Theorem 2.1 appears in [8]. Note that each block B is an $|L_B|$-regular graph, and all vertices of a single block that are not cut-vertices of G have the same list.

Henceforth let $k = \Delta(G)$, and let f denote a precoloring of W. In the setting of Theorem 1.1, which we call the *clique case*, f is a proper k-coloring of $G[W]$ using the set $[k]$ as colors. In the setting of Theorem 1.2, which we call the *list case*, the coloring f is any proper coloring of $G[W]$. We discuss both cases together as a list coloring problem by defining $L(v) = [k]$ for all $v \in V(G) - W$ in the clique case. Hence the "theorem statement" refers to both theorems together. Let $N_G(v)$ denote the neighborhood of a vertex v in a graph G.

CLAIM 1. *Let G be a graph with maximum degree k, precoloring f, lists L, and precolored set W. If G is a smallest counterexample to the theorem statement, then the following hold for the graph H defined by $H = G - W$ and the list assignment L_f on $V(H)$ defined by $L_f(v) = L(v) - f(N_G(v) \cap W)$.*

(a) *H is connected.*

(b) *Every component of H is a Gallai tree, and in every block the lists L_f on the non-cut-vertices are the same and have size equal to vertex degree.*

(c) *If* $v \in V(H)$*, then* $d_G(v) = k$.

Proof. (a) When W is a separating set in G, extension of the coloring to the various components of $G - W$ is independent, and deleting one does not violate the hypotheses of the theorem. By the minimality of G, we may therefore assume that H is connected.

(b) An L_f-coloring for H would permit the extension of the coloring for G, so there is no L_f-coloring for H. Since $|L(v)| = k$ and we lose at most one color for each lost neighbor, $|L_f(v)| \geq d_H(v)$ for all $v \in V(H)$. Hence L_f is supervalent, and H has no L_f-coloring, so Theorem 2.1 applies to H and immediately yields the claim.

(c) For $v \in V(H)$,

$$d_H(v) = |L_f(v)| = |L(v) - f(N_G(v) \cap W)| \geq |L(v)| - |N_G(v) \cap W|.$$

Since $d_G(v) = d_H(v) + |N_G(v) \cap W|$, we obtain $d_G(v) \geq |L(v)| = k$. Since $\Delta(G) = k$, equality holds. □

Henceforth we maintain the notation (k, f, L, W, H, L_f) and assumptions (G is a smallest counterexample) of Claim 1. By the *distance requirement*, we mean the hypothesis that the distance between components of $G[W]$ is at least 8 in general and is at least 10 when $k = 3$ and we are in the clique case.

Remarks. The computation in the proof of Claim 1(c) implies that the colors used on neighbors of v in W are distinct and appear in $L(v)$. By the distance requirement, $N_G(v) \cap W$ lies in a single component of $G[W]$. In the clique case, $|N_G(v) \cap W| \leq k-1$, since $K_{k+1} \not\subseteq G$. In the list case, $\delta(H) \geq k-1$, since W consists of isolated vertices.

We next consider the edges joining $V(H)$ and W. A *leaf block* in a graph H is a block of H containing at most one cut-vertex of H. For a block B in H, we henceforth let B' denote the set of vertices in B that are not cut-vertices of H.

CLAIM 2. *Let B be a leaf block of H in a smallest counterexample G, and let $m = |V(B)|$.*

(a) *The neighbors in W of vertices in B lie in the same component of $G[W]$; call it $Q(B)$.*
(b) *Every vertex in $Q(B)$ is adjacent to all or none of the set B' of non-cut-vertices in B.*
(c) *B is a complete graph, and $Q(B)$ has exactly $k-m+1$ vertices with neighbors in B and at most one vertex with no neighbors in B.*
(d) *H has more than one block.*

Proof. (a) This follows immediately from the distance requirement, since all vertices of B except possibly one have neighbors in W (by Claim 1(c)), and the distance between neighbors of adjacent vertices of B is at most 3.

(b) By Claim 1(b), the lists under L_f are the same for all $v \in B'$; let S be this common list. By Theorem 2.1(a), $|S| = d_H(v)$. By part (a), $N_G(v) \cap W$ lies in a single component of $G[W]$, so the colors assigned to its vertices by f are distinct. In the clique case, $L(v) = [k]$, so arriving at S requires each vertex of B' to lose the same colors from its list.

