The domination number of cubic Hamiltonian graphs

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Abstract

Let $\gamma(G)$ denote the domination number of a graph, and let $\mathscr{C}$ be the set of all Hamiltonian cubic graphs. Let

$$\bar{\gamma}(n) = \max\{\gamma(G) |\ G \in \mathscr{C} \quad \text{and} \quad |V(G)| = n\}\ ,$$

and

$$\underline{\gamma}(n) = \min\{\gamma(G) |\ G \in \mathscr{C} \quad \text{and} \quad |V(G)| = n\}\ .$$

Then, for $n \geq 4$, n even,

$$\bar{\gamma}(n) = \left\lfloor \frac{n+1}{3} \right\rfloor \quad \text{and} \quad \underline{\gamma}(n) = \left\lfloor \frac{n+2}{4} \right\rfloor\ .$$

1 Introduction

The *domination number* $\gamma(G)$ of a graph G is the least number of vertices needed to *dominate* G. Thus, if $N(v)$ denotes the closed neighbourhood of a vertex v, then

$$\gamma(G) = \min_{S \subseteq V(G)} \left\{ |S| : \ V(G) \subseteq \bigcup_{v \in S} N(v) \right\} .$$

Throughout let G be a Hamiltonian cubic graph, and let $n = |V(G)|$.

Some attention has been given to the relationship between the domination number of a graph G and its minimum degree $\delta(G)$. Blank [1] and later, independently, McCuaig and Shephard [4] showed that, apart from seven exceptional graphs, if $\delta(G) \geq 2$ then $\gamma(G) \leq \frac{2}{5}|V(G)|$. Then, in [5], Reed showed that if $\delta(G) \geq 3$, then $\gamma(G) \leq \frac{3}{8}|V(G)|$. Kawarabayashi, Plummer and Saito [3] have recently shown (as a special case of a more general result) that if G is a 2-edge-connected cubic graph of girth $3k$ then

$$\gamma(G) \leq \left(\frac{3k+2}{9k+3} \right) |V(G)|\ .$$

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This improves upon Reed's result when $k \geq 3$.

In [5] Reed also conjectured that if G is a connected cubic graph then $\gamma(G) \leq \lceil \frac{n}{3} \rceil$. In the very special case when G is Hamiltonian as well as cubic, we can select every third vertex of a Hamiltonian cycle, so Reed's conjecture is clearly true in this case. However, Plummer suggested to the authors that, in this very special case, the slightly stronger inequality $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$ was true. There is no difference between these conjectures if $n \equiv 0 \pmod 3$. We show that Plummer's conjecture is true if $n \equiv 1 \pmod 3$, but is false if $n \equiv 2 \pmod 3$.

Let $\mathscr{C}$ be the set of all Hamiltonian cubic graphs. Let

$$\bar{\gamma}(n) = \max\{\gamma(G) | \ G \in \mathscr{C} \quad \text{and} \quad |V(G)| = n\} \ .$$

The precise result we prove is:

Theorem 1. *For $n \geq 4$, n even,* $\quad \bar{\gamma}(n) = \lfloor \frac{n+1}{3} \rfloor$.

If $\quad \underline{\gamma}(n) = \min\{\gamma(G) | \ G \in \mathscr{C} \quad \text{and} \quad |V(G)| = n\}$, we also prove:

Theorem 2. *For $n \geq 4$, n even,* $\quad \underline{\gamma}(n) = \lfloor \frac{n+2}{4} \rfloor$.

We just noted that $\gamma(n) \leq \lceil \frac{n}{3} \rceil$ for all $n \geq 4$, and in [5] Reed showed that $\gamma(n) = \frac{n}{3} = \lfloor \frac{n+1}{3} \rfloor$ if $n \equiv 0 \pmod 3$. Therefore Theorem 1 follows from the following propositions.

Proposition 3. *If* $\quad n = 3k + 2, \quad$ *then* $\quad \overline{\gamma}(n) \geq \lfloor \frac{n+1}{3} \rfloor = k + 1$.

Proposition 4. *If* $\quad n = 3k + 1, \quad$ *then* $\quad \overline{\gamma}(n) \geq \lfloor \frac{n+1}{3} \rfloor = k$.

