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On Two Conjectures on Packing of Graphs

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For Béla Bollobás on his 60th birthday

In 1978, Bollobás and Eldridge [5] made the following two conjectures.

(C1) There exists an absolute constant $c > 0$ such that, if k is a positive integer and G_1 and G_2 are graphs of order n such that $\Delta(G_1), \Delta(G_2) \leqslant n - k$ and $e(G_1), e(G_2) \leqslant ckn$, then the graphs G_1 and G_2 pack.

(C2) For all $0 < \alpha < 1/2$ and $0 < c < \sqrt{1/8}$, there exists an $n_0 = n_0(\alpha, c)$ such that, if G_1 and G_2 are graphs of order $n > n_0$ satisfying $e(G_1) \leqslant \alpha n$ and $e(G_2) \leqslant c\sqrt{n^3/\alpha}$, then the graphs G_1 and G_2 pack.

Conjecture (C2) was proved by Brandt [6]. In the present paper we disprove (C1) and prove an analogue of (C2) for $1/2 \leqslant \alpha < 1$. We also give sufficient conditions for simultaneous packings of about $\sqrt{n}/4$ sparse graphs.

1. Introduction

One of the basic notions of graph theory is that of *packing*. Two graphs, G_1 and G_2, of the same order are said to *pack* if G_1 is a subgraph of the complement $\overline{G}_2$ of G_2, or,

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equivalently, G_2 is a subgraph of the complement $\overline{G}_1$ of G_2. The study of packings of graphs was started in the 1970s by Sauer and Spencer [13] and Bollobás and Eldridge [5].

In particular, Sauer and Spencer [13] proved the following result.

Theorem 1.1. *Suppose that G_1 and G_2 are graphs of order n such that $2\Delta(G_1)\Delta(G_2) < n$. Then G_1 and G_2 pack.*

The main conjecture in the area is the Bollobás–Eldridge–Catlin (BEC) conjecture (see [4, 3, 5, 10]) stating that *if G_1 and G_2 are graphs with n vertices, maximum degrees Δ_1 and Δ_2, respectively, and $(\Delta_1 + 1)(\Delta_2 + 1) \leqslant n + 1$, then G_1 and G_2 pack.* If true, this conjecture is a considerable extension of the Hajnal–Szemerédi theorem [12] on equitable colouring, which is itself an extension of the Corrádi–Hajnal theorem on equitable 3-colourings of graphs. Indeed, the Hajnal–Szemerédi theorem is the special case of the BEC conjecture when G_2 is a disjoint union of cliques of the same size [12]. The conjecture has also been proved when either $\Delta_1 \leqslant 2$ [1, 2], or $\Delta_1 = 3$ and n is huge [11]. The progress on the topic has been surveyed by Yap [16] and Wozniak [15].

The following two theorems are the main results of Bollobás and Eldridge [5].

Theorem 1.2. *Suppose that G_1 and G_2 are graphs with n vertices, $\Delta(G_1)$, $\Delta(G_2) < n - 1$, $e(G_1) + e(G_2) \leqslant 2n - 3$ and $\{G_1, G_2\}$ is not one of the following pairs: $\{2K_2, K_1 \cup K_3\}$, $\{\overline{K}_2 \cup K_3, K_2 \cup K_3\}$, $\{3K_2, \overline{K}_2 \cup K_4\}$, $\{\overline{K}_3 \cup K_3, 2K_3\}$, $\{2K_2 \cup K_3, \overline{K}_3 \cup K_4\}$, $\{\overline{K}_4 \cup K_4, K_2 \cup 2K_3\}$, $\{\overline{K}_5 \cup K_4, 3K_3\}$. Then G_1 and G_2 pack.*

Theorem 1.3. *For $0 < \alpha < 1/2$, there is an integer $n_0 = n_0(\alpha)$ such that, if G_1 and G_2 are graphs of order $n \geqslant n_0$ with $e(G_1) \leqslant \alpha n$ and $e(G_2) \leqslant \frac{1-2\alpha}{5} n^{3/2}$, then G_1 and G_2 pack.*

Let n be even, x be odd, $G_1(n)$ be a perfect matching on n vertices and $G_2(n, x)$ be the complete bipartite graph $K_{x,n-x}$. Since x is odd, the graphs $G_1(n)$ and $G_2(n, x)$ do not pack. Since $e(G_1(n)) = n/2$ and $e(G_2(n, x)) = x(n - x) < xn$, these examples show that the condition $\alpha < 1/2$ in Theorem 1.3 cannot be relaxed without imposing other restrictions on G_1 and/or G_2. However, Bollobás and Eldridge [5] could not find an example showing that the factor $(1 - 2\alpha)/5$ is close to optimal, and they were led to the following conjecture.

