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ON A GRAPH PACKING CONJECTURE BY BOLLOBÁS, ELDRIDGE AND CATLIN*

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Two graphs G_1 and G_2 of order n *pack* if there exist injective mappings of their vertex sets into $[n]$, such that the images of the edge sets are disjoint. In 1978, Bollobás and Eldridge, and independently Catlin, conjectured that if $(\Delta(G_1)+1)(\Delta(G_2)+1)\le n+1$, then G_1 and G_2 pack. Towards this conjecture, we show that for $\Delta(G_1),\Delta(G_2)\ge 300$, if $(\Delta(G_1)+1)(\Delta(G_2)+1)\le 0.6n+1$, then G_1 and G_2 pack. This is also an improvement, for large maximum degrees, over the classical result by Sauer and Spencer that G_1 and G_2 pack if $\Delta(G_1)\Delta(G_2)<0.5n$.

1. Introduction

Two n-vertex graphs G_1 and G_2 are said to *pack* if there exist injective mappings of their vertex sets onto $[n]=\{1,\dots,n\}$ such that the images of the edge sets do not intersect. In other words, G_1 and G_2 *pack* if G_1 is isomorphic to a subgraph of the complement of G_2. This concept leads to a natural generalization of a number of problems in extremal graph theory, such as existence of a fixed subgraph, equitable colorings, and Turán-type problems.

The study of extremal problems on packings of graphs was started in the 1970s by Bollobás and Eldridge [3,4], Sauer and Spencer [13], and Catlin [6].

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(See the surveys by Wozniak [14] and Yap [15] for later developments in this field.) In particular, Sauer and Spencer [13] proved the following result.

Theorem 1 (Sauer and Spencer). *Let G_1 and G_2 be n-vertex graphs with maximum degrees Δ_1 and Δ_2, respectively. If $2\Delta_1\Delta_2 < n$, then G_1 and G_2 pack.*

Kaul and Kostochka [12] gave a characterization of the pairs (G_1, G_2) of n-vertex graphs with $2\Delta_1\Delta_2 = n$ that do not pack.

The main conjecture in this area was posed in 1978 by Bollobás and Eldridge [4], and independently by Catlin [7].

Conjecture 1 (Bollobás, Eldridge, and Catlin). Let G_1 and G_2 be n-vertex graphs with maximum degrees Δ_1 and Δ_2. If $(\Delta_1+1)(\Delta_2+1) \le n+1$, then there is a packing of G_1 and G_2.

If true, this conjecture would be sharp, and a considerable extension of the Hajnal–Szemerédi Theorem [10] on equitable colorings. The Hajnal–Szemerédi Theorem is a special case of the conjecture when G_2 is the disjoint union of cliques of the same size. This conjecture has been proved in the case $\Delta_1 \le 2$ by Aigner and Brandt [1] and Alon and Fisher (for sufficiently large n) [2], and in the case when $\Delta_1 = 3$ and n is huge by Csaba, Shokoufandeh, and Szemerédi [8]. Bollobás, Kostochka and Nakprasit [5] proved a strengthening of the conjecture when G_1 is d-degenerate and $d < \Delta_1/40$. Eaton [9] showed that under the given condition, there is a near-packing of degree 1 of G_1 and G_2, that is, an embedding of the two graphs into a common vertex set such that the maximum degree of the subgraph defined by the edges common to both copies is 1.

In this paper, instead of proving the conjecture for another class of graphs we give a weaker bound for all graphs with high maximum degrees.

Problem 1. For a fixed $0 \le \epsilon \le 1$, and $|V(G_1)| = |V(G_2)| = n$ and

$$(\Delta_1 + 1)(\Delta_2 + 1) \le \frac{1}{2}n(1+\epsilon) + 1\,.$$

Do G_1 and G_2 pack?

The case $\epsilon = 0$ is essentially the Sauer–Spencer Theorem, while the case $\epsilon = 1$ is the Bollobás–Eldridge–Catlin conjecture. We show that if G_1 and G_2 satisfy the above condition for $\epsilon = 0.2$, and if Δ_1 and Δ_2 are relatively large, then G_1 and G_2 pack.

Theorem 2. *Let G_1 and G_2 be n-vertex graphs with maximum degrees Δ_1 and Δ_2, respectively. If $\Delta_1, \Delta_2 \geq 300$ and*

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq 0.6n + 1, \tag{1}$$

then G_1 and G_2 pack.

This improves the bound of the Sauer–Spencer Theorem for large Δ_1 and Δ_2 and thus partially answers Problem 4.4 in [11].

In the next section, we give some definitions and prove a basic lemma that is heavily used later. In Section 3 we derive some structural properties of a hypothetical minimal counterexample to our theorem. In Section 4 we prove the theorem modulo the main lemma, Lemma 2. Lemma 2 is proved in the last two sections.

2. Preliminaries

Without loss of generality, we fix a graph G_1 with $V(G_1) = V_1 = [n]$. Given G_2 with $V_2 = V(G_2)$ and $|V_2| = n$, we will try to find an injective $f : V_2 \to [n]$ extending to an embedding of G_2 in $\overline{G_1}$.

The result of each bijection $f : V_2 \to [n]$ will be viewed as a (multi)graph G with edges of two types. The vertex set is $[n]$, and two vertices u_1 and u_2 are connected by an edge in E_1 if $u_1u_2 \in E(G_1)$ and by an edge in E_2 if $f^{-1}(u_1)f^{-1}(u_2) \in E(G_2)$. Each vertex $u \in V(G)$ has two kinds of neighborhoods: $N_1(u) = \{v \in V(G) : uv \in E_1\}$ and $N_2(u) = \{v \in V(G) : uv \in E_2\}$. Let $N(u) = N_1(u) \cup N_2(u)$ and $d_1(u) = |N_1(u)|$ and $d_2(u) = |N_2(u)|$.

