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# On the Induced Ramsey Number $IR(P_3, H)$

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**Summary.** The induced Ramsey number  $IR(G, H)$  is the least positive integer  $N$  such that there exists an  $N$ -vertex graph  $F$  with the property that for each 2-coloring of its edges with red and blue, either some induced in  $F$  subgraph isomorphic to  $G$  has all its edges colored red, or some induced in  $F$  subgraph isomorphic to  $H$  has all its edges colored blue. In this paper, we study  $IR(P_3, H)$  for various  $H$ , where  $P_3$  is the path with 3 vertices. In particular, we answer a question by Gorgol and Luczak by constructing a family  $\{H_n\}_{n=1}^{\infty}$  such that  $\limsup_{n \rightarrow \infty} \frac{IR(P_3, H_n)}{IR_w(P_3, H_n)} > 1$ , where  $IR_w(G, H)$  is defined almost as  $IR(G, H)$ , with the only difference that  $G$  should be induced only *in the red subgraph* of  $F$  (not in  $F$  itself) and  $H$  should be induced only in the blue subgraph of  $F$ .

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## 1 Introduction

The *induced Ramsey number*,  $IR(G, H)$ , is the greatest positive integer  $N$  such that for each graph  $F$  on at most  $N - 1$  vertices, there exists a 2-coloring of its edges with red and blue such that no induced copy of  $G$  in  $F$  has all its edges red and no induced copy of  $H$  in  $F$  has all its edges blue. Say that a graph  $F$  is an *IR-graph for graphs  $G$  and  $H$* , if for each 2-coloring of edges of  $F$  with red and blue, either some induced in  $F$  subgraph isomorphic to  $G$  has all its edges colored red, or some induced in  $F$  subgraph isomorphic to  $H$  has all its edges colored blue. In these terms, the *induced Ramsey number*,  $IR(G, H)$ , is the least order of an *IR-graph for  $G$  and  $H$* . The fact that an *IR-graph* exists for each  $G$  and  $H$  and thus  $IR(G, H)$  is finite was proved independently

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by Deuber [Deu75], Erdős et al. [EHP75], and Rödl [Rödl73]. Estimating induced Ramsey numbers of graphs in different classes attracted considerable attention (see, e.g., [Deu75, GL02, HNR83, KPR98, LR96, HKL95, Neš95]). In particular, Haxell, Kohayakawa and Łuczak [HKL95] showed that the diagonal induced Ramsey numbers for paths and cycles grow linearly in terms of their lengths. However, there are very few exact results.

A characteristic similar to the induced Ramsey number is the *weak induced Ramsey number*,  $IR_w(G, H)$  – the least positive integer  $N$  such that there exists an  $N$ -vertex graph  $F$  with the property that for each 2-coloring of its edges with red and blue, either the red subgraph of  $F$  contains an induced (in this red graph) copy of  $G$ , or the blue subgraph of  $F$  contains an induced (in this blue graph) copy of  $H$ . Gorgol and Łuczak [GL02] gave an example of a pair of graphs for which the induced Ramsey number is greater than the weak induced Ramsey number. Namely, they showed that  $IR(P_3, P_4) = 7$  and  $IR_w(P_3, P_4) = 6$ , where  $P_k$  is the path with  $k$  vertices. They also asked whether there exists a sequence  $\{H_n\}_{n=1}^\infty$  of graphs such that for some graph  $G$ ,

$$\limsup_{n \rightarrow \infty} \frac{IR(G, H_n)}{IR_w(G, H_n)} > 1. \quad (1)$$

Among other results, Gorgol and Łuczak proved that for every  $n \geq 3$ ,

$$1.5n - 1 \leq IR(P_3, P_n) \leq 2n - 1 \quad \text{and} \quad 4n/3 \leq IR_w(P_3, P_n) \leq 5n/3.$$

In this paper, we estimate  $IR(P_3, H)$  for various graphs  $H$ . We give the general bound

$$IR(P_3, H) \leq |V(H)| + |E(H)| \quad (2)$$

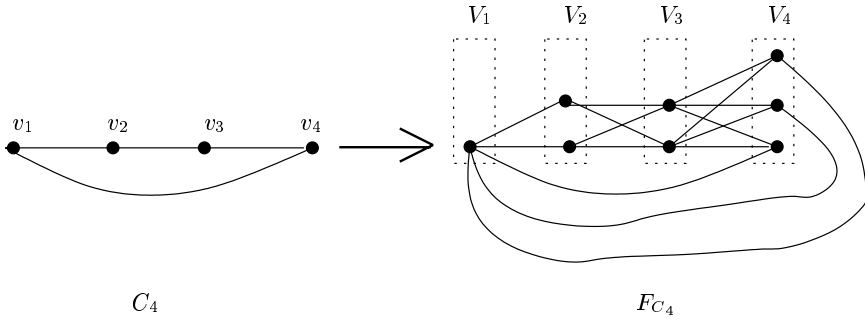
and show that this bound is sharp when  $H$  is the union of complete graphs. Then we refine bound (2) for graphs having vertices with equal neighborhoods and prove that this refined bound is sharp when  $H$  is any complete multipartite graph or the disjoint union of complete multipartite graphs. We also answer in the affirmative the above question of Gorgol and Łuczak by constructing a sequence  $\{H_n\}_{n=1}^\infty$  of graphs such that (1) holds for them with  $P_3$  in place of  $G$ .