In the list case, $|L(v)| = k$ for $v \in B'$, and $Q(B)$ consists of only one vertex. Also $K_{k+1} \not\subseteq G$ implies $d_H(v) < k$, so each $v \in B'$ loses one color from its list. Thus the one vertex of $Q(B)$ is adjacent to all of B'.

(c) If B is a cycle of length at least 5, then by Claim 1(c) and part (a), each vertex of B' has two neighbors in B and $k-2$ neighbors in $Q(B)$. By part (b), these neighbors in $Q(B)$ have degree at least $k-3+4$, since $|B'| \geq 4$. This degree would exceed $\Delta(G)$.

Hence B is a complete graph, by Claim 1(b). The vertices of B' have $m-1$ neighbors in H. By Claim 1(c), they have $k-m+1$ neighbors in $Q(B)$. By part (b), these are always the same $k-m+1$ vertices.

These $k-m+1$ vertices in $Q(B)$ have $k-m$ neighbors among themselves and at least $m-1$ neighbors in B', so they have at most one more neighbor in $Q(B)$.

(d) If H has only one block, then the first statement in part (c) makes it a complete graph, but the second then yields $K_{k+1} \subseteq G$, which is forbidden. □

3. Leaf blocks. We begin with a tool for studying the structure of leaf blocks of H. As before, B' is the set of vertices in B that are not cut-vertices of H.

CLAIM 3. *There is no partial extension of f to a partial coloring f' that gives the same color to two neighbors of an uncolored vertex of H.*

Proof. Let f' be such an extension, and let U be the set of vertices outside W to which f' assigns colors. Note that $f'(u) \in L_f(u)$ is required for all $u \in U$. Let $G' = G - W - U$; note that G' is an induced subgraph of H.

For $v \in V(G')$, let $L_{f'}(v) = L(v) - f(N_G(v) \cap (W \cup B'))$. By the same argument as for L_f, we have $|L_{f'}(v)| \geq d_{G'}(v)$ for all $v \in V(G')$. Also, if $x \in V(G')$ has neighbors with the same color under f', then $|L_{f'}(x)| > d_{G'}(x)$. By Theorem 2.1, G' then has an $L_{f'}$-coloring, which yields an L-coloring of G. Since G has no L-coloring, there is no such f'. ∎

When B is a leaf block of H, we let x_B denote the cut-vertex of G contained in B. If B has m vertices, then Claim 2(c) yields $|Q(B)| \in \{k-m+1, k-m+2\}$. Define B to have *Type* j when $|Q(B)| = k-m+j$. In the list case, always $|Q(B)| = 1$, which requires that $m = k$ and that $B \cong K_k$ and B has Type 1.

Let $Q'(B)$ denote the set of vertices in $Q(B)$ having neighbors in $V(B)$. If B has Type 1, then $Q'(B) = Q(B)$, and each vertex of $Q(B)$ may have one neighbor that is not in $B' \cup Q(B)$. If B has Type 2, then vertices of $Q'(B)$ have no such additional neighbors, and we let w_B denote the vertex of $Q(B) - Q'(B)$.

CLAIM 4. *If B is a leaf block of Type 2, then x_B has no neighbor in $Q(B)$.*

Proof. By Claim 2, $B' \cup Q'(B)$ is a clique of size k. Since also $Q'(B) \subseteq N_G(w_B)$, x_B has no neighbor in $Q'(B)$. Finally, since leaf blocks of Type 2 occur only in the clique case, every extension of f to B' uses on B' all the colors not used on $Q'(B)$, including $f(w_B)$. If x_B is adjacent to w_B, then we have formed a partial extension of f that is forbidden by Claim 3. ∎

CLAIM 5. *In the clique case, if B is a leaf block of Type 1 and $y \in N_H(x_B) - B'$, then y has no neighbor in $Q(B)$.*

Proof. Suppose that $wy \in E(G)$, where $w \in Q(B)$. Since w also has $k-1$ neighbors in $B' \cup Q(B)$, we conclude that $x_B w \notin E(G)$. Since $B' \cup Q(B)$ is a clique, we can extend f to B by using on B' the colors of $[k] - f(Q(B))$, and then we can use color $f(w)$ on x_B. This partial extension gives the same color to two neighbors of y, which violates Claim 3. Hence no such edge wy exists. ∎