For each such mapping f and $i, j \in \{1, 2\}$, a $(u, v; i, j)$-*link* is a path of length two from $u \in V(G)$ to $v \in V(G)$ passing through some vertex $w \in V(G)$ such that $uw \in E_i$ and $wv \in E_j$. A *link* is a $(u, v; i, j)$-link for some $u, v \in V(G)$ and $i, j \in \{1, 2\}$ with $i \neq j$.

A $(u_1, u_2, \ldots, u_k)$-*swap* replaces a mapping f with a mapping f' that differs from f only in that $(f')^{-1}(u_i) = f^{-1}(u_{i-1})$, for $1 \leq i \leq k$, where indices are treated modulo k.

The following lemma provides a basic tool that will be used repeatedly in the proof. It allows us to transform an embedding of G_2 by making 'vertex swaps' that do not increase the number of conflicting edges. In fact, often a good choice of the vertices to be swapped will lead to disappearing of conflicting edges. In the statement of the lemma, the indices sum modulo k.

Lemma 1. *Let G_1, G_2, f, G, links and vertex swaps be defined as above. Let $u_1, \ldots, u_k$ be vertices of G.*

(1) Suppose that there are no $(u_i, u_{i+1}; 2, 1)$*-links for any* i*. If there are no* $1 \le i < j \le k$ *such that* $u_i u_j \in E(G_2)$ *and* $u_{i+1}u_{j+1} \in E(G_1)$*, then a* $(u_1, \dots, u_k)$*-swap does not create new conflicting edges.*

(2) Suppose that there are no $(u_i, u_{i+1}; 1, 2)$*-links for any* i*. If there are no* $1 \le i < j \le k$ *such that* $u_i u_j \in E(G_1)$ *and* $u_{i+1}u_{j+1} \in E(G_2)$*, then a* $(u_1, u_k, \dots, u_2)$*-swap does not create new conflicting edges.*

Proof. Assume that G does contain a sequence $u_1, \dots, u_k$ satisfying conditions in (1) of the lemma. Consider the mapping f' that differs from f only in that $(f')^{-1}(u_i) = f^{-1}(u_{i+1})$ for $i = 1, \dots, k$ (indices are taken modulo k). Let $G' = G(f')$.

Since G has no $(u_i, u_{i+1}; 2, 1)$-links for any i, no conflicting edge in G' connects the set $U = \{u_1, \dots, u_k\}$ with $V(G') - U$. Because there are no $1 \le i < j \le k$ such that $u_i u_j \in E(G_2)$ and $u_{i+1}u_{j+1} \in E(G_1)$, no conflicting edge in G' appears inside U.

The proof for (2) is similar. ∎

This lemma is a useful generalization of the idea of the proof of Theorem 1 [13] by Sauer and Spencer.

3. Properties of hypothetical counterexamples

Suppose that Theorem 2 does not hold. Then for some n there is a critical pair (G_1, G_2) of n-vertex graphs G_1 and G_2 satisfying the conditions of the theorem. By the choice of the pair (G_1, G_2), there is a mapping $f: V(G_2) \to [n]$ such that some edge $e = u^* v^*$ of the resulting multigraph G is the only parallel edge in G.

Lemma 1 immediately yields that every vertex in G is within distance two of u^*:

Claim 1. *For every vertex* $v \in V(G) - \{u^*, v^*\}$*, there is either a* $(u^*, v; 1, 2)$*-link or a* $(u^*, v; 2, 1)$*-link.*

Proof. Suppose not. Then after a (u^*, v)-swap there are no parallel edges between u^* and v^*, and by Lemma 1, G_1 and G_2 pack. ∎

Thus, $V(G) - \{v^*\} = N_1(N_2(u^*)) \cup N_2(N_1(u^*))$. This allows us to partition $V(G)$ into meaningful subsets whose sizes can be estimated by counting arguments. Define $A_1 = N_2(N_1(u^*)) - N(u^*)$ and $B_1 = N_1(N_2(u^*)) - N(u^*)$. Let $A = A_1 - B_1, B = B_1 - A_1$ and $C = A_1 \cap B_1$.

Let $\sigma = \Delta_1 + \Delta_2$. By (1),

$$\Delta_1 \Delta_2 \le 0.6n - \sigma. \qquad (2)$$

By (2), for $i=1,2$ we have $\Delta_i \le (0.6n-\sigma)/\Delta_{3-i}$. Since $\Delta_1, \Delta_2 \ge 300$, we obtain that

$$\Delta_1, \Delta_2 \le 0.002n - 2 \;\text{ and }\; \sigma \le 0.004n - 4. \tag{3}$$

Also by (2),

$$|A_1|, |B_1| \le \Delta_1 \Delta_2 \le 0.6n - \sigma.$$

By Claim 1, $|A_1 \cup B_1| \ge n - \sigma$.

It follows that

$$|A| = |A_1 \cup B_1| - |B_1| \ge (n-\sigma) - (0.6n - \sigma) = 0.4n, \tag{4}$$

and similarly,

$$|B| = |A_1 \cup B_1| - |A_1| \ge (n-\sigma) - (0.6n - \sigma) = 0.4n. \tag{5}$$

By Claim 1, $V(G) - \{v^*\} = N_1(N_2(u^*)) \cup N_2(N_1(u^*))$. By definition, $N_1(N_2(u^*)) \cap N_2(N_1(u^*)) = C \cup \{u^*\}$, thus

$$\begin{aligned} |C| &= |N_1(N_2(u^*))| + |N_2(N_1(u^*))| - |N_1(N_2(u^*)) \cup N_2(N_1(u^*))| - 1 \\ &\le 2\Delta_1\Delta_2 - (n-1) - 1 \le 1.2n - 2\sigma - n = 0.2n - 2\sigma, \end{aligned} \tag{6}$$

it follows that

$$|A| + |B| \ge n - \sigma - |C| \ge 0.8n + \sigma. \tag{7}$$

Lemma 1 allows us to make the following observation about vertices in A and B.