The structure of the paper is as follows. In the next section we give upper bounds on  $IR(P_3, H)$  and prove that for some graphs they are exact. In the last section we answer the question of Gorgol and Łuczak [GL02].

## 2 Upper Bounds on $IR(P_3, H)$

Since  $P_3$  is a very simple graph,  $IR(P_3, H)$  grows at most linearly with the growth of  $|V(H)| + |E(H)|$ . A simple construction below proves this.

**Theorem 2.1.** *For every graph  $H$ ,  $IR(P_3, H) \leq |V(H)| + |E(H)|$ .*



**Fig. 1.** An  $IR$ -graph for  $P_3$  and  $C_4$

*Proof.* Given a graph  $H$ , we construct the associated graph  $F_H$  as follows. Let  $L = (v_1, v_2, \dots, v_n)$  be a list of all vertices of  $H$  written in some order. For  $i = 1, 2, \dots, n$ , let  $d_{H,L}(v_i)$  be the number of neighbors of  $v_i$  in  $\{v_1, \dots, v_{i-1}\}$ . The vertex set of  $F = F_{H,L}$  is  $V(F_{H,L}) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_n$ , where  $|V_i| = 1 + d_{H,L}(v_i)$ . For every edge  $(v_i, v_j)$  in  $H$ , we add all the edges between  $V_i$  and  $V_j$  in  $F_{H,L}$ . This completes the construction of  $F$ . Figure 1 illustrates this construction for  $C_4$  in a particular list.

Note that  $|V(F_{H,L})| = |V(H)| + \sum_{i=1}^n d_{H,L}(v_i) = |V(H)| + |E(H)|$ .

*Claim 2.2.* Each red-blue edge coloring of  $F_{H,L}$  contains either an induced red copy of  $P_3$  or an induced blue copy of  $H$  such that  $v_i \in V_i$  for every  $i$ .

*Proof.* We use induction on  $n = |V(H)|$ . The claim is trivially true for  $n = 1$ , in which case  $H = F_{H,L} = K_1$ . Suppose that the claim holds for each graph with less than  $n$  vertices. Consider a graph  $H$  with  $n$  vertices and let  $L = (v_1, v_2, \dots, v_n)$  be a list of all vertices of  $H$ . Let  $f$  be a red–blue edge coloring of  $F_{H,L}$ . Consider the graph  $H' = H - v_n$  and let  $L'$  be the list of vertices of  $H'$  obtained from  $L$  by deleting  $v_n$ . Then  $F_{H',L'} = F_{H,L} - V_n$ . Let  $f'$  be the edge coloring induced in  $H'$  by  $f$ . Assume that  $F_{H,L}$  has no induced red  $P_3$ . Then, as a subgraph of  $F_{H,L}$ , the graph  $F_{H',L'}$  also has no induced red  $P_3$ . Thus, by the induction hypothesis,  $F_{H',L'}$  has an induced blue copy of  $H'$  such that  $v_i \in V_i$  for  $i = 1, \dots, n-1$ . Let  $v_n$  have  $m$  neighbors in  $H$ . Then,  $|V_n| = m + 1$ . Let  $M$  be the set of  $m$  vertices in the induced blue copy  $\tilde{H}$  of  $H'$  in  $F_{H',L'}$  that need a new common neighbor to make the graph  $H$ . Each vertex in  $V_n$  is a potential candidate for this neighbor. If each of the  $m + 1$  vertices in  $V_n$  has at least one red edge leading to  $M$ , then by pigeonhole principle, some vertex in  $M$  has two neighbors in  $V_n$  with the corresponding edges being red. Since  $V_n$  forms an independent set in  $F_{H,L}$ , this gives an induced red copy of  $P_3$ , a contradiction. Hence, at least one of the vertices in  $V_n$ , has all of its edges to  $M$  in blue color, thereby giving us an induced blue copy of  $H$  with  $v_n \in V_n$ . This proves the claim and thus the theorem.  $\square$

The following simple fact observed in [GL02] will be used for lower bounds on  $IR(P_3, H)$ .

**Lemma 2.3.** *Let  $F$  and  $H$  be any graphs and  $f$  be any red–blue edge coloring of  $F$ . If an edge  $uv \in E(F)$  is colored red, then at most one of  $u$  and  $v$  can belong to an induced in  $F$  blue copy of  $H$ . As a consequence, any blue induced copy of  $H$  in  $F$  contains at most one vertex from each red clique in  $F$ .*

We now prove that the bound of Theorem 2.1 is sharp for the disjoint unions of complete graphs. The sign  $+$  between graphs below denotes the disjoint union of corresponding graphs.

**Theorem 2.4.** *For any positive integers  $n_1 \leq \dots \leq n_m$ ,*

$$IR(P_3, K_{n_1} + K_{n_2} + \dots + K_{n_m}) = \sum_{i=1}^m \frac{n_i(n_i + 1)}{2}.$$

*Proof.* Let  $H = K_{n_1} + K_{n_2} + \dots + K_{n_m}$ . The upper bound follows from Theorem 2.1. Choose an  $IR$ -graph  $F$  for  $P_3$  and  $H$  with fewest vertices.