Conjecture 1.4. *For all $0 < \alpha < 1/2$ and $0 < c < \sqrt{1/8}$, there exists an $n_0 = n_0(\alpha, c)$ such that, if G_1 and G_2 are graphs of order $n > n_0$ satisfying $e(G_1) \leqslant \alpha n$ and $e(G_2) \leqslant c\sqrt{n^3/\alpha}$, then the graphs G_1 and G_2 pack.*

This conjecture was proved by Brandt [6] in 1995. As the main result of this paper, we prove the following extension of this theorem of Brandt to the case when G_1 has αn edges, with $1/2 \leqslant \alpha < 1$.

Theorem 1.5. *Let $1/2 \leqslant \alpha < 1$ and $c > 0$ satisfy*

$$8\alpha c^2 < 1, \tag{1.1}$$

and put

$$\varepsilon = \frac{1}{4}\min\{1-\alpha,\ 1-8\alpha c^2\}. \tag{1.2}$$

Let G_1 and G_2 be graphs of order

$$n > (10/\varepsilon)^6 \tag{1.3}$$

such that $e(G_1) \leqslant \alpha n$, $e(G_2) \leqslant cn^{3/2}$, and $\Delta(G_2) < n - 1 - \frac{\sqrt{n}}{\sqrt{2\alpha(1-\alpha)}}$. Then G_1 and G_2 pack.

Observe that the only additional restriction in Theorem 1.5 is that each vertex in G_2 has at least $\frac{\sqrt{n}}{\sqrt{2\alpha(1-\alpha)}}$ non-neighbours. The example of $G_1(n)$ and $G_2(n,x)$ where x is the largest odd integer not exceeding $c\sqrt{n}$ shows that the factor $\sqrt{n}$ is unavoidable there.

The examples of a perfect matching and $G_2(n,x)$ also explain why Bollobás and Eldridge [5, p. 118] made the following conjecture.

Conjecture 1.6. *There exists an absolute constant $c > 0$ such that, if $k \geqslant 1$ and G_1 and G_2 are graphs of order n satisfying the conditions $\Delta(G_1), \Delta(G_2) \leqslant n-k$ and $e(G_1), e(G_2) \leqslant ckn$, then the graphs G_1 and G_2 pack.*

We shall disprove Conjecture 1.6; more precisely, we shall prove the following result.

Theorem 1.7. *Let k be a positive integer and q be a prime power. Then for every $n \geqslant q\frac{q^{k+1}-1}{q-1}$, there are graphs $G_1(n,k)$ and $G_2(n,q,k)$ of order n that do not pack and have the following properties:*

(a) *$G_1(n,k)$ is a forest with $n-k$ edges and maximum degree at most n/k;*
(b) *$G_2(n,q,k)$ is a $\frac{q^k-1}{q-1}$-degenerate graph with maximum degree at most $2n/q$.*

Theorem 1.7 not only disproves Conjecture 1.6, but also shows that Theorem 1.5 can not be extended even to $\alpha = 1$ without essential restrictions on the maximal degree of G_2.

The rest of the paper is organized as follows. In the next section we shall discuss properties of special enumerations of vertices in graphs; our proof of Theorem 1.5, which is to be given in Section 3, will be based on these enumerations. In Section 4 we shall make use of the proof of Theorem 1.5 to give conditions providing simultaneous packing of about $\frac{1}{4}\sqrt{n/\alpha^3}$ graphs of order n with at most αn edges each. More precisely, we shall prove the following result.

Theorem 1.8. *Let $\frac{1}{2} \leqslant \alpha < 1$,*

$$n > (50/(1-\alpha))^6, \tag{1.4}$$

and $m = \lceil \frac{1}{4}\sqrt{n/\alpha^3}\rceil$. Let $H_1, H_2, \ldots, H_m$ be graphs with n vertices and at most αn edges each. Then $H_1, H_2, \ldots, H_m$ pack.

In the final section, Section 5, we discuss counterexamples to Conjecture 1.6 and prove Theorem 1.7.

Note that the proofs of upper bounds are algorithmic, and so enable one to construct polynomial-time algorithms for packing graphs satisfying the conditions of Theorems 1.5 or 1.8.

2. Greedy and degenerate enumerations

Before embarking on the proof of Theorem 1.5, we introduce some notation and prove some auxiliary statements.