Claim 2. *For each $a \in A, b \in B$, there is an $(a,b;2,1)$-link.*

Proof. Assume for a contradiction that for some $a \in A$ and $b \in B$, there is no $(a,b;2,1)$-link. Note that there is no $(b,u^*;2,1)$-link and no $(u^*,a;2,1)$-link. Also, $au^* \notin E(G)$ and $bu^* \notin E(G)$, thus by Lemma 1, after an (a,b,u^*)-swap, G_1 and G_2 pack, a contradiction. ∎

Next, we need to partition C into more informative subsets. Let

$$A' = \{c \in C : \text{ there exists } a \in A \text{ such that } ac \notin E_2, \text{ and there is no } (a,c;2,1)\text{-link}\}$$

and

$$B' = \{c \in C : \text{ there exists } b \in B \text{ such that } bc \notin E_1, \text{ and there is no } (c,b;2,1)\text{-link}\}$$

and let $C' = C - A' - B'$.

A' and B' are defined in such a way that vertices in A' (respectively, B') behave 'similarly' to those in A (respectively, B).

Claim 3. *$A' \cap B' = \emptyset$. Moreover, for each $c_1 \in A'$, and for each $b \in B$, there is a $(c_1, b; 2, 1)$-link; and for each $c_2 \in B'$ and for each $a \in A$, there is an $(a, c_2; 2, 1)$-link.*

Proof. Suppose that $c \in A' \cap B'$. Then there exist $a \in A$ and $b \in B$ such that there is no $(c, b; 2, 1)$-link and there is no $(a, c; 2, 1)$-link and $ca \notin E_2$. Note that there is no $(u^*, a; 2, 1)$-link and there is no $(b, u^*; 2, 1)$-link, also, $u^*a, u^*b \notin E(G)$. Thus by Lemma 1, after a (u^*, a, c, b)-swap, we get a packing of G_1 and G_2.

Moreover, let $c_1 \in A'$ and $b \in B$. Then by the above argument, $c_1 \notin B'$. Suppose for a contradiction that there is no $(c_1, b; 2, 1)$-link. Note that, since $c_1 \notin B'$, $bc_1 \in E_1$. Also, by definition of A', there exists $a \in A$ such that there is no $(a, c_1; 2, 1)$-link and $c_1 a \notin E_2$. As in the argument above, we can now apply Lemma 1 to get a contradiction through a (u^*, a, c_1, b)-swap.

The proof for $c_2 \in B'$ is similar. ∎

4. Proof of Theorem 2

By symmetry, from now on we assume that $|B \cup B'| \geq |A \cup A'|$. Then by (7) we have

$$|B \cup B'| \geq 0.5(n - \sigma - |C'|). \tag{8}$$

The main idea of the proof is to count the number of pairs of vertices in $A \times B$ with a unique $(2,1)$-link between them, and get contradictory lower and upper bounds. If $|C'|$ is 'small', we will use vertices from B' as well.

Let

$$B_0 = \begin{cases} B \cup B', & \text{if } |C'| \leq \frac{1}{15} n, \\ B, & \text{otherwise.} \end{cases}$$

and let N denote the set of pairs (a, b), $a \in A$, $b \in B_0$ having a unique $(a, b; 2, 1)$-link.

Let M be the set of central vertices lying on those unique links. The size of M gives an upper bound on $|N|$:

$$|N| \leq |M| \Delta_1 \Delta_2. \tag{9}$$

We will bound the size of M by estimating its intersection with $B \cup B'$ and $A \cup A'$.

Lemma 2.
(10)
$$|M \cap (B \cup B')| \leq \begin{cases} \varDelta_1\varDelta_2 - |A| - |A'| - \frac{1}{2}|C'| + \frac{5\sigma}{2}, \text{ always;} \\ \varDelta_1\varDelta_2 - |A| - |A'| - |C'| + \frac{5\sigma}{2}, \text{ if } |B \cup B'| \leq \frac{8}{15}n - \sigma. \end{cases}$$

Furthermore,

$$|M \cap (A \cup A')| \leq \varDelta_1\varDelta_2 - |B| - |B'| - |C'| + 2.5\sigma. \tag{11}$$

This main lemma will be proved in the next two sections. By Claims 2 and 3, it directly leads to the following bound on $|M|$:

$$\begin{aligned} |M| &\leq |M \cap (B \cup B')| + |M \cap (A \cup A')| + |M \cap C'| + \sigma \\ &\leq \begin{cases} 2\varDelta_1\varDelta_2 - n + \frac{1}{2}|C'| + 7\sigma \leq 0.2n + 5\sigma + \frac{1}{2}|C'|, \text{ always;} \\ 2\varDelta_1\varDelta_2 - n + 7\sigma \leq 0.2n + 5\sigma, \text{ if } |B \cup B'| \leq \frac{8}{15}n - \sigma. \end{cases} \end{aligned} \tag{12}$$

Next, we give a lower bound on $|N|$ by a counting argument.

Lemma 3. $|N| \geq 0.4n(|B_0| - 0.6n + |B \cup B' \cup C'|)$.

Proof. Let $a \in A$. At most $\varDelta_1\varDelta_2$ $(2,1)$-links start at a. By Claims 2 and 3, at least $(|B \cup B'| - |B_0|) + (|C'| - \sigma)$ of them land in $(B \cup C) - B_0$. It follows that for at most

$$\varDelta_1\varDelta_2 - (|B \cup B'| - |B_0|) - |C'| + \sigma - |B_0| = \varDelta_1\varDelta_2 - |B \cup B'| - |C'| + \sigma$$

vertices $b \in B_0$, an $(a, b; 2, 1)$-link is not unique.

There are $|A||B_0|$ pairs $(a, b) \in A \times B_0$, and each of them is connected by a $(2,1)$-link. Therefore,

$$\begin{aligned} |N| &\geq |A||B_0| - |A|(\varDelta_1\varDelta_2 - |B \cup B'| - |C'| + \sigma) \\ &= |A|(|B_0| + |B \cup B' \cup C'| - \varDelta_1\varDelta_2 - \sigma) \\ &\geq 0.4n(|B_0| - 0.6n + |B \cup B' \cup C'|). \end{aligned} \tag{13}$$ ∎

Now, using the lower and the upper bounds on $|N|$, we will get a contradiction. This would complete the proof of the theorem, modulo the proof of Lemma 2.