CLAIM 6. *Let B_1 and B_2 be distinct leaf blocks in H such that $Q(B_1) = Q(B_2)$.*

(a) *If $Q'(B_1) \cap Q'(B_2) = \varnothing$, then B_1 and B_2 have Type 2 with k vertices, and $|Q(B_1)| = 2$.*

(b) *If $Q'(B_1) \cap Q'(B_2) \neq \varnothing$, then B_1 and B_2 have Type 1 with two vertices, and $|Q(B_1)| = k-1$ (also $Q'(B_1) = Q'(B_2) = Q(B_1)$).*

(c) *The condition in the hypothesis arises only in the clique case.*

(d) *There is no third leaf block B_3 with $Q(B_3) = Q(B_1)$.*

(e) $x_{B_1} \neq x_{B_2}$.

Proof. (a) By Claim 2(c), each $Q(B_i)$ has at most one vertex not in $Q'(B_i)$, and it is the only candidate for $Q'(B_{3-i})$. Since $|B'_i \cup Q'(B_i)| = k$, the sizes are as claimed.

(b) Consider $u \in Q'(B_1) \cap Q'(B_2)$. For $i \in \{1, 2\}$, since $u \in Q'(B_i)$, there is at most one neighbor of u outside $B'_i \cup Q'(B-i)$, and such a neighbor exists in B'_{3-i}. Hence B_i has Type 1, and $Q'(B_i) = Q(B_i)$. Also $|B'_{3-i}| = 1$, so B_{3-i} has two vertices and $Q(B_{3-i}) = k-1$.

(c) The conclusions above yield $|Q(B_i)| > 1$, which occurs only in the clique case. The two possibilities are shown in Figure 4.

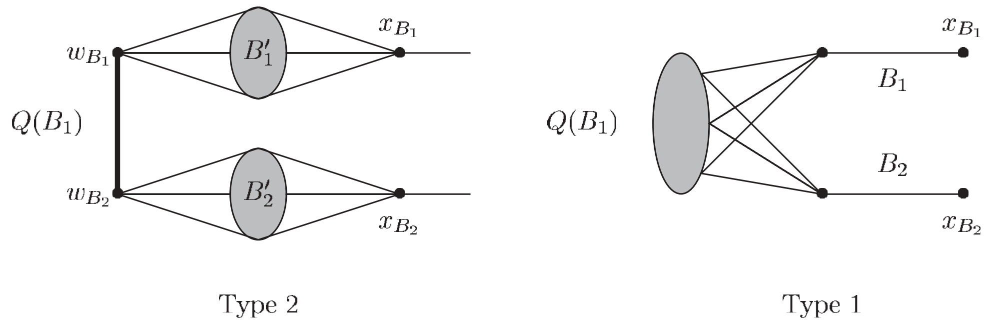

FIG. 4. *Leaf blocks B_1 and B_2 with $Q(B_1) = Q(B_2)$.*

(d) If such a B_3 exists and B_1 has Type 2, then applying part (a) to the pairs $\{B_1, B_2\}$ and $\{B_1, B_3\}$ gives $2k-1$ neighbors to w_{B_1}. If B_1 has Type 1, then applying part (b) to these pairs gives $k+1$ neighbors to each vertex of $Q(B_1)$.

(e) Suppose that $x_{B_1} = x_{B_2}$. If these blocks have Type 2, then $d_H(x_B) \geq 2k-2$, which exceeds k when $k > 2$. If they have Type 1, then let b_i be the unique vertex of B_i'. Both b_1 and b_2 have neighborhood $Q(B) \cup \{x_B\}$. Since we are in the clique case, there is only one choice of color when extending f to b_1 and b_2. Since these both neighbor x_B, Claim 3 implies that no such partial extension exists. ∎

4. Remote blocks. The *block-cutpoint graph* of a graph H has a vertex for each block in H and a vertex for each cut-vertex of H, and a cut-vertex v is adjacent to a block B if $v \in V(B)$. The block-cutpoint graph of a connected graph H is a tree, and its leaves correspond to blocks in H.

We continue to discuss a smallest counterexample G, with notation as defined in the preceding section. Let T be the block-cutpoint tree of H. We define a *remote block* in H to be a block corresponding to a vertex of maximum eccentricity in T. Our strategy will be to work our way in from a remote block, restricting the structure of H as we go.