We will make  $n_m$  attempts to color the edges of  $F$ . Let  $f_1$  be the coloring of all the edges of  $F$  with blue. Since  $F$  is an  $IR$ -graph for  $P_3$  and  $H$ , there is an induced copy  $H_1$  of  $H$ . Recall that  $|V(H_1)| = n_1 + n_2 + \dots + n_m$ . Color the edges of  $H_1$  with red and all other edges with blue. This is  $f_2$ . Again, by the choice,  $F$  contains an induced copy  $H_2$  of  $H$ . Let  $H_{1,2} = H_1 - V(H_2)$ . Since all the edges of  $H_1$  are red, by Lemma 2.3, at most one vertex from each clique in  $H_1$  belongs to  $V(H_2)$ . Hence  $|V(H_{1,2})| \geq (n_1 - 1) + \dots + (n_m - 1)$  and

$$|V(F)| \geq |V(H_{1,2})| + |V(H_2)| \geq \sum_{i=1}^m (n_i + (n_i - 1)).$$

Color the edges of  $H_2$  and of  $H_{1,2}$  with red and all other edges of  $F$  with blue. This is  $f_3$ . Again, by the choice,  $F$  contains an induced copy  $H_3$  of  $H$ . And we do this way  $m$  times in total.

In general, after the  $k$ th attempt, we have a new blue induced in  $F$  copy  $H_k$  of  $H$  and  $k - 1$  partially destroyed copies of  $H$ : for  $i = 1, \dots, k - 1$ , let  $H_{i,k} = H_i - V(H_{i+1}) - V(H_{i+2}) - \dots - V(H_k)$ . The subgraphs  $H_{i,k}$  are vertex disjoint from each other and from  $H_k$ . Furthermore, by Lemma 2.3, for every  $i = 1, \dots, k - 1$ , each clique in  $H_{i,k-1}$  has at most one vertex in common with  $H_k$ . Therefore,

$$|V(H_{i,k})| \geq \max\{0, n_1 - (k - i)\} + \dots + \max\{0, n_m - (k - i)\}$$

and hence

$$|V(F)| \geq \sum_{i=1}^k \sum_{j=1}^m \max\{0, n_j - (k - i)\} = \sum_{j=1}^m \sum_{l=0}^{k-1} \max\{0, n_j - l\}. \quad (3)$$

Thus, after the  $n_m$ th attempt, (3) yields

$$|V(F)| \geq \sum_{j=1}^m \sum_{l=0}^{n_m} \max\{0, n_j - l\} = \sum_{j=1}^m \sum_{l=0}^{n_j} (n_j - l) = \sum_{j=1}^m \frac{n_j(n_j + 1)}{2}.$$

This proves the theorem. □

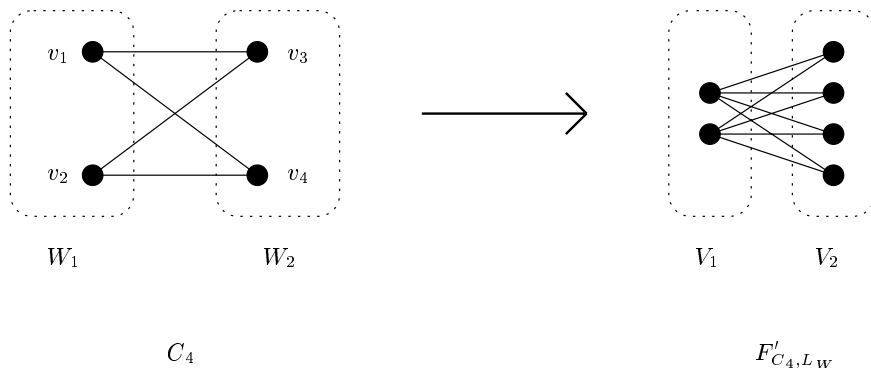
The bound might be sharp for some other graphs, but it is not sharp for graphs having vertices with the same nontrivial neighborhood, like complete multipartite graphs. For such graphs, we modify the bound.

Let  $H$  be a graph. Say that vertices  $v$  and  $w$  are *equivalent* (and write  $v \sim w$ ) if their neighborhoods are the same. In particular, equivalent vertices are not adjacent. Partition  $V(H)$  into the equivalence classes  $V(H) = W_1 \cup \dots \cup W_s$  so that  $|W_1| \leq \dots \leq |W_s|$ . Let  $L_W = (v_1, v_2, \dots, v_n)$  be a list of vertices of  $H$  such that it first encounters vertices in  $W_1$ , then in  $W_2$ , and so on. By the construction, all “degrees to the left”  $d_{H, L_W}(v_i)$  are the same for vertices in the same equivalence class.

**Theorem 2.5.** *For every graph  $H$ , and any choice of  $w_j \in W_j$  for  $j = 1, \dots, s$ ,*

$$IR(P_3, H) \leq |V(H)| + \sum_{j=1}^s d_{H, L_W}(w_j).$$

*Proof.* Given a graph  $H$ , we construct the graph  $F' = F'_{H, L_W}$  as follows. The vertex set of  $F'$  is  $V(F'_{H, L_W}) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_s$ , where  $|V_j| = |W_j| + d_{H, L_W}(w_j)$ . For every edge  $(w_i, w_j)$  in  $H$ , we add all the edges between  $V_i$  and  $V_j$  in  $F'_{H, L_W}$ . This construction is illustrated for the graph  $C_4$  in Fig. 2 (compare with Fig. 1).