Case 1. $|C'| \geq \frac{1}{15}n$. Then $|B_0| = |B| \geq 0.4n$ and $|N| \geq 0.4n(-0.2n + |B \cup B' \cup C'|)$. Since $|C'| \geq \frac{1}{15}n$, we have $|B \cup B'| \leq (0.6n - \sigma) - |C'| \leq 8/15n - \sigma$. Hence by (12), $|M| \leq 0.2n + 5\sigma$. Combining with (8), we obtain

$$0.4n\Big(-0.2n + \frac{1}{2}(n - \sigma - |C'|) + |C'|\Big) \leq |N| \leq |M|\Delta_1\Delta_2 \leq (0.2n + 5\sigma)(0.6n - \sigma).$$

Since $|C'| \geq \frac{1}{15}n$, we get

$$0.2n\Big(\frac{2}{3}n - \sigma\Big) \leq (0.2n + 5\sigma)(0.6n - \sigma).$$

Opening the parentheses, we come to

$$\frac{n^2}{75} - 3n\sigma + 5\sigma^2 \leq 0.$$

But this inequality does not hold for $0 < \sigma < 0.004n$, a contradiction to (3).

Case 2. $|C'| < \frac{1}{15}n$ and $|B \cup B'| \leq \frac{8}{15}n - \sigma$. Then $B_0 = B \cup B'$ and by (8),

$$|N| \geq 0.4n(2|B \cup B'| + |C'| - 0.6n) \geq 0.4n(0.4n - \sigma).$$

Together with (12), we have

$$0.4n(0.4n - \sigma) \leq |N| \leq |M|\Delta_1\Delta_2 \leq (0.2n + 5\sigma)(0.6n - \sigma),$$

which yields

$$0.04n^2 - 3.2n\sigma + 5\sigma^2 \leq 0.$$

Again, this inequality does not hold for $0 < \sigma \leq 0.004n$, a contradiction to (3).

Case 3. $|C'| < \frac{1}{15}n$ and $|B \cup B'| \geq \frac{8}{15}n - \sigma$. Then again $B_0 = B \cup B'$, and by (13)

$$|N| \geq 0.4n(2|B \cup B'| + |C'| - 0.6n) \geq 0.4n\Big(\frac{16}{15}n - 2\sigma + |C'| - 0.6n\Big) = 0.4n\Big(\frac{7}{15}n - 2\sigma + |C'|\Big).$$

Therefore

$$0.4n\Big(\frac{7}{15}n - 2\sigma + |C'|\Big) \leq |N| \leq |M|\Delta_1\Delta_2 \leq (0.2n + 5\sigma + 0.5|C'|)(0.6n - \sigma),$$

which yields

$$\frac{1}{15}n^2 - 3.6n\sigma + 5\sigma^2 \leq |C'|(-0.5\sigma - 0.1n) \leq 0.$$

Again, this inequality cannot hold for $0 < \sigma \leq 0.004n$, a contradiction to (3).

5. Proof of Lemma 2, Part I

In this section, we prove (10). Let $t = \Delta_1\Delta_2 - |A| - |A'| - |C'| + 2.5\sigma$. The following lemma gives bounds on $|B|$ and $|B \cup B'|$ in terms of t.

Lemma 4. *If* $|B \cup B'| \leq \frac{8}{15}n - \sigma$, *then*
(a) $|B| \geq 2.5t + 1.5\sigma$, *and*
(b) $|B| \geq 2t + |C'| + 0.5\sigma$.
Furthermore, if $|C'| < \frac{1}{15}n$, *then*
(c) $|B \cup B'| \geq 2.5(t + |C'|)$.

Proof. Recall that

$$(14) \qquad \begin{aligned} t &= \Delta_1\Delta_2 - |A| - |A'| - |C'| + 2.5\sigma \\ &\leq 0.6n - \sigma - (n - |B \cup B'| - \sigma) + 2.5\sigma \\ &\leq |B \cup B'| - 0.4n + 2.5\sigma. \end{aligned}$$

Hence statement (a) of the lemma is true if the inequality $|B| \geq 2.5(|B \cup B'| - 0.4n + 2.5\sigma) + 1.5\sigma$ holds. This is equivalent to

$$0.4n \geq \frac{3}{5}|B \cup B'| + \frac{2}{5}|B'| + 3.1\sigma.$$

Since $|B \cup B'| \leq \frac{8}{15}n - \sigma$ (and hence $|B'| \leq \frac{2}{15}n - \sigma$, by (5)), the RHS of the last inequality is at most

$$\frac{3}{5}\left(\frac{8}{15}n - \sigma\right) + \frac{2}{5}\left(\frac{2}{15}n - \sigma\right) + 3.1\sigma = \frac{28n}{75} + 2.1\sigma.$$

Thus to prove (a), it is sufficient to prove that

$$2.1\sigma \leq 0.4n - \frac{112n}{300} = \frac{8n}{300},$$

which is true when $\sigma \leq 0.004n$. This proves (a).

Similarly, statement (b) of the lemma is true if the inequality $|B| \geq 2(|B \cup B'| - 0.4n + 2.5\sigma) + |C'| + 0.5\sigma$ holds. This is equivalent to

$$0.8n \geq |B \cup B' \cup C'| + |B'| + 5.5\sigma.$$

Since $|B \cup B' \cup C'| \leq 0.6n - \sigma$ and $|B'| \leq \frac{2}{15}n - \sigma$, we come to the inequality $0.8n \geq \frac{11}{15}n + 3.5\sigma$ which holds if $\sigma \leq 0.004n$. This proves (b).

Finally, (c) reduces to $|B \cup B'| \geq 2.5(|B \cup B'| - 0.4n + 2.5\sigma + |C'|)$, which is equivalent to

$$n \geq 1.5|B \cup B' \cup C'| + |C'| + 6.25\sigma.$$

As above $|B\cup B'\cup C'|\le 0.6n-\sigma$, and in addition we have $|C'|<\frac{1}{15}n$. Thus we need

$$n \geq 1.5(0.6n-\sigma)+\frac{1}{15}n+6.25\sigma,$$

which is true for $\sigma\le 0.004n$. ∎

Proof of Statement (10) of Lemma 2.