Let $v_1, v_2, \ldots, v_n$ be an enumeration of the vertices of a graph G. For $1 \leqslant i \leqslant n$, let $G(i)$ be the subgraph of G induced by the vertices $v_i, v_{i+1}, \ldots, v_n$; thus $G(1) = G$ and $G(n)$ consists of the single vertex v_n. We call $v_1, v_2, \ldots, v_n$ a *greedy enumeration* of the vertices or, somewhat loosely, a *greedy order* on G, if $d_{G(i)}(v_i) = \Delta(G(i))$ for every i, $1 \leqslant i \leqslant n$, *i.e.*, the vertex v_i has maximal degree in $G(i)$. Similarly, the enumeration and order are *degenerate* if $d_{G(i)}(v_i) = \delta(G(i))$ for every i, $1 \leqslant i \leqslant n$, *i.e.*, the vertex v_i has minimal degree in $G(i)$. Note that if $v_1, v_2, \ldots, v_n$ is a greedy order on G then $v_i, v_{i+1}, \ldots, v_n$ is a greedy order on $G(i)$, and an analogous assertion holds for the degenerate order. Another simple observation is that $v_1, v_2, \ldots, v_n$ is a greedy order on G if and only if it is a degenerate order on the complement $\overline{G}$. Needless to say, a graph may have numerous greedy orders and degenerate orders.

For a graph G, set

$$\varphi(G) = \sum_{v \in V(G)} \frac{1}{1 + d_G(v)}.$$

The result below is a slight extension of an inequality due to Caro [7] and Wei [14], first published in [8], implying a weak form of Turán's theorem. We formulate it in the usual way, for the complement of the graph, *i.e.*, for finding a large independent set rather than a complete subgraph.

Theorem 2.1. *Let $v_1, v_2, \ldots, v_n$ be a greedy enumeration of the vertices of a graph G, and set $\ell = \lceil \varphi(G) \rceil$. Then the last ℓ vertices form an independent set. Equivalently, if $d_{G(i)}(v_i) \geqslant 1$ then $G(i)$ has an independent set of at least $\varphi(G)$ vertices.*

Proof. We apply induction on the number of edges of G. If there are no edges then $\varphi(G) = n$ and the entire vertex set is independent, as required. Suppose that G has $m > 0$ edges and the result holds for graphs with fewer edges. Write d for the maximal degree of G, *i.e.*, for the degree of v_1, and let $u_1, u_2, \ldots, u_d$ be the neighbours of v_1. Then

$$\begin{aligned}
\varphi(G(2)) &= \varphi(G(1)) - \frac{1}{d+1} + \sum_{i=1}^{d} \left(\frac{1}{d(u_i)} - \frac{1}{d(u_i)+1} \right) \\
&= \varphi(G(1)) - \frac{1}{d+1} + \sum_{i=1}^{d} \frac{1}{d(u_i)(d(u_i)+1)} \\
&\geqslant \varphi(G(1)) - \frac{1}{d+1} + d\, \frac{1}{d(d+1)} = \varphi(G(1)) = \varphi(G).
\end{aligned}$$

By the induction hypothesis, the last $\lceil\varphi(G(2))\rceil \geqslant \lceil\varphi(G)\rceil = \ell$ vertices of $v_2, v_3, \ldots, v_n$ form an independent set of $G(2)$, and so of G, completing the proof. □

We shall also need the following simple but somewhat technical lemma concerning greedy orders.

Lemma 2.2. *Let α, γ and ε be positive numbers satisfying $\gamma \leqslant \alpha \leqslant 1-2\varepsilon$ and $k_0 \leqslant (1-\gamma-\varepsilon/2)n-1$ a nonnegative integer. Let $v_1, v_2, \ldots, v_n$ be an enumeration of the vertices of a graph G with m edges with the following properties:*
(i) $e(G_{k_0+1}) \leqslant m(1-\frac{2k_0(\alpha+\varepsilon)}{n(\alpha-\gamma+\varepsilon/2)})$;
(ii) *the enumeration $v_{k_0+1}, v_{k_0+2}, \ldots, v_n$ is greedy.*
Then there is an index i, $k_0 \leqslant i \leqslant (1-\gamma-\varepsilon/2)n$, such that

$$\Delta(G(i+1)) = d_{G(i+1)}(v_{i+1}) < \frac{2m(n-i)(\alpha+\varepsilon)}{n^2(\alpha-\gamma+\varepsilon/2)}. \tag{2.1}$$

Proof. Suppose that the assertion is false. Then for $k = \lceil(1-\gamma-\varepsilon/2)n\rceil$ we have