Proposition 5. *If* $\quad n = 3k + 1, \quad$ *then* $\quad \overline{\gamma}(n) \leq k$.

2 Proof of Proposition 3

For $k \geq 1$ and $1 \leq i \leq k$, let S_i be the graph depicted in Figure 1 with vertex set $\{a_{i-1}, b_i, c_i, \ a_i, \ a'_{i-1}, \ b'_i, \ c'_i, \ a'_i\}$ and edge set

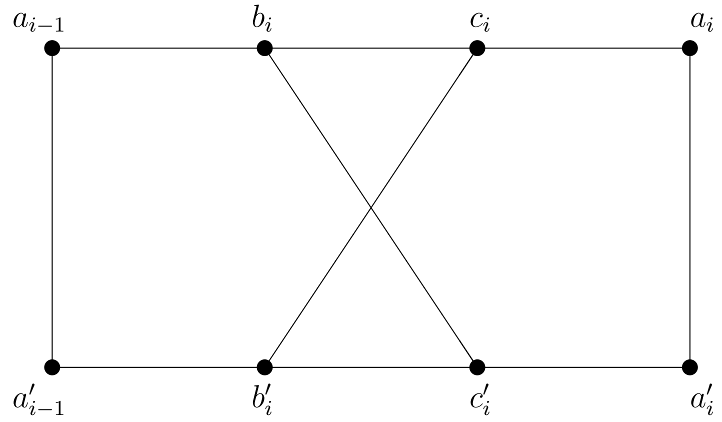

Figure 1

$\{a_{i-1}\, b_i,\, b_i\, c_i,\, c_i\, a_i,\, a'_{i-1}\, b'_i,\, b'_i\, c'_i,\, c'_i\, a'_i,\, a_{i-1}\, a'_{i-1},\, a_i\, a'_i,\, b_i\, c'_i,\, b'_i\, c_i\}$. Let $H(6k+2)$ be the graph $S_1 \cup ... \cup S_k$, let $H_1(6k+2)$ be $H(6k+2) \cup \{a_0\, a_k,\, a'_0\, a'_k\}$ and let $H_2(6k+2)$ be $H(6k+2) \cup \{a_0\, a'_k,\, a_0\, a'_k\}$.

Clearly, $H_1(6k+2)$ and $H_2(6k+2)$ are cubic Hamiltonian graphs. We shall show that $\gamma\,(H_1(6k+2)) = \gamma\,(H_2(6k+2)) = 2k+1 \geq \lceil \frac{6k+2}{3} \rceil$. Then Proposition 3 follows.

We may easily check that $\gamma\,(H_1(8)) = \gamma\,(H_2(8)) = 3$. Suppose Proposition 3 is not true. Then there is a smallest integer k such that, for some $H \in \{H_1(6k+2),\, H_2(6k+2)\}$, $\gamma(H) \leq 2k$. Since $\gamma\,(H_1(8)) = \gamma\,(H_2(8)) > 2$, it follows that $k \geq 2$.

Let D be a dominating set of cardinality $2k$ of H. For $0 \leq i \leq k$, let $A_i = D \cap \{a_i,\, a'_i\}$ and, for $1 \leq i \leq k$, let $X_i = D \cap \{b_i,\, b'_i,\, c_i,\, c'_i\}$.

Lemma 6. *For $0 \leq i \leq k-1$, if $|A_i| = 2$ then $|A_{i+1}| = 0$ (i.e. $A_{i+1} = \emptyset$), and, for $1 \leq i \leq k$, if $|A_i| = 2$ then $|A_{i-1}| = 0$.*

Proof. Suppose $|A_i| = 2$ and $0 \leq i \leq k-1$.

Case 1. $|A_{i+1}| = 2$.

Let $\tilde{H}$ be obtained from H by deleting b_{i+1}, b'_{i+1}, c_{i+1}, c'_{i+1}, and identifying a_i with a_{i+1} and a'_i with a'_{i+1}. Then $\tilde{H} \in \{H_1\,(6(k-1)+2),\; H_2\,(6(k-1)+2)\}$ and $\tilde{H}$ has a dominating set $\tilde{D}$ obtained from D by identifying a_i with a_{i+1} and a'_i with a'_{i+1} of cardinality at most $2(k-1)$. This contradicts the minimality of k.