Suppose (10) does not hold. Choose $\varGamma\subset M\cap(B\cup B')$ with

$$|\varGamma| = \begin{cases} t+\frac{1}{2}|C'|, \text{ if } |B\cup B'| \geq \frac{8}{15}n-\sigma\,; \\ t, \text{ otherwise.} \end{cases}$$

Let $\varGamma_B=\varGamma\cap B$ and $\varGamma_{B'}=\varGamma\cap B'$, and let $t_B=|\varGamma_B|$ and $t_{B'}=|\varGamma_{B'}|$.

For each $x\in\varGamma$, we choose $a(x)\in A$ and $b(x)\in B_0$ such that x is the central vertex on the unique $(a,b;2,1)$-link. If $b(x)\in B'$, then by the definition of B', there exists a $b_1(x)\in B$ such that $b(x)b_1(x)\notin E_1$ and there is no $(b(x),b_1(x);2,1)$-link.

Define the auxiliary graph $\mathcal{B}$ as follows: $V(\mathcal{B})=B\cup B'$ and $E(\mathcal{B})=G_{\mathcal{B}}\cup D_{\mathcal{B}}$, where

$$D_{\mathcal{B}} = \left\{\overrightarrow{xy} : x,y\in B\cup B',\ \text{there is an } (x,y;1,2)\text{-link}\right\}$$

and

$$G_{\mathcal{B}} = \{xy : x\in\varGamma, y\in B\cup B', xy\in E_2 \text{ or } a(x)y\in E_2 \text{ or } b(x)y\in E_1 \text{ or } b_1(x)y\in E_1\},$$

where $b_1(x)y\in E_1$ applies only when $b(x)\in B'$.

We call edges in $G_{\mathcal{B}}$ and $D_{\mathcal{B}}$ *G-edges* and *D-edges*, respectively.

We can bound the out-degrees of the vertices in $\mathcal{B}$ as follows.

Claim 4. (c1) *The out-degree of each vertex in B in $\mathcal{B}$ is at most*

$$d=\varDelta_1\varDelta_2-|A|-|A'|-(|C'|-\varDelta_1)=t-2.5\sigma+\varDelta_1.$$

(c2) *The out-degree of each vertex in B' in $\mathcal{B}$ is at most*

$$d'=\varDelta_1\varDelta_2-|A|-(|A'|-\varDelta_1)=t-2.5\sigma+\varDelta_1+|C'|.$$

Proof. For every vertex $u\in B\cup B'$, the number of $(1,2)$-links starting from u is at most $\varDelta_1\varDelta_2$, at least $|A|$ of those land in A by Claims 2 and 3.

To prove (c1), observe that by Claim 3 and the definition of C', at least $|A'|+|C'|-\varDelta_1$ $(1,2)$-links starting at u land in $A'\cup C'$, thus at most $\varDelta_1\varDelta_2-(|A|+|A'|+|C'|-\varDelta_1)$ of those $(1,2)$-links land in $B\cup B'$.

By the definition of B', there exists $x \in B$ such that $ux \notin E_1$ and $\overrightarrow{xu} \notin D_{\mathcal{B}}$ when $u \in B'$. To prove (c2), it suffices to show that for each $a' \in A' - N_1(x)$, there is a $(u, a'; 1, 2)$-link.

Suppose not. Then by the definition of A', there is $a'' \in A$ such that $a'a'' \notin E_2$ and there is no $(a', a''; 1, 2)$-link. Note that $a'a'' \notin E_2$, $a'x \notin E_1$, $ux \notin E_1$ and $u^*a'', u^*x \notin E(G)$, thus by Lemma 1, after a (u^*, a'', a', u, x)-swap, there is no conflicting edges, a contradiction. ∎

Therefore, the number of G-edges incident with $B - \Gamma$ and D-edges (directed) from $B-\Gamma$ is at most $(|B| - t_B)d + 2\sigma|\Gamma|$, and the number of G-edges incident with $(B \cup B') - \Gamma$ and D-edges (directed) from $(B \cup B') - \Gamma$ is at most

$$(|B| - t_B)d + (|B'| - t_{B'})d' + 2\sigma|\Gamma|. \tag{15}$$

In the rest of the proof, we will prove that we have more G-edges or D-edges incident with $(B \cup B') - \Gamma$ than is given by (15). The obtained contradiction will prove the lemma. To do this, we will show that we may assign $|\Gamma|$ distinct G-edges or D-edges to each vertex in $(B \cup B') - \Gamma$.

If each $u \in (B \cup B') - \Gamma$ has either a G-edge or a D-edge to every vertex of Γ, we are done. Otherwise, consider each vertex $x \in (B \cup B') - \Gamma$ not having $|\Gamma|$ edges to Γ.

Let $\Gamma_x = \{y \in \Gamma : xy \notin G_{\mathcal{B}}, \overrightarrow{xy} \notin D_{\mathcal{B}}\}$. If $\Gamma_x = \emptyset$ then x has $|\Gamma|$ edges to Γ.

For each vertex $y \in \Gamma_x$, let $Z_y = \{z \in (B \cup B') - \Gamma - \{x\} : yz \notin E, \overrightarrow{yz} \notin D_{\mathcal{B}}\}$.

Next, we prove two technical claims that will be helpful in the final counting argument.

Claim 5. *Given $z \in Z_y$ such that $a(y)z \notin E_2$, and if $b(y) \in B'$ implies that $b(y)z \notin E_2$ and $b_1(y)z \notin E_1$, then $\overrightarrow{zx} \in D_{\mathcal{B}}$.*

Proof. Assume $\overrightarrow{zx} \notin D_{\mathcal{B}}$ for some $z \in Z_y$ satisfying the conditions in Claim 5. Then $\overrightarrow{xy}, \overrightarrow{yz}, \overrightarrow{zx} \notin D_{\mathcal{B}}$ and $yz \notin E, xy \notin E_2$. By Lemma 1, a (y, x, z)-swap creates no new conflicting edges. Note then after the (y, x, z)-swap, there is no $(a(y), b(y); 2, 1)$-link, since $xa(y), za(y) \notin E_2$ and $xb(y) \notin E_1$.