**Fig. 2.** Another  $IR$ -graph for  $P_3$  and  $C_4$

Note that  $|V(F'_{H, L})| = |V(H)| + \sum_{j=1}^s d_{H, L_W}(w_j)$ .

*Claim 2.6.* For each red-blue edge coloring of  $F'_{H, L_W}$ , it contains either an induced red copy of  $P_3$  or an induced blue copy of  $H$  such that for every  $j = 1, \dots, s$ , each  $w_i \in W_j$  belongs to  $V_j$ .

*Proof.* The proof is by induction on  $s$  and is very similar to that of Claim 2.2. The claim is trivially true for  $s = 1$ , in which case  $H$  and  $F'$  are equal edgeless graphs. Suppose that the claim holds for each graph with less than  $s$  equivalence classes. Consider a graph  $H$  with  $s$  equivalence classes. Let  $f$  be a red–blue edge coloring of  $F'$ . Consider the graph  $H' = H - W_s$  and let  $L'_W$  be the list of vertices of  $H'$  obtained from  $L_W$  by deleting  $W_s$ . Then  $F'_{H',L'_W} = F'_{H,L_W} - W_s$ . Assume that  $F'_{H,L_W}$  has no induced red  $P_3$ . Then, as a subgraph of  $F'_{H,L_W}$ , the graph  $F'_{H',L'_W}$  also has no induced red  $P_3$ . Thus, by the induction hypothesis,  $F'_{H',L'_W}$  has an induced blue copy of  $H'$  such that for every  $j = 1, \dots, s-1$  each  $w_i \in W_j$  belongs to  $V_j$ . Let  $w_s$  have  $m$  neighbors in  $H$ . Then,  $|V_s| = m + |W_s|$ . Let  $M$  be the set of  $m$  vertices in the induced blue copy  $\tilde{H}$  of  $H'$  in  $F'_{H',L'_W}$  that need  $|W_s|$  new common neighbors to make the graph  $H$ . Each of the  $m + |W_s|$  vertices in  $V_n$  is a potential candidate for such a neighbor. If at least  $m + 1$  of the  $m + |W_s|$  vertices in  $V_s$  have at least one red edge leading to  $M$ , then, by pigeonhole principle, some vertex in  $M$  has two neighbors in  $V_s$  with the corresponding edges being red. Since  $V_s$  forms an independent set in  $F'_{H,L_W}$ , this gives an induced red copy of  $P_3$ , a contradiction. Hence, at least  $|W_s|$  of the vertices in  $V_s$ , have all their edges to  $M$  in blue, thereby giving us an induced blue copy of  $H$  with every vertex of  $W_s$  in  $V_s$ . This proves the claim and thus the theorem.  $\square$

*Remark.* For graphs with large equivalence classes, the bound of Theorem 2.5 is significantly better than that of Theorem 2.1. For example, Theorem 2.1 yields  $IR(P_3, K_{m,m}) \leq 2m + m^2$ , while Theorem 2.5 gives a stronger bound of  $IR(P_3, K_{m,m}) \leq 3m$ . In fact, the bound of Theorem 2.5 is tight for complete multipartite graphs.

**Theorem 2.7.** *Let  $n_1 \leq n_2 \leq \dots \leq n_s$  be positive integers and  $H = K_{n_1, n_2, \dots, n_s}$ . Then*

$$IR(P_3, H) = sn_1 + (s - 1)n_2 + \dots + n_s = \sum_{i=1}^s n_i(s + 1 - i).$$

*Proof.* The upper bound follows from Theorem 2.5. Choose an  $IR$ -graph  $F$  for  $P_3$  and  $H$  with fewest vertices.

We will make  $s$  attempts to color the edges of  $F$ . Let  $f_1$  be the coloring of all the edges of  $F$  with blue. Since  $F$  is an  $IR$ -graph for  $P_3$  and  $H$ , there is an induced copy  $H_1$  of  $H$ . Recall that  $|V(H_1)| = n_1 + n_2 + \dots + n_s$ . Let  $H'_1$  be a spanning subgraph of  $H_1$  which is the disjoint union of  $n_1$  cliques of size  $s$ , and  $n_2 - n_1$  cliques of size  $(s - 1)$ , and so on all the way down to  $n_s - n_{s-1}$  cliques of size 1. Color the edges of  $H'_1$  with red and all other edges with blue. This is  $f_2$ . Again, by the construction and the choice of  $F$ , it contains an induced copy  $H_2$  of  $H$ . Let  $H_{1,2} = H_1 - V(H_2)$ . Since all the edges of  $H'_1$  are red, by Lemma 2.3, the set  $V(H'_1) \cap V(H_2)$  is independent in  $H'_1$ . Recall that  $H'_1$  has  $n_s$  disjoint cliques. Hence  $|V(H_{1,2})| \geq n_1 + \dots + n_{s-1}$  and

$$|V(F)| \geq |V(H_{1,2})| + |V(H_2)| \geq n_s + 2 \sum_{i=1}^{s-1} n_i.$$

Let  $H'_2$  be a subgraph of  $H_2$  isomorphic to  $H'_1$ . Color the edges of  $H'_2$  and of  $H_{1,2}$  with red and all other edges of  $F$  with blue. This is  $f_3$ . Again,  $F$  contains an induced copy  $H_3$  of  $H$ . And we do this way  $s$  times in total.