Case 2. $|A_{i+1}| = 1$.

We may suppose that $A_{i+1} = \{a_{i+1}\}$. Then D must contain a vertex that dominates c'_{i+1} (or possibly coincides with c'_{i+1}) in S_{i+1}. Therefore, if $\tilde{H}$ is constructed from H as in Case 1, then $\gamma(\tilde{H}) \leq 2(k-1)$, a contradiction. Therefore $|A_{i+1}| \neq 1$.

It follows that $|A_{i+1}| = 0$.

The argument showing that, if $1 \le i \le k$ and $|A_i| = 2$, then $A_{i-1} = \emptyset$ is similar. □

Lemma 7. *If* $0 \le i \le k-1$ *and* $|A_i| = 1$ *then* $|A_{i+1}| \ne 1$. *Equivalently, if* $1 \le i \le k$ *and* $|A_i| = 1$ *then* $|A_{i-1}| \ne 1$.

Proof. For some i, $0 \le i \le k-1$, suppose that $|A_i| = |A_{i+1}| = 1$. Then one of $\{b_{i+1},\, c'_{i+1},\, b'_{i+1}, c_{i+1}\}$ lies in D. We construct a graph H^* by deleting b_{i+1}, b'_{i+1}, c_{i+1}, c'_{i+1} and identifying the vertex of $D \cap A_i$ with the vertex of $D \cap A_{i+1}$, and the vertex of $A_i \backslash D$ with the vertex of $A_{i+1} \backslash D$. Then H^* is isomorphic to one of $H_1\,(6(k-1)+2)$ and $H_2\,(6(k-1)+2)$. Since $\gamma(H^*) \le 2(k-1)$, we have a contradiction. Therefore $|A_{i+1}| \ne 1$. □

Lemma 8. *For* $1 \le i \le k$, $|X_i| \le 1$.

Proof. Suppose that, for some i, $|X_i| \ge 2$. Consider the graphs $\tilde{H}$ and $\tilde{H}'$ obtained by deleting b_i, b'_i, c_i, c'_i and identifying a_{i-1} with a_i, and a'_{i-1} with a'_i, or a_{i-1} with a'_i, and a'_{i-1} with a_i respectively. All vertices of $\tilde{H}$ and $\tilde{H}'$ apart from the two new vertices are dominated by $D\backslash X_i$. Hence if $|X_i| \ge 3$ then $(D\backslash X_i) \cup \{a_i\}$ is a dominating set of cardinality at most $2(k-1)$. If $|X_i| = 2$, then at least two of a_{i-1}, a'_{i-1}, a_i and a'_i are dominated by $D\backslash X_i$. Thus in this case, the set $D\backslash X_i$ is dominating either in $\tilde{H}$ or $\tilde{H}'$, and its cardinality is at most $2(k-1)$. Since each of $\tilde{H}$ and $\tilde{H}'$ is isomorphic to one of $H_1\,(6(k-2)+2)$ and $H_2\,(6(k-2)+2)$, we have a contradiction against the minimality of k. Therefore $|X_i| \le 1$. □

Lemma 9. *For* $1 \le i \le k-1$, $A_i \ne \emptyset$.

Proof. Suppose $A_i = \emptyset$ for some i, $1 \le i \le k-1$. By Lemma 8, $|X_i| \le 1$, so b_{i+1} and b'_{i+1} must be dominated by the same vertex. This is only possible if $X_{i+1} \subseteq \{c_{i+1},\, c'_{i+1}\}$. Therefore $X_i \cap \{b_{i+1},\, b'_{i+1}\} = \emptyset$. Therefore a_i and a'_i must be dominated by c_i and c'_i respectively, contradicting Lemma 8. □

Lemma 10. $k \le 2$.

Proof. Suppose $k \ge 3$. By Lemma 9, $|A_1| \ge 1$ and $|A_2| \ge 1$.

Case 1. Suppose $|A_1| = 1$. Then, by Lemma 6, $|A_2| \le 1$, so $|A_2| = 1$. But this contradicts Lemma 7.

Case 2. Suppose $|A_1| = 2$. Then, by Lemma 6, $A_2 = \emptyset$, contradicting Lemma 9. □

Lemma 11. $k \ne 2$.