If $b(y) \in B$, then since $\overrightarrow{a(y)u^*}, \overrightarrow{u^*b(y)}, \overrightarrow{b(y)a(y)} \notin D_{\mathcal{B}}$ and $a(y)u^*, b(y)u^* \notin E(G)$, by Lemma 1, an $(a(y), b(y), u^*)$-swap gives a packing of G_1 and G_2.

If $b(y) \in B'$, then since $xb_1(y) \notin E_1$, $b(y)z \notin E_2$ and $b_1(y)z \notin E_1$, after the (y, x, z)-swap, there is still no $(b(y), b_1(y); 2, 1)$-link. Therefore, after a $(u^*, a(y), b(y), b_1(y))$-swap, by Lemma 1, G_1 and G_2 pack. ∎

Claim 6. *For any subset $\Gamma'_x \subseteq \Gamma_x$, there is some vertex $y_0 \in \Gamma'_x$ with the out-degree at least $0.5(|\Gamma'_x| - 2\sigma)$ in $\mathcal{B}[\Gamma'_x]$.*

Proof. Consider any vertex $y\in\Gamma'_x$. We claim that for each $z\in\Gamma'_x-N_1(b(y))-N_2(a(y))-N_1(b_1(y))-N_2(b(y))$, either $\overrightarrow{yz}$ or $\overrightarrow{zy}$ is a D-edge. Otherwise, since $zb(y)\notin E_1, za(y)\notin E_2$, after a (y,z)-swap, there is no $(a(y),b(y);2,1)$-link.

Then if $b(y)\in B$, after a $(u^*,a(y),b(y))$-swap, by Lemma 1, there are no conflicting edges anymore. This is a contradiction.

If $b(y)\in B'$, since $zb(y)\notin E_1$ and $zb_1(y)\notin E_2$, after the (y,z)-swap, there is still no $(b(y),b_1(y);2,1)$-link. Therefore, after an $(u^*,a(y),b(y),b_1(y))$-swap, by Lemma 1, G_1 and G_2 pack. This is also a contradiction.

We may assume $|\Gamma'_x|\geq 2\sigma$. Therefore, the sum of the out-degrees of vertices in Γ'_x is at least $0.5|\Gamma'_x|(|\Gamma'_x|-2\sigma)$. Then there is some vertex $y_0\in\Gamma'_x$ such that the out-degree of y_0 is at least $0.5(|\Gamma'_x|-2\sigma)$. ∎

Now, we do the final computations, considering two cases.

Case 1. $|B\cup B'|\geq\frac{8}{15}n-\sigma$. Then $|C'|\leq\frac{1}{15}n$. By Lemma 4, $|B\cup B'|\geq 2.5(t+|C'|)$.

Claim 7. *$\Gamma_x=\emptyset$, that is, each $x\in(B\cup B')-\Gamma$ has $|\Gamma|$ edges to Γ.*

Proof. Suppose that $\Gamma_x\neq\emptyset$. By Claim 6, choose $y\in\Gamma_x$ with the out-degree at least $0.5(|\Gamma_x|-2\sigma)$ in $\mathcal{B}[\Gamma_x]$. Then y has at most $d'-0.5(|\Gamma_x|-2\sigma)$ D-edges to $(B\cup B')-\Gamma$. By Claim 5, there are $|Z_y|-\Delta_2-\sigma$ D-edges from Z_y to x.

Note that $|Z_y|\geq|B\cup B'|-|\Gamma|-\sigma-(d'-0.5(|\Gamma_x|-2\sigma))$. To assign $|\Gamma|$ D-edges to x, it suffices to require

$$|B\cup B'|-|\Gamma|-\sigma-(d'-0.5(|\Gamma_x|-2\sigma))-\Delta_2-\sigma\geq|\Gamma_x|, \tag{16}$$

that is,

$$\begin{aligned}|B\cup B'|&\geq|\Gamma|+d'+|\Gamma_x|-0.5(|\Gamma_x|-2\sigma)+2\sigma+\Delta_2\\&=t+d'+0.5|\Gamma_x|+0.5|C'|+\sigma+\Delta_2\\&=2t+0.5|\Gamma_x|+1.5|C'|-0.5\sigma.\end{aligned}$$

It suffices that

$$|B\cup B'|\geq 2.5t+1.5|C'|,$$

and this is true since $|B\cup B'|\geq 2.5(t+|C'|)$. ∎

Thus, in this case, we have assigned $|\Gamma|$ distinct G-edges or D-edges to each vertex in $(B\cup B')-\Gamma$.

Now, consider the edges incident with $(B\cup B')-\Gamma$. We have

$$\begin{aligned}&(|B\cup B'|-|\Gamma|)|\Gamma|\leq(|B|-t_B)d+(|B'|-t_{B'})d'+2\sigma|\Gamma|\\&\quad\leq(|B|-t_B)(t-2.5\sigma+\Delta_1)+(|B'|-t_{B'})(t-2.5\sigma+|C'|+\Delta_1)+2\sigma|\Gamma|\\&\quad=(|B\cup B'|-|\Gamma|)(t-2.5\sigma+\Delta_1)+|C'|(|B'|-t_{B'})+2\sigma|\Gamma|\\&\quad\leq(|B\cup B'|-|\Gamma|)t+|C'|(|B'|-t_{B'})-(|B\cup B'|-|\Gamma|)(1.5\sigma)+2\sigma|\Gamma|.\end{aligned}$$

Since $|B\cup B'|-|\Gamma|\geq 2.5(t+|C'|)-(t+0.5|C'|)\geq 1.5|\Gamma|$,

$$-(|B\cup B'|-|\Gamma|)(1.5\sigma)+2\sigma|\Gamma|\leq 0.$$

From the above inequality, we have

$$\frac{1}{2}(|B\cup B'|-|\Gamma|)|C'|\leq |C'|(|B'|-t_{B'}).$$

It follows that

$$|B|\leq t_B+(|B'|-t_{B'})\leq t+|B'|. \tag{17}$$

By (14),

$$t\leq |B\cup B'|-0.4n+2.5\sigma\leq \Delta_1\Delta_2-0.4n+2.5\sigma\leq 0.2n+1.5\sigma.$$

Thus (17) together with (6) implies

$$0.4n\leq (0.2n+1.5\sigma)+(0.2n-2\sigma),$$

a contradiction.