$$\begin{aligned}
e(G(k+1)) &= e(G(k_0+1)) - \sum_{i=k_0+1}^{k} \Delta(G(i)) \\
&\leqslant m\left(1-\frac{2k_0(\alpha+\varepsilon)}{n(\alpha-\gamma+\varepsilon/2)}\right) - \sum_{i=k_0+1}^{k} \frac{2m(n+1-i)(\alpha+\varepsilon)}{n^2(\alpha-\gamma+\varepsilon/2)} \\
&\leqslant m - \sum_{i=1}^{k} \frac{2m(n+1-i)(\alpha+\varepsilon)}{n^2(\alpha-\gamma+\varepsilon/2)} \\
&= m - \frac{2m(\alpha+\varepsilon)}{n^2(\alpha-\gamma+\varepsilon/2)}\left(\binom{n+1}{2} - \binom{n+1-k}{2}\right) \\
&< m - \frac{m(\alpha+\varepsilon)}{n^2(\alpha-\gamma+\varepsilon/2)}(n^2-(n-k)^2) \\
&\leqslant m\left(1-\frac{(\alpha+\varepsilon)(1-(\gamma+\varepsilon/2)^2)}{(\alpha-\gamma+\varepsilon/2)}\right) = \rho m,
\end{aligned} \tag{2.2}$$

say. To arrive at a contradiction and so complete the proof, we shall show that $\rho < 0$. To this end, set $\delta = \gamma + \varepsilon/2$, and note that

$$\rho(\alpha-\gamma+\varepsilon/2) = \delta(\delta(\alpha+\varepsilon)-1). \tag{2.3}$$

Since, by assumption, $\delta > 0$ and

$$\delta(\alpha+\varepsilon) \leqslant (\alpha+\varepsilon/2)(\alpha+\varepsilon) < 1,$$

identity (2.3) implies that ρ is indeed negative, completing our proof. □

We shall also use the following fact observed by several authors.

Claim 2.3. *Suppose that we are packing the vertices of a graph G_1 in the reverse degenerate order into (the complement of) a graph G_2 of order N and maximal degree D_2. Suppose that* *we have already packed j vertices and a vertex $w \in V(G_1)$ has x neighbours among these j vertices. If*

$$j + xD_2 < N, \tag{2.4}$$

then we can also find a legal placement for w.

Proof. We cannot place w at the j vertices of G_2 that we have already used and into G_2-neighbours of the images of the x neighbours of w. However, w can be mapped into every other vertex of G_2. □

3. Proof of Theorem 1.5

Let G_1 and G_2 be graphs of order $n > (10/\varepsilon)^6$ such that $e(G_1) \leqslant \alpha n$, $e(G_2) \leqslant cn^{3/2}$, and $\Delta(G_2) < n - 1 - \frac{\sqrt{n}}{\sqrt{2\alpha(1-\alpha)}}$. Since $\alpha \geqslant 1/2$, condition (1.1) yields that $c < 1/2$. Since the greater is c, the stronger is the assertion, we may assume that

$$\frac{1}{3} < c < \frac{1}{2}. \tag{3.1}$$

Observe that, by (1.2),

$$8(\alpha + \varepsilon)c^2 < 1 - 2\varepsilon \quad \text{and} \quad \alpha + 2\varepsilon < 1. \tag{3.2}$$

Let $T_1, \ldots, T_t$ be the components of G_1 that are trees (including isolated vertices) with $v(T_1) \leqslant \cdots \leqslant v(T_t)$, where we write $v(H) = |V(H)|$ for the order of a graph H. Let $G_1^* = G_1 - T_1 - \cdots - T_t$. In other words, let G_1^* be the union of the components of G_1 containing cycles. Suppose that G_1^* has exactly γn vertices. Then it has at least γn edges and hence $\gamma \leqslant \alpha$. Since $e(G_1) \leqslant \alpha n$,

$$t \geqslant (1 - \alpha)n. \tag{3.3}$$

It is trivial to check that the following assertion holds.

Claim 3.1. *For every $1 \leqslant j < t$, we have*
(a) $\sum_{i=1}^{j} v(T_i) \leqslant \frac{1-\gamma}{1-\alpha} j$, *and*
(b) $v(T_j) \leqslant \frac{n(1-\gamma)}{t-j+1}$.

Let $w_1, w_2, \ldots, w_n$ be a degenerate order of the vertices of G_1 with the additional condition that first we list vertices in T_1, then those in T_2, and so on, and we enumerate the vertices in G_1^* only after having enumerated all vertices in $T_1, \ldots, T_t$. Let $u_1, u_2, \ldots, u_n$ be a greedy order of the vertices of G_2. Let k_0' be the maximal k such that $\deg_{G_2(k)}(u_k) > \frac{(1-\alpha)^2}{20} n$. Since $e(G_2) \geqslant \sum_{i=1}^{k_0'} \deg_{G_2(k)}(u_k)$, we have

$$k_0' < \frac{20c}{(1-\alpha)^2}\sqrt{n} \leqslant \frac{10}{(1-\alpha)^2}(0.1\varepsilon)^3 n \leqslant 0.01(1-\alpha)n. \tag{3.4}$$

Claim 3.2. *For $j = 1, \ldots, k_0'$, there is a set $U_j \subset V(G_2)$ such that*
(i) $U_j \supset \{u_1, \ldots, u_j\}$,

(ii) $|U_j| = \sum_{i=1}^{j} v(T_i)$,
(iii) *there exists a packing of* $G_1[V(T_1) \cup \cdots \cup V(T_j)]$ *and* $G_2[U_j]$.