Proof. Suppose $k = 2$. By Lemma 9, $|A_1| \geq 1$.

Case 1. $|A_1| = 1$.

By Lemma 7, $|A_0| \neq 1$ and $|A_2| \neq 1$. By Lemma 6, $|A_0| \neq 2$ and $|A_2| \neq 2$. Therefore $A_0 = A_2 = \emptyset$. In order that a_0, a_0', a_2, a_2' be dominated, it is necessary that b_1, b_1', c_2, $c_2' \in D$. But then $\gamma(H) = 5 > 2k$, contradicting the definition of k.

Case 2. $|A_1| = 2$.

By Lemma 6, $A_0 = A_2 = \emptyset$, and we get a contradiction as in Case 1. □

We conclude that Proposition 3 is true.

3 Proof of Proposition 4

Since any cubic graph has even order, and since $n \equiv 1 \pmod 3$, it follows that $n \equiv 4 \pmod 6$. If $n = 4$, then $\gamma(K_4) = 1 = \left\lfloor \frac{n+1}{3} \right\rfloor$. Now suppose that $n > 4$. Then $n \geq 10$. Let $n = 6k + 4$, where $k \geq 1$. Take the graph $H_1(6k + 2)$ constructed in Section 2 and insert two further vertices v_1 and v_1' in the edges $a_0\, b_1$ and $a_0'\, b_1'$ respectively, and add an edge $v_1\, v_1'$. We obtain a cubic Hamiltonian graph G with $6k + 4$ vertices. Suppose that D is a dominating set of G. If $\{v_1, v_2\} \notin D$ then D dominates $H_1(6k + 2)$, so $|D| \geq 2k + 1$. Similarly if $v_1 \in D$, $v_2 \notin D$ then $(D \backslash \{v_1\}) \cup \{a_0\}$ dominates $H_1(6k + 2)$, and if $v_1, v_2 \in D$ then $(D \backslash \{v_1, v_2\}) \cup \{a_0, a_0'\}$ dominates $H_1(6k + 2)$. Thus $|D| \geq 2k + 1$.

Therefore, for all $n \geq 4$, if $n \equiv 1 \pmod 3$ then $\overline{\gamma}(n) \geq \left\lfloor \frac{n+1}{3} \right\rfloor$. □

4 Proof of Proposition 5

We need to show that if $n = 3k + 1$ and G is a cubic Hamiltinian graph of order n, then $\gamma(G) \leq k$. Suppose to the contrary that $\gamma(G) \geq k + 1$. Fix a Hamiltonian cycle H of G.

An *arc* of H is a path P contained by H; the number of edges in the arc P is its *length*; we shall denote the length by $|P|$. If an arc P has x edges and $x \equiv i \pmod 3$, where $0 \leq i \leq 2$, then we say that P is an i-arc. An edge of G which is not an edge of H is a *chord*.

If A, B, C, D are four vertices on H and AB and CD are chords and A, C, B, D occur in that order going round H, then the chords AB and CD are said to *cross*. If AC, CB, BD, DA are a-, c-, b-, d-arcs respectively, then $ACBD$ is an $(acbd)$-partition of H. Clearly, $a + b + c + d \equiv 1 \pmod 3$ and an

($acbd$)-partition is also a (πa, πc, πb, πd)-partition for any cyclic permutation π of $acbd$.

We first note that no chord of G separates H into two 2-arcs. For if AB were such a chord and P were one of the 2-arcs, then $P \cup AB$ has $3x$ edges for some integer x, and has a dominating set of x vertices including A. The other arc is dominated by A and $k-x$ vertices, so $\gamma(G) = k$, a contradiction.

Thus each chord separates H into a 0-arc and a 1-arc.

It follows that no two crossing chords AB and CD give an ($acbd$)-partition $(D\backslash\{v_1, v_2\}) \cup \{a_0, a'_0\}$ with two adjacent 1's, or an adjacent 0 and 2, counting d as being adjacent to a. Therefore the only possible partitions are a (0001)-partition, a (0121)-partition and a (1222)-partition.

In fact a (1222)-partition cannot occur. Before showing this, we need the following Lemma.