Case 2. $|B\cup B'|\leq \frac{8}{15}n-\sigma$. By Lemma 4, $|B|\geq \max\{2.5t+1.5\sigma, 2t+|C'|+0.5\sigma\}$.

Claim 8. $\Gamma_x\cap\Gamma_B=\emptyset$.

Proof. Let $t_x=|\Gamma_x\cap\Gamma_B|$ and assume $t_x\neq 0$. By Claim 6, choose $y_1\in\Gamma_x\cap\Gamma_B$ with the out-degree in $\mathcal{B}[\Gamma_x\cap\Gamma_B]$ at least $0.5(t_x-2\sigma)$. Then y_1 has at most $d-0.5(t_x-2\sigma)$ D-edges to $(B\cup B')-\Gamma$. By Claim 5, there are $|Z_{y_1}|-\Delta_2-\sigma$ D-edges from Z_{y_1} to x.

Note that

$$|Z_{y_1}|\geq |B\cup B'|-t-\sigma-(d-0.5(t_x-2\sigma))\geq |B|-t_B-\sigma-(d-0.5(t_x-2\sigma)).$$

To assign t D-edges to x, it suffices to require

$$|B|-t_B-\sigma-(d-0.5(t_x-2\sigma))-\Delta_2-\sigma\geq |\Gamma_x|=t_x+|\Gamma_x\cap B'|,$$

that is,

$$|B|\geq 2\sigma+d+\Delta_2+t_B+|\Gamma_x\cap B'|+t_x-0.5(t_x-2\sigma). \tag{18}$$

Since $t_B+|\Gamma_x\cap B'|\leq t$, (18) holds if

$$|B|\geq 1.5t+d+3\sigma+\Delta_2=2.5t+1.5\sigma.$$

Since $|B|\geq 2.5t+1.5\sigma$, we are done. ∎

Claim 9. *$\Gamma_x = \emptyset$, that is, each $x \in (B \cup B') - \Gamma$ has $|\Gamma|$ edges to Γ.*

Proof. Assume that $\Gamma_x \neq \emptyset$. By Claim 6, there exists $y_2 \in \Gamma_x \cap B'$ with the out-degree at least $0.5(|\Gamma_x \cap B'| - 2\sigma)$ in $\mathcal{B}[\Gamma_x \cap B']$. Then y_2 has at most $d' - 0.5(|\Gamma_x \cap B'| - 2\sigma)$ D-edges to $B - \Gamma$.

By Claim 5, there are $|Z_{y_2}| - \Delta_2 - \sigma$ D-edges from Z_{y_2} to x.

Note that $|Z_{y_2}| \geq |B| - t_B - \sigma - (d' - 0.5(|\Gamma_x \cap B'| - 2\sigma))$. By Claim 8, to assign t edges to x, it suffices to require

$$|Z_{y_2}| - \Delta_2 - \sigma \geq |B| - t_B - \sigma - (d' - 0.5(|\Gamma_x \cap B'| - 2\sigma)) - \Delta_2 - \sigma \geq t - t_B. \tag{19}$$

To get (19), it suffices that

$$|B| \geq t + d' + 2\sigma + \Delta_2 = 2t + |C'| + 0.5\sigma.$$

Since $|B| \geq 2t + |C'| + 0.5\sigma$, the above inequality holds. ∎

Thus, in this case, we have assigned $|\Gamma|$ distinct G-edges or D-edges to each vertex in $(B \cup B') - \Gamma$.

Now consider the edges incident with $B - \Gamma$. We have

$$(|B| - t_B)t \leq (|B| - t_B)d + 2\sigma t,$$

which yields $|B| < t_B + 2\sigma t / t - d$. Since $t - d \geq 2.5\sigma - \Delta_1 \geq 1.5\sigma$ and

$$|B| \geq 2.5t \geq t_B + 1.5t \geq t_B + \frac{4t}{3} \geq t_B + t\frac{2\sigma}{t-d},$$

we have a contradiction. This completes the proof of (10) in Lemma 2. ∎

6. Proof of Lemma 2, Part II

We need to prove (11). The proof mostly repeats part of the previous section and is simpler, since by (8), $|A \cup A'|$ is always less than $0.5n$. But there is some asymmetry between A and B. So we mostly will refer to Section 5 but will give definitions and prove a couple of claims to be on the safe side.

Let $t' = \Delta_1 \Delta_2 - |B| - |B'| - |C'| + 2.5\sigma$. Symmetrically to Lemma 4, we have the following fact.

Lemma 4′. *(a)* $|A| \geq 2.5t' + 1.5\sigma$, *and*
(b) $|A| \geq 2t' + |C'| + 0.5\sigma$.

The proof repeats that of Lemma 4. Recall again that we do not need a statement symmetrical to (c) of Lemma 4, since $|A \cup A'| < 0.5n$.

So, suppose that (11) does not hold. Choose $\Gamma \subset M \cap (A \cup A')$ with $|\Gamma| = t'$. Let $\Gamma_A = \Gamma \cap A$ and $\Gamma_{A'} = \Gamma \cap A'$, and let $t_A = |\Gamma_A|$ and $t_{A'} = |\Gamma_{A'}|$.

For each $x \in \Gamma$, we choose $a(x) \in A, b(x) \in B_0$ such that x is the central vertex lying on the unique $(a, b; 2, 1)$-link. If $b(x) \in B'$, then by the definition of B', there exists $b_1(x) \in B$ such that $b(x)b_1(x) \notin E_1$ and there is no $(b(x), b_1(x); 2, 1)$-link.