CLAIM 7. *A remote block in H intersects only one other block in H.*

Proof. Let B be a remote block in H. If x_B lies in two non-remote blocks, then B is not remote, so at most one block containing x_B is non-remote. If at least two blocks other than B contain x_B, then at least one is a remote block C. Since neighbors of x_B in B and C have neighbors in W, the distance requirement yields $Q(C) = Q(B)$. Now $x_C = x_B$ contradicts Claim 6(e). ∎

When B is a remote block in H, we let $F(B)$ denote the other block sharing x_B. At this point $F(B)$ may be a complete graph or an odd cycle.

CLAIM 8. *Let B be a remote block in H. If C is a leaf block in H, and $d_H(x_B, x_C)$ is* 1 *when B has Type* 1 *and is at most* 3 *when B has Type* 2*, then $F(B) \cong K_2$, unless $k = 3$ and $F(B) \cong K_3$, as on the left in Figure* 5*.*

Proof. By the distance requirement, $Q(B) = Q(C)$. Now Claim 6 implies that B and C have the same Type and that this is the clique case. If B and C have Type 2, then Claim 6(a) yields $|V(B)| = k$. Hence x_B has $k-1$ neighbors in B and only one in $F(B)$, as desired.

Hence we may assume that B and C have Type 1. By Claim 6(b), $|V(B)| = |V(C)| = 2$. Since $x_B x_C \in E(H)$, Claim 5 implies that x_C has no neighbor in $Q(B)$. Thus x_C has $k-1$ neighbors in $F(B)$, so $|V(F(B))| \geq k$.

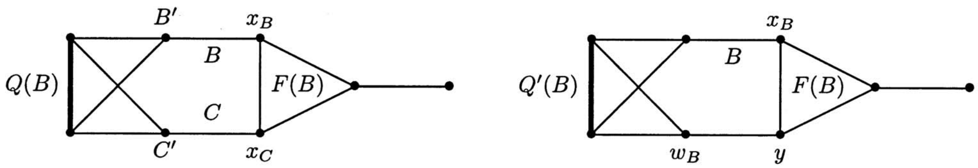

FIG. 5. *Exceptions in Claims* 8 *and* 9 *when* $k = 3$.

Claim 6(b) also yields $|Q(B)| = k - 1$. In addition to $k - 2$ neighbors within $Q(B)$, each vertex of $Q(B)$ has neighbors in B' and C'. Indeed, one neighbor is added for each neighbor of x_B or x_C in $F(B)$ that belongs to a leaf block. Hence there are no such vertices other than x_B and x_C. Also, since B is a remote block, $F(B)$ has at most one vertex belonging to a non-remote block other than $F(B)$. Since $\{x_B, x_C\}$ has at least two additional neighbors when $|V(F(B))| \geq 4$, the requirement of $|V(F(B))| \geq k$ yields $k = 3$. In that remaining case, the configuration is as on the left in Figure 5. □

CLAIM 9. *If* B *is a remote block in* H*, then* $F(B) \cong K_2$*, except that* $F(B)$ *may have three vertices as in Figure* 5 *in the clique case with* $k = 3$.

Proof. In the list case, $|Q(B)| = 1$, which forces $|V(B)| = k$. By Claim 2(c), $B \cong K_k$. Hence x_B has only one neighbor outside B. This argument applies to all leaf blocks in the list case, not just the remote ones.

Now consider the clique case. Let y be a vertex of $F(B)$ other than x_B. If B has Type 1, then Claim 5 and the distance requirement imply that y has no neighbors in W. Since $d_G(y) = k$ (by Claim 1(c)), y is a cut-vertex of H. Since B is remote, at most one vertex of $F(B)$ belongs to a non-remote block other than $F(B)$. Hence if x_B has more than one neighbor in $F(B)$, then Claim 8 implies that in fact it has only one such neighbor, unless $k = 3$ and the configuration is as on the left in Figure 5.

Otherwise, B has Type 2. Since $d_G(y) = k$ (by Claim 1(c)), and y has at most one neighbor in $Q(B)$ (namely, w_B), we conclude that y is a cut-vertex of H unless it has $k - 1$ neighbors in $F(B)$ and is adjacent to w_B.