Lemma 12. *Given a graph G, suppose that an edge XY is subdivided by three vertices U, V, W so that X, U, V, W, Y occur in that order, producing a graph G^*. Then $\gamma(G^*) \leq \gamma(G) + 1$.*

Proof. Let D be a dominating set of cardinality $\gamma(G)$ of G. If $X, Y \notin D$ or $\{X, Y\} \subseteq D$, then $D \cup \{V\}$ dominates G^*. If $|D \cap \{X, Y\}| = 1$ then we may suppose that $X \in D$. In that case $D \cup \{W\}$ dominates G^*. Thus $\gamma(G^*) \leq \gamma(G) + 1$. □

Suppose that AB and CD are crossing chords giving a (1222)-partition with the arcs DA, AC, CB, BD being 1-, 2-, 2-, 2-arcs respectively. If these arcs have length 1 or 2 then G has 7 vertices and is dominated by 2 vertices, B and C. If $3k + 1 > 7$ then repeated application of Lemma 12 shows that $\gamma(G) \leq k$, a contradiction. This establishes:

Claim 1. All partitions are (0001)-partitions or (0121)-partitions.

Claim 1 has two consequences.

Claim 2. Let AB be a chord with a 0-arc and let C be a vertex on the 0-arc of AB such that $|AC| \equiv 1 \pmod 3$. If the chord CD crosses AB then A is on the 1-arc of CD.

Proof. Since $|AC| \equiv 1 \pmod 3$ and $AC \cup CB$ is a 0-arc, $|CB| \equiv 2 \pmod 3$, so by Claim 1, $|AD| \equiv 0 \pmod 3$. □

Claim 3. Let AB be a chord with a 0-arc and let C be a vertex on the 1-arc of AB such that $|AC| \equiv 2 \pmod 3$. Then the chord CD does not cross AB.

Proof. Since $|AC| \equiv 2 \pmod 3$, $|CB| \equiv 2 \pmod 3$ also. By Claim 1, CD does not cross AB. □

¿From Claim 1 we also deduce the following lemma.

Lemma 13. *Let AB be a chord with a 0-arc and let A, A_1, A_2, …, A_s, B be the vertices of its 0-arc. If the chords $A_1 C_1$ and $A_s C_s$ cross AB, then they do not cross each other.*

Proof. Suppose $A_1 C_1$ and $A_s C_s$ cross each other and AB. Then the vertices A, A_1, A_s, B, C_1, C_s are on H in this order. Since $|A A_1| \equiv 1 \pmod 3$ and $|A_1 B| \equiv 2 \pmod 3$, by Claim 1 applied to AB and $A_1 C_1$, $|BC_1| \equiv 1 \pmod 3$. Similarly $|AC_s| \equiv 1 (\mathrm{mod}\, 3)$. Thus $A_1 C_1$ and $A_s C_s$ yield a (1222)-partition of H, contradicting Claim 1. □

Now choose a shortest 1-arc AB in H. Then $|AB| \geq 4$. There are two distinct vertices, C, D, on the arc AB such that CD is a 0-arc. To see this, let C be a vertex on AB such that the path AC has two edges. Then by Claim 3, the chord on C, say CD, does not cross AB. By the definition of AB, the chord CD is a 0-arc.

Let KL be a shortest 0-arc with both vertices on the arc AB. Let the vertices of KL be, in order, K, K_1, K_2, …, K_s, L. Let $K_1 D_1$ and $K_s D_s$ be the chords starting at K_1 and K_s respectively. By the minimality of KL and AB, each of $K_1 D_1$ and $K_s D_s$ cross KL. By Claim 2, the arc of $K_1 D_1$ containing K is a 1-arc, and the arc of $K_s D_s$ containing L is a 1-arc. By Lemma 13, $K_1 D_1$ and $K_s D_s$ do not cross. By the minimality of AB, each of $K_1 D_1$ and $K_s D_s$ crosses AB. Thus A, K, K_1, K_s, L, B, D_s, D_1 occur in this order going round H. This is illustrated in Figure 2.