Proof. Suppose that the claim is proved for $j' \leqslant j-1 \leqslant k_0'-1$. Assume that the vertices of T_j are $w_{z-y+1}, w_{z-y+2}, \ldots, w_z$. Let m be the smallest index such that $u_m \notin U_{j-1}$. By the induction assumption, $m \geqslant j$. Identify u_m with w_z and denote $v_0 = u_m$. To prove the claim, it is enough to find for every $i = 1, \ldots, y-1$ a vertex $v_i \in V(G_2) - U_{j-1} - \{v_0, \ldots, v_{i-1}\}$ not adjacent to the vertex $v_{i'}$, $i' < i$ that was identified with a neighbour $w_{z-i'}$ of w_i. Then we can identify v_i with w_{z-i} and continue.

Case 1: $j \leqslant 2c\sqrt{n}$. Then by Claim 3.1(a), $|U_{j-1} \cup \{v_0, \ldots, v_{i-1}\}| \leqslant \frac{j}{1-\alpha}$. Since, under conditions of the theorem, $v_{i'}$ has at least $\frac{\sqrt{n}}{\sqrt{2\alpha}(1-\alpha)}$ non-neighbours, it has a non-neighbour in $V(G_2) - U_{j-1} - \{v_0, \ldots, v_{i-1}\}$.

Case 2: $j > 2c\sqrt{n}$. Then $\deg_{G_2(j)}(v_{i'}) \leqslant \deg_{G_2(j)}(u_j) \leqslant \frac{cn^{1.5}}{j} < n/2$ and by Claim 3.1(a), $|U_{j-1} \cup \{v_0, \ldots, v_{i-1}\}| < k_0' \frac{1-\gamma}{1-\alpha}$. By (3.4), the last expression is at most $0.01n$. Again, we can choose v_i as needed. □

Let $U = U_{k_0'}$ be a set provided by the claim above. We reorder the vertices $u_1', u_2', \ldots, u_n'$ of G_2 as follows: first we enumerate the vertices of U in any order, and then enumerate the vertices of $G_2 - U$ in a greedy order. We will denote $k_0 = |U|$.

Claim 3.3. $\varphi(G_2 - U) = \sum_{v \in V(G_2-U)} \frac{1}{1 + d_{G_2-U}(v)} \geqslant \frac{n}{1+2c\sqrt{n}}$.

Proof. Let $H = G_2 - U$. Since φ is convex,

$$\varphi(H) \geqslant \frac{v(H)}{1 + \frac{2e(H)}{v(H)}}.$$

Recall that $v(H) = n - k_0 \geqslant n - \frac{k_0'(1-\gamma)}{1-\alpha}$ and $e(H) \leqslant cn\sqrt{n} - \frac{k_0'(1-\alpha)^2}{20(1-\gamma)}n$. Thus, to prove the claim, we will verify that

$$\frac{n - \frac{k_0'}{1-\alpha}}{1 + \frac{2(cn\sqrt{n} - \frac{k_0'(1-\alpha)^2}{20}n)}{n - \frac{k_0'}{1-\alpha}}} \geqslant \frac{n}{1+2c\sqrt{n}}. \tag{3.5}$$

Multiplying both parts of (3.5) by the product of the denominators, opening the parentheses in the left-hand side, and cancelling n in both parts, we get

$$2cn\sqrt{n} - \frac{k_0'}{1-\alpha} - \frac{2ck_0'\sqrt{n}}{1-\alpha} \geqslant \frac{2n^2}{n - \frac{k_0'}{1-\alpha}}\left(c\sqrt{n} - \frac{k_0'(1-\alpha)^2}{20}\right).$$

Multiplying both parts of the last inequality by $n - \frac{k_0'}{1-\alpha}$, cancelling $2cn^2\sqrt{n}$ in both parts and dividing the rest by $\frac{-k_0'}{1-\alpha}$ we obtain that (3.5) is equivalent to

$$2cn\sqrt{n} + \left(n - \frac{k_0'}{1-\alpha}\right)(1 + 2c\sqrt{n}) \leqslant 0.1n^2(1-\alpha)^3,$$

which is weaker than

$$1 + 4c\sqrt{n} \leqslant 0.1n(1-\alpha)^3. \tag{3.6}$$

By (3.1), (1.2), and (1.3), inequality (3.6) holds. □

The main difficulties of packing below are: (1) packing vertices of G_2 of very high degree; (2) packing cyclic components of G_1, (3) packing big components of G_1 that are trees, and (4) finishing the packing when there is not much freedom.