This requires that $F(B) \cong K_k$ or that $k = 3$ and $F(B)$ is an odd cycle. In either case, x_B has $k - 1$ neighbors in $F(B)$ and only one in B, so $|V(B)| = 2$ and $|Q'(B)| = k - 1$ (by Claim 2(c)). Hence w_B has at most one neighbor outside $Q'(B)$, so at most one vertex of $F(B)$ fails to be a cut-vertex. Also, at most one vertex of $F(B)$ belongs to a non-remote block in H other than $F(B)$. Hence some vertex of $F(B)$ within distance 2 of x_B belongs to a remote block and Claim 8 finishes the proof, unless $k = 3$ and $F(B) \cong K_3$. In that remaining case, we may again have $F(B)$ with three vertices, as on the right in Figure 5. □

The two exceptional configurations in Figure 5 are essentially the same. In the clique case with $k = 3$, only the colors 1, 2, 3 can be used. When $Q'(B)$ is precolored, the common neighbors of these two vertices must have the third color. Thus it does not matter whether w_B in Figure 5 is precolored or not; either way, every extension uses $f(Q'(B))$ on $\{x_B, y\}$, and the third color is forced on the remaining vertex of $F(B)$. However, in transforming the problem we must avoid decreasing the distance between components of $G[W]$; hence we may assume that the exceptional case occurs only in Type 1, as on the left in Figure 5.

This exceptional case is in fact the building block and argument used in the example of Figure 2, showing that distance 9 is not enough for the extension theorem when $k = 3$.

5. Nearly remote blocks. Working in from a remote block B in H, we now consider the less remote vertex in $F(B)$. Based on Claim 9, we say that $F(B) \cong K_2$ is the *usual case*, while $F(B) \cong K_3$ with $k = 3$ is the *exceptional case*, which we may assume occurs only when B has Type 1.

In both the usual and exceptional cases, let y_B denote the unique vertex of $F(B)$ that is farthest from $Q'(B)$. Also define a *branching path* to be a path in H whose edges lie in distinct blocks.

CLAIM 10. *If B is a remote block of H, then H cannot have two leaf blocks reached from $F(B)$ along branching paths in H that exit y_B on different edges and have length at least* 3. *The same conclusion holds when the edges leaving y_B are in different blocks and the paths have length at least* 2.

Proof. Suppose that the claim fails, and C_1 and C_2 are two such leaf blocks. If two blocks of H are joined by a branching path in H of length l, then the distance between them in the block-cutpoint tree T is $2l$. Depending on whether the paths from $F(B)$ to C_1 and C_2 depart from y_B using edges of the same block B^* (solid edges) or different blocks B_1 and B_2 (dashed edges), the subgraph of T consisting of the paths among B, C_1, and C_2 is as shown in Figure 6. For any block of H, the distance to one of $\{C_1, C_2\}$ in T will exceed the distance to B. This contradicts the remoteness of B, so there is no such pair $\{C_1, C_2\}$. The same argument holds in the dashed case with paths shorter by one block and cut-vertex. ☐

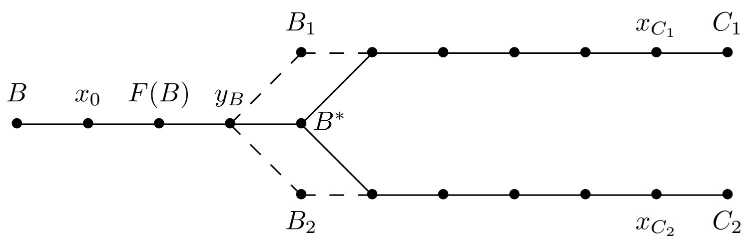

FIG. 6. *Portion of T involving distant blocks from B.*

CLAIM 11. *If B is a remote block of H in the usual case, and C is a leaf block of H, then $d_H(B, C) \geq 4$.*

Proof. Suppose that $d_H(B, C) \leq 3$. Since the cut-vertex contained in a leaf block of H has distance at most 2 from W, the distance requirement yields $Q(B) = Q(C)$. By Claim 6, this occurs only in the clique case, with B and C occurring as B_1 and B_2 in Figure 4. Claim 6(e) implies that $d_H(B, C) \geq 1$.