In general, after the  $k$ th attempt, we have a new induced in  $F$  blue copy  $H_k$  of  $H$  and  $k - 1$  partially destroyed copies of  $H$ : for  $i = 1, \dots, k - 1$ , let  $H_{i,k} = H'_i - V(H_{i+1}) - V(H_{i+2}) - \dots - V(H_k)$ , where  $H'_i$  is a spanning subgraph of  $H_i$  which is a disjoint union of  $n_s$  cliques isomorphic to  $H'_1$ . The subgraphs  $H_{i,k}$  are vertex disjoint from each other and from  $H_k$ . Furthermore, by Lemma 2.3, for every  $i = 1, \dots, k - 1$ ,  $H'_i - V(H_{i,k})$  is the union of  $k - i$  independent sets in  $H'_i$ . By the construction of  $H'_1$ , such union can contain at most  $n_s + n_{s-1} + \dots + n_{s-k+i+1}$  vertices. Therefore,  $|V(H_{i,k})| \geq n_1 + \dots + n_{s-k+i}$  and hence

$$|V(F)| \geq \sum_{i=1}^k \sum_{j=1}^{s-k+i} n_j = n_s + 2n_{s-1} + \dots + (k - 1)n_{s-k+2} + k \sum_{i=1}^{s-k+1} n_i. \quad (4)$$

Thus, after the  $s$ th attempt, (4) yields

$$|V(F)| \geq n_s + 2n_{s-1} + \dots + sn_1.$$

This proves the theorem. □

In fact, the bound of Theorem 2.5 is exact for all disjoint unions of multipartite graphs.

**Theorem 2.8.** *Let  $n_{1,1} \leq n_{1,2} \leq \dots \leq n_{1,s_1}$ ,  $n_{2,1} \leq n_{2,2} \leq \dots \leq n_{2,s_2}, \dots$ ,  $n_{m,1} \leq n_{m,2} \leq \dots \leq n_{m,s_m}$  be positive integers. Let  $H$  be the disjoint union of the complete multipartite graphs  $H_1 = K_{n_{1,1}, n_{1,2}, \dots, n_{1,s_1}}$ ,  $H_2 = K_{n_{2,1}, n_{2,2}, \dots, n_{2,s_2}}, \dots$ ,  $H_m = K_{n_{m,1}, n_{m,2}, \dots, n_{m,s_m}}$ . Then*

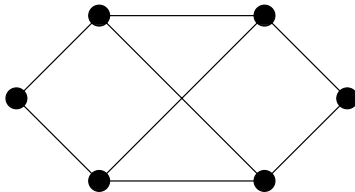
$$IR(P_3, H) = \sum_{i=1}^m IR(P_3, H_i).$$

The upper bound immediately follows from Theorem 2.5 and the proof of the lower bound practically repeats that of Theorem 2.7 only with more subscripts, so we leave it to the reader.

### 3 Weak Versus Ordinary

As it was mentioned in the introduction, Gorgol and Łuczak [GL02] proved that  $IR_w(P_3, P_4) = 6 < IR(P_3, P_4) = 7$ . To prove the upper bound on  $IR_w(P_3, P_4)$ , they made the following observation.

*Claim 3.1.* For each red-blue coloring of the edges of the graph  $F_1$  in Fig. 3 such that the red subgraph has no induced  $P_3$ , the blue subgraph has an induced path connecting the vertices of degree two. In particular, it contains an induced in blue subgraph  $P_4$  starting at any vertex of degree two.



**Fig. 3.** The Gorgol-Luczak example

Recall a couple of definitions. Let  $F$  be a graph. For a set  $T \subset V(F)$ , let  $o(F - T)$  be the number of *odd* components of  $F - T$ , i.e. components of odd order, and let  $\text{def}(T) = o(F - T) - |T|$  be called *the deficiency of  $T$* . The *deficiency of  $F$*  is

$$\text{def}(F) = \max_{W \subseteq V(F)} \text{def}(W) = \max_{W \subseteq V(F)} \{o(F - W) - |W|\}.$$

Let  $\pi(F)$  denote the size of a maximum matching in  $F$ . By Berge–Tutte Formula,  $\text{def}(F) = |V(F)| - 2\pi(F)$ .

The main result of this section confirming that (1) holds is the following.

**Theorem 3.2.** *For a positive integer  $k$ , let  $H_k$  be the vertex disjoint union of  $k$  paths  $P_4$ . Then  $IR_w(P_3, H_k) \leq 6k$  and  $IR(P_3, H_k) \geq 6.1k$ . In particular,  $\frac{IR(P_3, H_k)}{IR_w(P_3, H_k)} \geq 1 + 1/60$  for each positive integer  $k$ .*

*Proof.* The upper bound on  $IR_w(P_3, H_k)$  is easy: we let  $F_k$  be the disjoint union of  $k$  copies of the 6-vertex graph in Fig. 3. As observed by Gorgol and Luczak, for any red-blue coloring of the edges of  $F_k$ , each copy either contains induced in red  $P_3$  or induced in blue  $P_4$ . Thus, the whole  $F_k$  either contains induced in red  $P_3$  or induced in blue  $H_k$ . The lower bound on  $IR(P_3, H_k)$  needs more work.