Our strategy will be the following.

Step 1: Map $V(T_1 \cup \cdots \cup T_{k_0'})$ onto U.

Step 2: Find some k_1, $k_0 \leqslant k_1 \leqslant (1-\gamma-\varepsilon/2)n + \frac{1}{1-\alpha}$ so that the maximum degree of $G_2(k_1+1)$ is moderate.

Step 3: Map the vertices of G_1^* into (the complement of) $G_2(k_1+1)$.

Step 4: Map the vertices of $T_t, T_{t-1}, \ldots, T_{1+\lfloor 3n(1-\alpha)/4 \rfloor}$ into some of the remaining free vertices of G_2.

Step 5: Complete the packing by arranging the vertices of the remaining tree-components of G_1 in the rest of G_2.

Step 1 will take care of difficulty (1), Steps 2 and 3 handle (2), and at Step 4 we overcome (3).

We can complete Step 1 by Claim 3.2. Note that G_2 with the enumeration $u_1', \ldots, u_n'$ satisfies condition (ii) of Lemma 2.2 and k_0 satisfies the restrictions in this lemma. Suppose that condition (i) fails for G_2 and k_0, *i.e.*, that

$$e(G_2 - U) > e(G_2)\left(1 - \frac{2k_0(\alpha+\varepsilon)}{n(\alpha-\gamma+\varepsilon/2)}\right).$$

Then the number $\tilde{e}(U)$ of edges in G_2 incident with U is less than

$$cn^{3/2}\frac{2k_0(\alpha+\varepsilon)}{n(\alpha-\gamma+\varepsilon/2)} < c\sqrt{n}\,\frac{2k_0}{\alpha-\gamma+\varepsilon/2}.$$

On the other hand, by the definition of k_0', $\tilde{e}(U) > k_0'\frac{(1-\alpha)^2}{20}n$, and by Claims 3.1 and 3.2, $k_0' \geqslant k_0\frac{1-\alpha}{1-\gamma}$. Thus if condition (i) fails for G_2 and k_0, then

$$k_0\frac{1-\alpha}{1-\gamma}\frac{(1-\alpha)^2}{20}n < c\sqrt{n}\,\frac{2k_0(1-\gamma)}{\alpha-\gamma+\varepsilon/2}$$

and hence

$$\begin{aligned}\sqrt{n} &< \frac{40c(1-\gamma)}{(1-\alpha)^3(\alpha-\gamma+\varepsilon/2)} < \frac{20((1-\alpha)+(\alpha-\gamma))}{(1-\alpha)^3(\alpha-\gamma+\varepsilon/2)}\\ &< \frac{20}{(1-\alpha)^2(\alpha-\gamma+\varepsilon/2)} + \frac{20}{(1-\alpha)^3}\\ &\leqslant \frac{20}{(1-\alpha)^2\varepsilon/2} + \frac{20}{(1-\alpha)^3} \leqslant \frac{60}{(1-\alpha)^2\varepsilon} < \frac{15}{\varepsilon^3}.\end{aligned}$$

This contradicts (1.3).

Therefore, G_2 with the enumeration $u_1', \ldots, u_n'$ satisfies the conditions of Lemma 2.2. This lemma implies that there is an index $k_0 \leqslant k_1 \leqslant (1-\gamma-\varepsilon/2)n$ such that the maximal degree $D = \Delta(H)$ of the graph $H = G_2(k_1 + 1)$ satisfies

$$D \leqslant \frac{2c(n-k_1)(\alpha+\varepsilon)}{\sqrt{n}(\alpha-\gamma+\varepsilon/2)}. \tag{3.7}$$

This completes Step 2. Note that the right-hand side of (3.7) is at most $\frac{4c(\alpha+\varepsilon)}{\varepsilon}\sqrt{n}$ and hence (3.7) together with (3.2) yields

$$D \leqslant \frac{4c(\alpha+\varepsilon)}{\varepsilon}\sqrt{n} \leqslant \frac{\sqrt{n}}{2c\varepsilon} \leqslant \frac{3\sqrt{n}}{2\varepsilon}. \tag{3.8}$$

Also, by Theorem 2.1 and Claim 3.3, for

$$\ell = \left\lceil \frac{n}{2c\sqrt{n}+1} \right\rceil, \tag{3.9}$$

the set $L = \{u'_{n-\ell+1}, u'_{n-\ell+2}, \ldots, u'_n\}$ of the last ℓ vertices of G_2 forms an independent set in G_2.