If B is Type 1, then Claim 6(b) implies that x_B has one neighbor in B and none in $Q(B)$, and Claim 9 implies that x_B has one neighbor in $F(B)$. Hence $d_G(x_B) = 2 < k$, which contradicts Claim 2. We conclude that B and C have Type 2 as on the left in Figure 4, and the block sharing x_C with C is a single edge. (In particular, Type 1 for such blocks B and C occurs only in the exceptional case.)

Since $d_G(v) = k$ for all $v \in V(H)$, every vertex in H has a neighbor in W or is a cut-vertex of H. If we follow a branching path from y_B starting in a block incident to y_B, we eventually reach a vertex of a leaf block. Claim 6(d) and the distance requirement imply that a branching path reaching a leaf block other than B or C takes at least three steps from y_B. By Claim 10, there is at most one such leaf block. Hence y_B has at most one neighbor not in $F(B)$ or along its path to C.

By Claim 9, $F(B) \cong K_2$. Since $d_H(B, C) \geq 1$, no leaf block is incident to y_B.

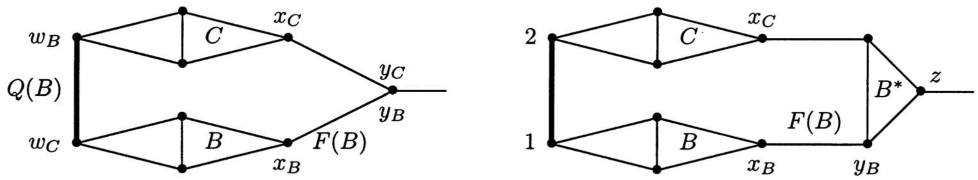

FIG. 7. *Exclusion of leaf blocks near a remote block B.*

Hence y_B has $k-1$ neighbors in H other than x_B. If $k \geq 4$, then we can pick two of them not along the path to C, which we have just shown cannot occur. Hence we may assume that $k = 3$.

If $d_H(B, C) = 1$, then $y_B = x_C$. Now $H = B \cup F(B) \cup C$ and $Q(B) = W$ and the precoloring extends. Hence $d_H(B, C) > 1$, and y_B lies in no leaf block.

If $d_H(B, C) = 2$, then $y_B \in N_H(x_B) \cap N_H(x_C)$. Since y_B has exactly 1 neighbor in the blocks it shares with each of x_B and x_C, it has one other neighbor in H, as shown on the left in Figure 6. In this case, we replace the configuration with the exceptional case on the left in Figure 5. That is, we delete one vertex from each of C' and B', make the remaining vertex of each adjacent to all of $Q(B)$, and add the edge $x_B x_C$. In both configurations, the distance from $Q(B)$ to other vertices of W is the same, and in each case every proper extension of f must give y_B the only color not in $f(Q(B))$. Hence G is a counterexample if and only if the smaller graph is a counterexample. By the minimality of G, we may thus exclude the configuration on the left in Figure 7.

Finally, suppose that $d_H(B, C) = 3$. Now x_C is not a neighbor of y_B but has distance 2 from it. If y_B lies in two blocks other than $F(B)$, then the one not leading to C begins a long enough branching path to contradict Claim 10 with C. Hence y_B lies in only one block other than $F(B)$; call it B^*. Since $d_H(y_B) = 3$, B^* is a triangle, and the vertex z in B^* that is not on the path to C is a cut-vertex of H, as shown on the right in Figure 7. In this situation, the coloring can be extended from $Q(B)$ to put any of the three colors on z. Hence we may delete the vertices in this figure other than z and its neighbor outside B^*, extend the coloring from $W - Q(B)$ to the rest of G, and then extend the coloring from $Q(B)$ to agree with it. This excludes this configuration. $\square$

CLAIM 12. *If B is a remote block of H for a minimal counterexample G, then $k = 3$ and y_B belongs to exactly one block of H other than $F(B)$.*

Proof. Let B be a remote block of H for a minimal counterexample G. Together, Claims 11 and 10 imply in the usual case that y_B has at most one neighbor in H outside $F(B)$ that is a cut-vertex of H. By Claim 9, y_B has only one neighbor in $F(B)$.

In the clique case, if B has Type 1, then Claim 9 implies that x_B has $k-1$ neighbors in the k-clique $B' \cup Q(B)$. Hence only one vertex of $Q(B)$ can have a neighbor outside B, and it has only one such neighbor. This also holds in the list case or in the clique case when B has Type 2.