For a contradiction, suppose that  $IR(P_3, H_k) = (6 + \varepsilon)k$ , where  $\varepsilon < 0.1$  (and possibly is negative). By definition,  $\varepsilon k$  is an integer. Consider a graph  $F$  with  $N = (6 + \varepsilon)k$  vertices such that for each red-blue coloring of its edges, either some induced in  $F$  subgraph isomorphic to  $P_3$  has all its edges colored red, or some induced in  $F$  subgraph isomorphic to  $H_k$  has all its edges colored blue.

Lemma 2.3 implies the next simple observation.

*Claim 3.3.*      $2k \leq \pi(F) \leq (2 + \varepsilon)k$ .



*Proof.* If  $\pi(F) < 2k$ , then  $F$  itself does not contain  $H_k$  which has a matching of size  $2k$ . Hence, by coloring all the edges of  $F$  in blue, we avoid both red  $P_3$  and blue  $H_k$  (even non-induced). This contradicts the choice of  $F$ .

If  $F$  has a matching  $M$  with  $|M| > (2 + \varepsilon)k$ , then we color the edges of  $M$  red and all other edges blue. We do not have red  $P_3$  at all. If we have a blue induced copy  $H'$  of  $H_k$  in  $F$ , then by Lemma 2.3, at most  $|M|$  vertices incident to the edges of  $M$  can belong to  $V(H')$ . Hence

$$\begin{aligned} |V(H') \cup V(M)| &= |V(H')| + |V(M)| - |V(H') \cap V(M)| \\ &\geq 4k + |M| > 4k + (2 + \varepsilon)k = N, \end{aligned}$$

a contradiction. □

Among the sets  $W \subset V(F)$  such that  $\text{def}(F) = o(F - W) - |W|$  (i.e., among the sets of maximum deficiency), choose a set  $X$  of the maximum cardinality. The maximality of cardinality implies that all components of the graph  $F - X$  are odd. Then by Claim 3.3,

$$(2 - \varepsilon)k \leq \text{def}(X) \leq (2 + \varepsilon)k. \tag{5}$$

Let  $A_1$  denote the set of components of  $F - X$  that are cliques and  $V_1$  be the set of vertices in all components in  $A_1$ . Similarly, let  $A_2$  denote the set of components of  $F - X$  that are not cliques and  $V_2$  be the set of vertices in all components in  $A_2$ . Furthermore, let  $x = |X|$  and for  $i = 1, 2$ , let  $a_i = |A_i|$  and  $v_i = |V_i|$ .

Since  $V(F) = X \cup V_1 \cup V_2$ , we have

$$x + v_1 + v_2 = (6 + \varepsilon)k. \tag{6}$$

The following is the left inequality in (5) rewritten using the names of quantities at hand:

$$(2 - \varepsilon)k \leq a_1 + a_2 - x. \tag{7}$$

*Claim 3.4.*

$$4k \leq 2x + \frac{a_2 + v_2}{2}. \tag{8}$$

*Proof.* Color with red all edges in components in  $A_1$  and a maximum matching in each component in  $A_2$ . Since  $X$  is a set of maximum deficiency, every odd component of  $F - X$  (and in particular every component in  $A_2$ ) has a matching saturating all but one vertex. Color all other edges of  $F$  with blue. Since every component of the obtained red graph is a clique, we have no red induced  $P_3$ . Hence, by the choice of  $F$  we have an induced blue copy  $H'$  of  $H_k$ . Note that by Lemma 2.3,  $H'$  can have at most one vertex in each component in  $A_1$ . Moreover, if a  $P_4$  has a vertex  $w$  in a component  $C \in A_1$ , then all neighbors

of  $w$  in this  $P_4$  are in  $X$ . Hence  $V_1 \cup X$  can contain at most  $2x$  vertices of  $H'$ . Again by Lemma 2.3, each component  $C \in A_2$  has at most  $(1 + |V(C)|)/2$  vertices of  $H'$ . Since  $|V(H')| = 4k$ , this proves the claim.  $\square$

If we add to Equation (6) Inequality (8) multiplied by 2 and Inequality (7) multiplied by 3, then we get

$$8k - 4\epsilon k + v_1 \leq 3a_1 + 4a_2. \tag{9}$$

Since  $a_1 \leq v_1$ , (9) yields the following:

$$4k - 2\epsilon k \leq a_1 + 2a_2. \tag{10}$$

*Claim 3.5.*  $F$  has an independent set  $T$  with  $|T| = 4k - 2\epsilon k$ .

*Proof.* Compose the independent set  $T'$  by taking a vertex from each component in  $A_1$  and taking two non-adjacent vertices from each component in  $A_2$ . Then  $|T'| = a_1 + 2a_2$  and by (10), this is at least  $4k - 2\epsilon k$ . Now, let  $T$  be any subset of  $T'$  of size  $4k - 2\epsilon k$ .  $\square$

From now on, we fix in  $F$  an independent set  $T$  with  $|T| = 4k - 2\epsilon k$  and let  $S = V(F) - T$ . Note that  $|S| = 2k + 3\epsilon k$ .

*Claim 3.6.* The size of a maximum matching in the subgraph  $F(S)$  induced by  $S$  in  $F$  is at most  $3\epsilon k$ .