Now, we identify the last ℓ vertices of G_1 with vertices in L. Since L is an independent set, this identification is 'legal' so far: no edge of G_1 is identified with an edge of G_2. If $w_{n-\ell}$ is not in G_1^*, then Step 3 is done, otherwise we continue as follows. We place the vertices $w_{n-\ell}, w_{n-\ell-1}, \ldots, w_{(1-\gamma)n+1}$ one by one into the rest of G_2, the 'middle' of G_2, namely $M = V(G_2(k_1+1)) - L$. We show now that all these vertices can be placed into M to give us a packing of G_1^* into (the complement of) G_2.

Suppose that we have placed the vertices $w_{n-\ell}, w_{n-\ell-1}, \ldots, w_{n-j+1}$ into M, and the next vertex to be placed, w_{n-j}, has x neighbours w_h with $h > n-j$. Since $w_1, w_2, \ldots, w_n$ is a degenerate order of the vertices of G_1, the subgraph $G_1(n-j)$ has minimal degree x. Furthermore, as G_1^* has γn vertices, we find that

$$jx + 2(\gamma n - j) \leqslant 2e(G_1) \leqslant 2\alpha n,$$

and so

$$x \leqslant 2 + 2(\alpha-\gamma)n/j. \tag{3.10}$$

By Claim 2.3, we have a legal placement for w_{n-j} provided that

$$n - k_1 - j - xD > 0. \tag{3.11}$$

Thus, to complete Step 3, it suffices to check that (3.11) holds.

Suppose that (3.11) is false. Then, by (3.10) and (3.7), we have

$$n - k_1 - j \leqslant D\left(2 + \frac{2(\alpha-\gamma)n}{j}\right) < 2D + \frac{2c(\alpha+\varepsilon)(n-k_1)2(\alpha-\gamma)n}{\sqrt{n}(\alpha-\gamma+0.5\varepsilon)j}. \tag{3.12}$$

Add j to both parts of (3.12) and divide both parts by $n-k_1$. Taking into account (3.8) and the fact that $k_1 \leqslant n(1-\gamma-\varepsilon/2)$, we get

$$1 < \frac{2D+j}{n-k_1} + \frac{4c(\alpha+\varepsilon)(\alpha-\gamma)n}{\sqrt{n}(\alpha-\gamma+0.5\varepsilon)j} \leqslant \frac{3\sqrt{n}/\varepsilon + j}{n(\gamma+0.5\varepsilon)} + \frac{4c(\alpha+\varepsilon)\sqrt{n}}{j}. \tag{3.13}$$

Consider the right-hand side of (3.13) as the function $f(j)$. This is a convex function of j (when other parameters are fixed). Since $\ell < j \leqslant \gamma n$, by (3.9), it is enough to check that

$f(j) \leqslant 1$ for $j = \frac{n}{2c\sqrt{n}+1}$ and $j = \gamma n$. Taking (1.1) into account, we get

$$f\left(\frac{n}{2c\sqrt{n}+1}\right) \leqslant \frac{3\sqrt{n}/\varepsilon + 2\sqrt{n}}{n(\gamma + 0.5\varepsilon)} + \frac{4c(\alpha+\varepsilon)\sqrt{n}(1+2c\sqrt{n})}{n}$$
$$\leqslant \frac{\frac{3}{\varepsilon}+2}{0.5\varepsilon\sqrt{n}} + \frac{4c(\alpha+\varepsilon)}{\sqrt{n}} + 8c^2(\alpha+\varepsilon).$$

By (3.2) and (1.3), the last expression is at most

$$\left(\frac{3}{\varepsilon}+2\right)(0.1\varepsilon)^2 + \frac{(0.1\varepsilon)^3}{2c} + 1 - 2\varepsilon < \frac{3\varepsilon}{100} + \frac{\varepsilon^2}{50} + \frac{2\varepsilon^3}{1000} + 1 - 2\varepsilon < 1.$$

Now,

$$f(\gamma n) = \frac{3}{\varepsilon\sqrt{n}(\gamma+0.5\varepsilon)} + \frac{\gamma}{\gamma+0.5\varepsilon} + \frac{4c(\alpha+\varepsilon)}{\gamma\sqrt{n}}.$$

If $\gamma \geqslant 0.1\varepsilon^2$ then, by (3.2) and (1.3), the last expression is at most

$$\frac{6}{\varepsilon^2\sqrt{n}} + \frac{1}{1+0.5\varepsilon} + \frac{5}{c\varepsilon^2\sqrt{n}} \leqslant \frac{6\varepsilon}{1000} + 1 - \frac{0.5\varepsilon}{1+0.5\varepsilon} + \frac{15\varepsilon}{1000} < 1.$$