Define the auxiliary graph $\mathcal{A}$ as follows: $V(\mathcal{A}) = A \cup A'$ and $E(\mathcal{A}) = G_{\mathcal{A}} \cup D_{\mathcal{A}}$, where

$$D_{\mathcal{A}} = \left\{\overrightarrow{xy} : x, y \in A \cup A',\ \text{there is } (x, y; 2, 1)\text{-link}\right\}$$

and

$$G_{\mathcal{A}} = \{xy : x \in \Gamma, y \in A \cup A', xy \in E_2 \text{ or } a(x)y \in E_2 \text{ or } b(x)y \in E_1 \text{ or } b_1(x)y \in E_1\},$$

where $b_1(x)y \in E_1$ applies only when $b(x) \in B'$.

We call the edges in $G_{\mathcal{A}}$ and $D_{\mathcal{A}}$ *G-edges* and *D-edges*, respectively. The following claim is proved exactly as Claim 4.

Claim 4′. (c1′) *The out-degree of each vertex in A in $\mathcal{A}$ is at most*

$$d = \Delta_1\Delta_2 - |B| - |B'| - (|C'| - \Delta_2) = t' - 2.5\sigma + \Delta_2.$$

(c2′) *The out-degree of each vertex in A' in $\mathcal{A}$ is at most*

$$d' = \Delta_1\Delta_2 - |B| - (|B'| - \Delta_2) = t' - 2.5\sigma + \Delta_2 + |C'|.$$

Therefore the number of G-edges incident with $A - \Gamma$ and D-edges (directed) from $A - \Gamma$ is at most $(|A| - t_A)d + 2\sigma|\Gamma|$.

We now show that we may assign $|\Gamma|$ $(= t')$ distinct G-edges or D-edges to each vertex in $A - \Gamma$.

If each $u \in A - \Gamma$ has either a G-edge or a D-edge to every vertex of Γ, then we are done. Otherwise, consider each vertex $x \in A - \Gamma$ not having t' such edges to Γ.

Let $\Gamma_x = \{y \in \Gamma : xy \notin G_{\mathcal{A}}, \overrightarrow{xy} \notin D_{\mathcal{A}}\}$.

For each vertex $y \in \Gamma_x$, let $Z_y = \{z \in A \cup A' - \Gamma - \{x\} : yz \notin E, \overrightarrow{yz} \notin D_{\mathcal{A}}\}$.

Claim 5′. *If $z \in Z_y$ such that $b(y)z \notin E_1$, then $\overrightarrow{zx} \in D_{\mathcal{A}}$.*

Proof. Assume $\overrightarrow{zx} \notin D_{\mathcal{A}}$ for some $z \in Z_y$ with $b(y)z \notin E_1$. Then there is no $(x,y;2,1)$-link, no $(y,z;2,1)$-link, no $(z,x;2,1)$-link, and $yz \notin E$, $xy \notin E_2$. By Lemma 1, a (x,y,z)-swap creates no new conflicting edges. Note that, after the (x,y,z)-swap, there is no $(a(y),b(y);2,1)$-link, since by the definition of $G_{\mathcal{A}}$, $xa(y) \notin E_2$ and $xb(y) \notin E_1$, and by our assumption $b(y)z \notin E_1$. Recall that G_1 stays fixed while the embedding of G_2 changes under a swap operation.

If $b(y) \in B$, then since there is no $(u^*, a(y);2,1)$-link, no $(a(y),b(y);2,1)$-link, no $(b(y),u^*;2,1)$-link, and $a(y)u^*, b(y)u^* \notin E(G)$, by Lemma 1, a $(u^*, a(y), b(y))$-swap gives a packing of G_1 and G_2, a contradiction.

If $b(y) \in B'$, then since $yb(y) \notin E_2$, $b_1(y)x \notin E_1$ and $b(y)x \notin E_2$ (since there is no $(x,y;2,1)$-link), after the (x,y,z)-swap, there is still no $(b(y),b_1(y);2,1)$-link. By Lemma 1, G_1 and G_2 pack after an $(u^*,a(y),b(y),b_1(y))$-swap, since there are no $(2,1)$-links between consecutive vertices, no edges from u^* to any other vertices, and $b(y)b_1(y) \notin E_1$, a contradiction. ∎

Claim 6′. *For any subset $\Gamma'_x \subseteq \Gamma_x$, there is some vertex $y_0 \in \Gamma'_x$ with the out-degree at least $0.5(|\Gamma'_x| - 2\sigma)$ in $\mathcal{A}[\Gamma'_x]$.*

Proof. Consider any vertex $y \in \Gamma'_x$. We claim that for each $z \in \Gamma'_x - N_1(b(y)) - N_2(a(y)) - N_1(b_1(y)) - N_2(b(y))$, either $\overrightarrow{yz}$ or $\overrightarrow{zy}$ is a D-edge.

If not, then after a (y,z)-swap, there is no $(a(y),b(y);2,1)$-link, since $zb(y) \notin E_1$ and $za(y) \notin E_2$.

If $b(y) \in B$, then as above, a $(u^*,a(y),b(y))$-swap gives a packing of G_1 and G_2. This is a contradiction.

If $b(y) \in B'$, then after the (y,z)-swap there is still no $(b(y),b_1(y);2,1)$-link, since $zb(y) \notin E_2$ and $zb_1(y) \notin E_1$. Therefore, as above, the $(u^*,a(y),b(y), b_1(y))$-swap gives a packing of G_1 and G_2. This is a contradiction. The sum of the out-degrees of vertices in Γ'_x is at least $0.5|\Gamma'_x|(|\Gamma'_x| - 2\sigma)$. Then there is some vertex $y_0 \in \Gamma'_x$ such that the out-degree of y_0 is at least $0.5(|\Gamma'_x| - 2\sigma)$ in Γ'_x. ∎

The rest of the proof simply repeats Case 2 of the main proof in Section 5. Again, we do not need Case 1 because $|A \cup A'| < 0.5n$.

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