The neighbors of y_B outside $F(B)$ that are not cut-vertices must have a neighbor in $Q(B)$. Hence there is at most one vertex that is of that type or equals y_B. We have shown that y_B together with its neighbors outside $F(B)$ includes only two vertices and that y_B belongs to only one block of H other than $F(B)$.

These remarks yield the conclusion that $k = 3$ in the usual case, but also $k = 3$

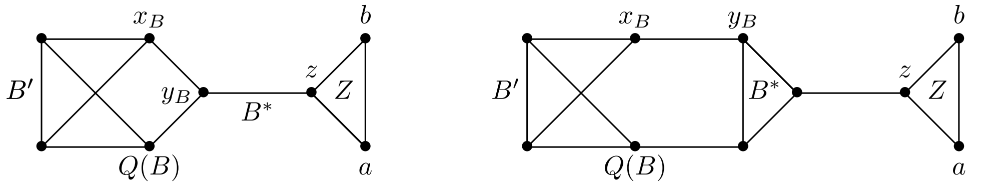

FIG. 8. *List case and clique Type* 1 *usual case when* $k = 3$.

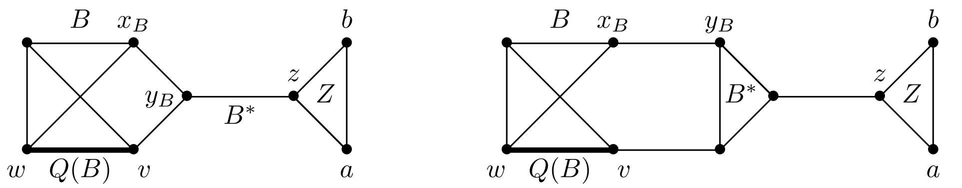

FIG. 9. *Clique Type* 2 *usual case when* $k = 3$.

in the exceptional case. □

When B is a remote block of H in the remaining case ($k = 3$), we let B^* denote the block of H other than $F(B)$ that contains y_B.

CLAIM 13. *There is no minimal counterexample G.*

Proof. Otherwise, Claim 12 yields $k = 3$. Let B be a remote block of H.

For the list case and for the usual clique case with B having Type 1, there are two remaining configurations, depending on whether the one vertex of $Q(B)$ is adjacent to y_B or to a vertex of B^* other than y_B. These configurations appear in Figure 8, where the additional vertices z, a, and b forming block Z are defined.

By the distance requirement, a and b have no neighbors in W; hence they are cut-vertices of H. Since $d_G(a, Q(B)) \le 4$, the distance requirement for $k = 3$ implies that every leaf block reached from a via a branching path along the block other than Z has distance at least 4 from a in G (it may be two steps more to W). The same is true of leaf blocks reached from b. These leaf blocks have distance at least 9 from Z in T, and the path P joining them in T passes through Z. On the other hand, $d_T(B, Z) \le 8$ via a path reaching P at Z. This contradicts the remoteness of B, so these cases do not occur. (This argument is not valid when the distance threshold is only 8.)

The usual clique case with B having Type 2 is very similar to that above. We merely relabel the picture as in Figure 9. We have $|Q(B)| = 2$. Let w be the neighbor of x_B in $Q(B)$, and let v be the nonneighbor of x. We have v adjacent to y_B or to a neighbor of y_B in B^*. Since again $d_G(a, Q(B)) \le 4$, the previous argument still works.

We have reduced the problem to the exceptional clique case with B having Type 1. Expanding the picture on the left in Figure 5 yields the configuration shown in Figure 10; it is a relabeling of those on the right in Figures 8 and 9. The argument mirrors those in the earlier cases. Note that $Q(B)$ is now one step farther from a and b, so the constraint from the distance requirement is weaker. However, B is now one step closer to a and b, so the remoteness argument is strengthened by the same amount that it is weakened. Again we contradict the remoteness of B. □

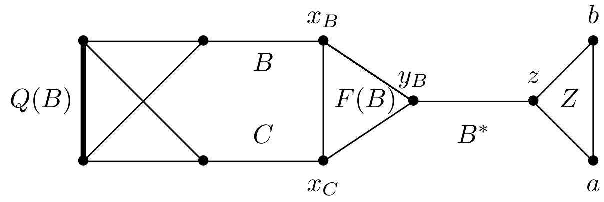

FIG. 10. *The exceptional case.*

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