The Pagenumber of Spherical Lattices is Unbounded

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Abstract. Although the pagenumber of planar ordered sets – even for a planar lattices remains unknown we give a sequence (L_n) of spherical lattices, where the pagenumber of L_n is at least n. Notice that the covering graph for each member of this sequence is planar.

1 Introduction

We say that *a covers b* (or *b covered by a*) in the ordered set P, and write $a \succ b$ (or $b \prec a$), if $a > b$ and whenever $a > c \geq b$, then $c = b$. Also, we say that a is an *upper cover* of b, or b is a *lower cover* of a, or (a, b) is an edge in P.

The covering graph of P, $cov(P)$, is the graph whose vertices are the elements of P, and a pair $\{a, b\}$ forms an edge in $cov(P)$ if $a \succ b$ or $a \prec b$. It is possible to draw $cov(P)$ on the Cartesian plane in such a way that the y-coordinate of a is less than the y-coordinate of b if $a \prec b$ and the edge

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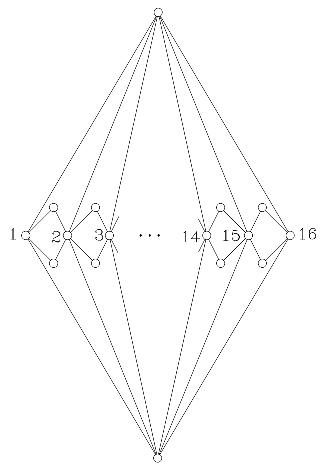

Figure 1: Four-page planar ordered set.

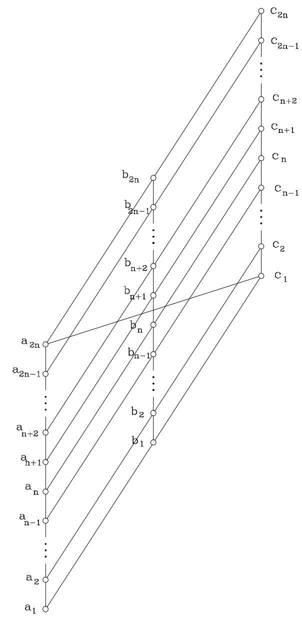

Figure 2:

(a, b) does not pass any other element of P. We call such drawing an *upward drawing* of P. An ordered set P is *planar* if there exists an upward drawing of P without edge crossings.

A *book embedding* of a graph G consists of an embedding of its nodes along the spine of a book, and an embedding of its edges on pages so that edges embedded on the same page do not intersect. The *pagenumber* of $G, page(G)$, is the minimum number of pages needed, taken over all permutations of the vertices of G.

The *pagenumber* of an ordered set $P, page(P)$, is the pagenumber of the graph $cov(P)$ taken over only the permutations of the vertices of P which form linear extensions (A total ordering of the elements of an ordered set P is called a *linear extension* of P, if it is consistent with the ordering of P.).

The pagenumber was first defined for graphs by **Bernhart** and **Kainen** [2], who showed that the one-page graphs are exactly the outerplanar graphs. **Yannakakis** [7], showed that $page(G) \leq 4$ for every planar graph G, and this upper bound is achieved.

The pagenumber for ordered sets was introduced by **Nowakowski** and **Parker** [4], who showed that $page(P) = 1$ if and only if $cov(P)$ is a forest.

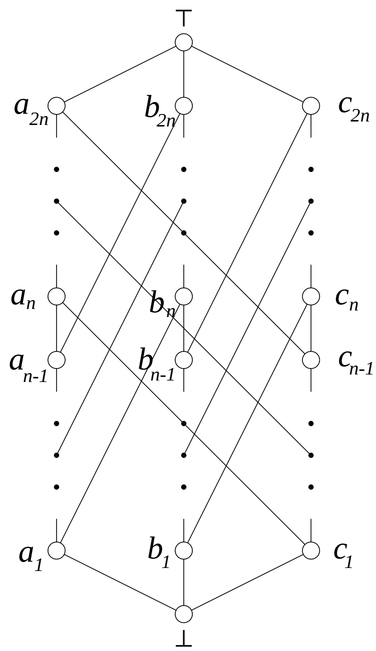

Figure 3:

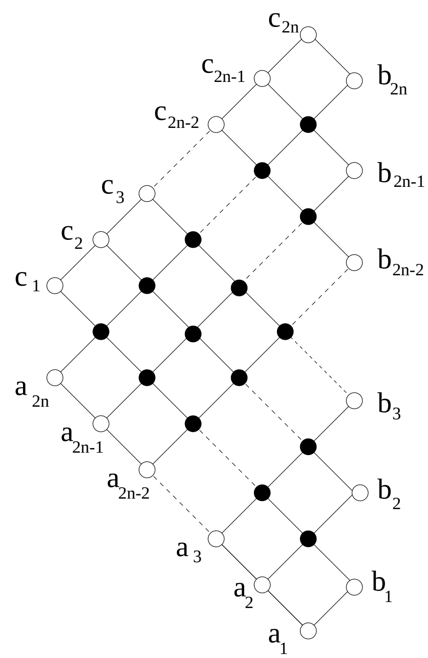

Figure 4:

Also, they derived a general lower bound on the pagenumber of ordered sets and upper bounds for special classes of ordered sets. **Hung** [3] showed that there exists a 48-element planar ordered set which requires four pages (see Figure 1 (This is the smallest known four-page planar ordered set.)). Moreover, no planar ordered set with pagenumber five is known.

For each positive integer n, **Heath** and **Pemmaraju** [5] gave a $6n$-vertex spherical ordered set P (see Figure 2) with a planar covering graph such that $page(P) \geq n$. (An ordered set is *spherical* if it has an upward drawing on the surface of the sphere such that all arcs are strictly increasing northward on the sphere, and no pair of arcs cross (see [6].) Figure 4 illustrates a spherical ordered set.)

Examples of Heath and Pemmaraju are not lattices. The smallest lattices containing these ordered sets as suborders (so called MacNeille completions) are illustrated in Figure 4. Since these lattices are graded planar lattices, their pagenumber is at most two (see [1]).

In this note we give a sequence L_n (illustrated in Figure 3) of spherical lattices which has unbounded pagenumber. The covering graph for each member of this sequence is planar. The importance of this sequence comes from the fact that a spherical ordered set is "almost" planar.

Theorem 1 *For each positive integer n, $page(L_n) \geq n$ where L_n is the (spherical) lattice illustrated in Figure 6.*

Proof. Let L be a linear extension of the lattice L_n. Since $a_n \parallel b_{n+1}$ in L_n, either $a_n < b_{n+1}$ or $b_{n+1} < a_n$ in L.

Suppose $a_n < b_{n+1}$ in L. As $a_1 < a_2 < \ldots < a_n$ and $b_{n+1} < b_{n+2} < \ldots < b_{2n}$ in L_n, we have in L: $a_1 < a_2 < \ldots < a_n < b_{n+1} < b_{n+2} < \ldots < b_{2n}$, which means that L contains the n-twist $\{(a_i, b_{n+i}) : 1 \leq i \leq n\}$.
Hence, $page(L_n, L) \geq n$.

Similarly, if $b_n < c_{n+1}$ or $c_n < a_{n+1}$ in L, then $page(L_n, L) \geq n$. Thus if $page(L_n, L) < n$, then $a_n > b_{n+1} > b_n > c_{n+1} > c_n > a_{n+1}$ in L, a contradiction.

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