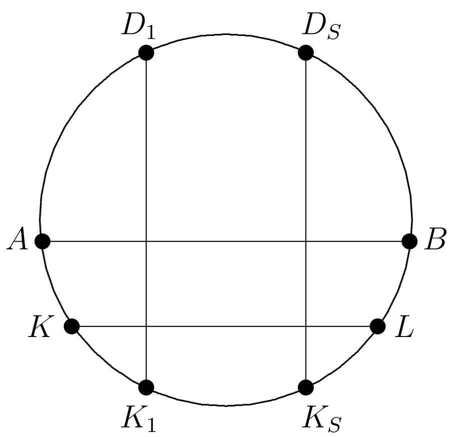

Figure 2

Since $|KL| \equiv 0 \pmod 3$ and $|AB| \equiv 1 \pmod 3$, it follows that $|AK| + |BL| \equiv 1 \pmod 3$.

Because of the symmetry, we need only consider two cases.

Case 1 $|AK| \equiv |LB| \equiv 2 \pmod 3$.

¿From simple arithmetic, it follows that
$|K\,K_1| \equiv |K_1\,K_s| \equiv |K_s\,L| \equiv |B\,D_s| \equiv |D_s\,D_1| \equiv |D_1\,A| \equiv 1 \pmod 3$.
In the case when all these sizes are 1 and 2, there are 10 vertices, and G is dominated by K, B and D_s. If $3k+1 > 10$ then repeated applications of Lemma 12 shows that G is dominated by k vertices in this case, a contradiction.

Case 2. $|AK| \equiv 0 \pmod 3$ and $|LB| \equiv 1 \pmod 3$.

By simple arithmetic we have
$|K\,K_1| \equiv |K_1\,K_s| \equiv |K_s\,L| \equiv |D_1\,D_s| \equiv |BL| \equiv 1 \pmod 3$,
$|LK| \equiv |A\,D_1| \equiv 0 \pmod 3$ and $|B\,D_s| \equiv 2 \pmod 3$. But then D_s, A, K_s, B mark a (1222)-partition, contradicting Claim 1.

In every case, our hypothesis that $\gamma(G) \geq k+1$ leads to a contradiction, so $\gamma(G) \leq k$, as asserted. □

5 Proof of Theorem 2

We construct a Hamiltonian cubic graph G with $\gamma(G) = \left\lfloor \frac{n+2}{4} \right\rfloor$ by identifying the pendent edges of the graphs in Figure 3.

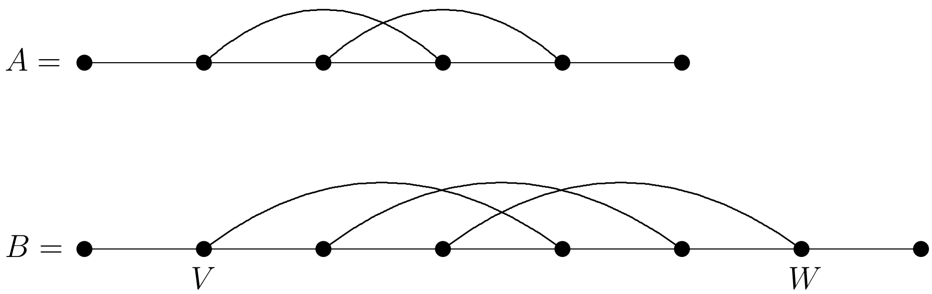

Figure 3

If $4|n$ we take $\frac{n}{4}$ copies of A identifying one pendent edge of one copy with a pendent edge of another, and the other pendent edge of the first copy with a pendent edge of a third copy (if $n \geq 12$), and so on, so as to form a cycle of such graphs. If $n \equiv 2 \pmod 4$ we take a copy of B and $\frac{1}{4}(n-6)$ copies of A, indentifying edges and forming a cycle of graphs, similarly. We find a dominating set of cardinality $\left\lfloor \frac{n+2}{4} \right\rfloor$ by taking one of the two central vertices from each copy of A, and by taking V and W from B.

Clearly if G is a cubic Hamiltonian graph, for each $v \in V(G)$, $|N(v)| = 4$, so $\gamma(G) \leq \left\lfloor \frac{n+2}{4} \right\rfloor$.

Thus $\underline{\gamma}(n) = \left\lfloor \frac{n+2}{4} \right\rfloor$ when n is even, as asserted. □

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