*Proof.* Suppose that there is a matching  $M_0$  in  $F(S)$  with  $|M_0| > 3\epsilon k$ . Color the edges of  $M_0$  with red and the remaining edges with blue. By the choice of  $F$ , it contains a blue induced subgraph  $C_0$  isomorphic to  $H_k$ . Since the independence number of  $H_k$  is  $2k$ , at most  $2k$  vertices of  $C_0$  are in  $T$  and hence at least  $2k$  vertices of  $C_0$  should be in  $S$ . But by Lemma 2.3,  $S$  contains at most  $|S| - |M_0| < (2k + 3\epsilon k) - 3\epsilon k = 2k$  vertices of  $C_0$ , a contradiction.  $\square$

If  $F$  does not contain induced copies of  $H_k$ , then we color all its edges blue and get a coloring contradicting the choice of  $F$ . Otherwise, choose an induced copy  $C_1$  of  $H_k$ . Denote  $B_1 = S \cap V(C_1)$  and  $D_1 = T \cap V(C_1)$ . Since  $T$  is independent,  $|B_1| \geq 2k$ . Let  $|B_1| = 2k + \alpha_1 k$ , where  $\alpha_1 \geq 0$ .

Since  $H_k$  is the union of  $k$  copies of  $P_4$ , it has the unique perfect matching, containing two edges in each copy of  $P_4$ . We will call this matching *principal*. Color the edges of the principal matching in  $C_1$  red and all other edges of  $F$  blue. By the choice of  $F$ , we still have a blue induced in  $F$  subgraph  $C_2$  isomorphic to  $H_k$ . Similarly to above, let  $B_2 = S \cap V(C_2)$ ,  $D_2 = T \cap V(C_2)$ , and  $|B_2| = 2k + \alpha_2 k$ , where  $\alpha_2 \geq 0$ .

Observe that each vertex in  $V(C_1)$  is incident to a red edge. Therefore by Lemma 2.3, each vertex in  $D_1 \cap D_2$ , has a neighbor (using a red edge) in  $|B_1 - B_2|$ . This gives us

$$\begin{aligned} |D_1 \cap D_2| &\leq |B_1 - B_2| \\ &\leq |S - B_2| = 2k + 3\epsilon k - (2k + \alpha_2 k) = 3\epsilon k - \alpha_2 k \end{aligned} \quad (11)$$

and hence

$$\begin{aligned} |D_1 \cup D_2| &= |D_1| + |D_2| - |D_1 \cap D_2| \\ &\geq 2k - \alpha_1 k + 2k - \alpha_2 k - (3\epsilon - \alpha_2)k = k(4 - 3\epsilon - \alpha_1). \end{aligned} \quad (12)$$

Also, we note that

$$|T - (D_1 \cup D_2)| \leq 4k - 2\epsilon k - k(4 - 3\epsilon - \alpha_1) = \epsilon k + \alpha_1 k \quad (13)$$

and

$$\begin{aligned} |B_1 \cap B_2| &\geq |B_1| + |B_2| - |S| \\ &= 2k + \alpha_1 k + 2k + \alpha_2 k - k(2 + 3\epsilon) = (2 + \alpha_1 + \alpha_2 - 3\epsilon)k. \end{aligned} \quad (14)$$

Let  $D'$  be the set of vertices in  $D_1 \cup D_2$  that have two neighbors in  $B_1 \cap B_2$ .

*Claim 3.7.* The subgraph of  $F$  induced by  $D' \cup (B_1 \cap B_2)$  contains a matching  $M$  that saturates  $D'$ .

*Proof.* We first observe that in the graph  $H_k$ , every edge connecting  $B_1 \cap B_2$  with  $D_1 \cup D_2$  belongs either to  $E(C_1)$  or to  $E(C_2)$  and each vertex is adjacent to exactly one vertex of degree two. By the definition, each vertex in  $D'$  has two neighbors in  $B_1 \cap B_2$ . In particular, this means that it has two neighbors either in  $C_1$  or in  $C_2$ . This means that each  $w \in B_1 \cap B_2$  has at most two neighbors in  $D_1 \cup D_2$ . Hence, by Hall's Theorem, the claimed matching  $M$  exists.  $\square$

Now we color the edges of the matching  $M$  provided by Claim 3.7 with red and all other edges with blue. By the choice of  $F$ , it contains a blue induced subgraph  $C_3$  isomorphic to  $H_k$ . We will say that a component of  $C_3$  (which is a  $P_4$ ) is in  $N$ -shape, if it has exactly two vertices in  $S$  and these vertices are not adjacent. The illustration in Fig. 4 explains why the name is used. Note that each component of  $C_3$  that is not in  $N$ -shape has at least one edge with both ends in  $S$ . If we take an edge with both ends in  $S$  from each such component, they will form a matching of size  $k - y$ , where  $y$  is the number of components of  $C_3$  in  $N$ -shape. Then Claim 3.6 yields that  $y \geq k - 3\epsilon k$ .

Let  $B_3 = S \cap V(C_3)$ ,  $D_3 = T \cap V(C_3)$ , and  $|B_3| = 2k + \alpha_3 k$ . Since  $y \geq k - 3\epsilon k$ , there are at least  $k - 3\epsilon k$  vertices of  $D_3$  that have two neighbors in  $B_3$  (using blue edges). Let  $Z$  be the set of vertices of  $D_3$  that have two neighbors in  $B_3 \cap B_1 \cap B_2$ . Since by (14),

$$\begin{aligned} |B_3 - (B_1 \cap B_2)| &\leq |S - (B_1 \cap B_2)| \\ &\leq (2 + 3\epsilon)k - (2 + \alpha_1 + \alpha_2 - 3\epsilon)k = (6\epsilon - \alpha_1 - \alpha_2)k, \end{aligned} \quad (15)$$

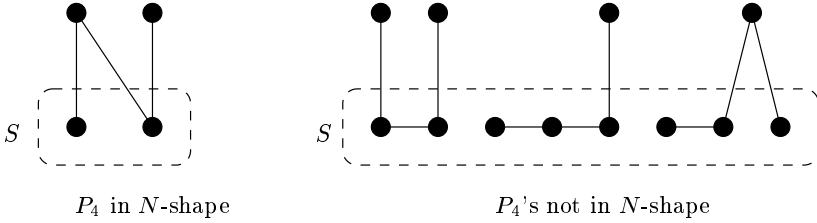


Fig. 4. Possibilities for copies of  $P_4$

we have

$$\begin{aligned}
 |Z| &\geq y - |B_3 - (B_1 \cap B_2)| \\
 &\geq k(1 - 3\varepsilon) - (6\varepsilon - \alpha_1 - \alpha_2)k = k(1 - 9\varepsilon + \alpha_1 + \alpha_2). \tag{16}
 \end{aligned}$$

Observe that  $Z \cap (D_1 \cup D_2) = \emptyset$ , because every vertex of  $Z$  has two neighbors in  $B_3 \cap B_1 \cap B_2$ , but every vertex of  $D_1 \cup D_2$  that has two neighbors in  $B_1 \cap B_2$  got one of its incident edges colored red and hence cannot be in  $C_3$ . Thus by (13),

$$|Z| \leq |T - (D_1 \cup D_2)| \leq \varepsilon k + \alpha_1 k.$$

Comparing with (16), we get  $1 - 9\varepsilon + \alpha_1 + \alpha_2 \leq \varepsilon + \alpha_1$ . It follows that,  $\varepsilon \geq 1/10$ , a contradiction. This completes our proof of the theorem.  $\square$

*Remark.* Our graphs  $H_k$  are not connected. A family of connected graphs with similar properties is as follows. Let  $H'_k$  be obtained from  $H_k$  by adding a new vertex  $z$  and connecting it by an edge with an end  $z_i$  of each of the  $k$  copies of  $P_4$  (see Fig. 5). Since  $H_k$  is an induced subgraph of  $H'_k$ ,  $IR(P_3, H'_k) \geq IR(P_3, H_k) \geq (6 + \frac{1}{10})k$ .

For the upper bound on  $IR_w(P_3, H'_k)$ , consider the graph  $F'_k$  obtained from the graph  $F_{k+1}$  by adding a vertex  $y$  adjacent to a vertex  $y_i$  of degree two in every component  $C_i$ ,  $i = 1, \dots, k + 1$  of  $F_{k+1}$  (see Fig. 5). By construction,  $F'_k$  has  $6k + 7$  vertices. Let  $f$  be a red-blue coloring of the edges of  $F'_k$ . If  $F'_k$  does not have an induced red  $P_3$ , at most one edge incident with  $y$  is red. The remaining  $k$  edges  $yy_i$  are blue and by Claim 3.1 inside each of the corresponding  $k$  components  $C_i$  we have an induced  $P_4$  starting at  $y_i$ . Thus,  $IR_w(P_3, H'_k) \leq |V(F'_k)| = 6k + 7$  and

$$\limsup_{k \rightarrow \infty} \frac{IR(P_3, H'_k)}{IR_w(P_3, H'_k)} \geq \lim_{k \rightarrow \infty} \frac{6.1k}{6k + 7} = 1 + 1/60.$$

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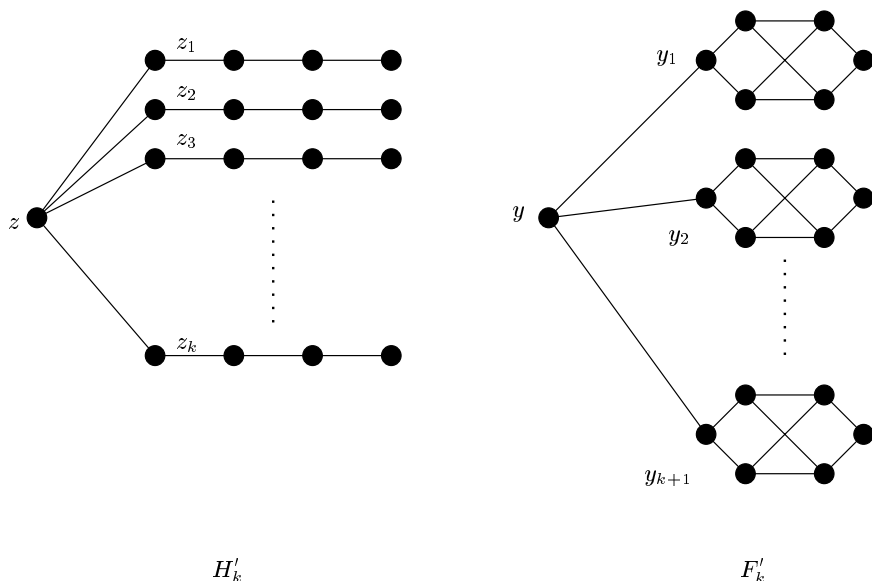


Fig. 5. The graphs  $H'_k$  and  $F'_k$

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