Suppose that $\gamma < 0.1\varepsilon^2$. Since $\gamma n > \ell$, we obtain by (3.9) and (3.2) that

$$f(\gamma n) \leqslant \frac{6}{\varepsilon^2\sqrt{n}} + \frac{0.1\varepsilon^2}{0.1\varepsilon^2+0.5\varepsilon} + \frac{4c(\alpha+\varepsilon)\sqrt{n}(1+2c\sqrt{n})}{n}$$
$$\leqslant \frac{6}{\varepsilon^2\sqrt{n}} + \frac{\varepsilon}{5} + \frac{4c(\alpha+\varepsilon)}{\sqrt{n}} + 8c^2(\alpha+\varepsilon)$$
$$\leqslant \frac{6}{\varepsilon^2\sqrt{n}} + \frac{\varepsilon}{5} + \frac{1}{2c\sqrt{n}} + (1-2\varepsilon) \leqslant \frac{8\varepsilon}{1000} + 1 - 1.8\varepsilon < 1.$$

This finishes Step 3.

Let G_2' denote the subgraph of G_2 induced by the vertices not used as the images of vertices in G_1^*, and in $T_1, \dots, T_{k_0'}$. Then by (3.4) and Claim 3.1,

$$n_2' = |V(G_2')| \geqslant (1-\gamma)n - k_0 \geqslant (1-\gamma)\left(n - \frac{k_0'}{1-\alpha}\right)$$
$$\geqslant (1-\gamma)\left(n - \frac{0.01(1-\alpha)n}{1-\alpha}\right) \geqslant 0.99(1-\alpha)n. \tag{3.14}$$

By the definition of k_0', the maximum degree D' of G_2' is at most $\frac{(1-\alpha)^2}{20}n$. Since the subgraph G_1' of G_1 induced by $V(T_t \cup T_{t-1} \cup \dots \cup T_{1+\lfloor 3n(1-\alpha)/4\rfloor})$ is 1-degenerate, we can apply Claim 2.3 with $x = 1$. The claim implies that we can complete Step 4 provided

$$n_2' > D' + |V(G_1')|. \tag{3.15}$$

Applying (3.4), we have

$$n_2' - |V(G_1')| \geqslant \sum_{i=k_0'+1}^{\lfloor 3n(1-\alpha)/4\rfloor} v(T_i) \geqslant \frac{3(1-\alpha)}{4}n - k_0' - 1 > 0.74(1-\alpha)n - 1 > \frac{2(1-\alpha)}{3}n.$$

Taking into account that $D' \leqslant \frac{(1-\alpha)^2}{20}n$, we get (3.15).

Remarks. **(1)** Any vertex in a tree could be made the last vertex in a degenerate order. In particular, we can make the last a vertex of maximum degree.

(2) Packing each tree, we can start from identifying a vertex of the highest degree in this tree with an available vertex of the smallest degree in G_2.

Finally, let G_2'' denote the subgraph of G_2 induced by the vertices not yet used as the images of vertices in G_1. Then, as in the previous paragraph,

$$n_2'' = |V(G_2'')| > \frac{2(1-\alpha)}{3}n$$

and $\Delta(G_2'') \leqslant D' \leqslant \frac{(1-\alpha)^2}{20}n$. Let $G_1'' = T_{k_0'+1} \cup T_{k_0'+2} \cup \cdots \cup T_{\lfloor 3n(1-\alpha)/4 \rfloor}$. By Claim 3.1(b), the maximum degree D_1 of G_1'' is less than $\frac{4}{1-\alpha}$. Therefore,

$$D_1 \cdot D' \leqslant \frac{4}{1-\alpha} \cdot \frac{(1-\alpha)^2}{20} n = \frac{(1-\alpha)}{5} n < \frac{n_2''}{2}.$$

Thus, by Theorem 1.1, G_1'' and G_2'' pack. This proves Theorem 1.5.

4. Packing many graphs

In this section, we use Theorem 1.5 to show that one can pack many graphs if each of these graphs has at most αn edges. First, we look again into the proof of Theorem 1.5.

Lemma 4.1. *Let α, c, n and G_1 and G_2 satisfy the conditions of Theorem 1.5. Let $H = G_1 \cup G_2$ be the graph with $V(H) = V(G_1) = V(G_2)$, $E(H) = E(G_1) \cup E(G_2)$ obtained by packing G_1 and G_2 as described in the proof of Theorem 1.5. Then $\Delta(H) \leqslant \max\{\alpha n + 0.04(1-\alpha)n, \Delta(G_2) + 2/(1-\alpha)\}$.*

Proof. Suppose that the lemma is false. Then there is a vertex v with $\deg_H(v) > \max\{\alpha n, \Delta(G_2)\} + 2/(1-\alpha)$. We may assume that v is the result of identifying $w_i \in V(G_1)$ with $u_j \in V(G_2)$.

Case 1: $\deg_{G_2}(u_j) > 0.5\alpha n + 2$. If $j > k_0$, then by (3.4) and the definition of k_0',

$$\deg_{G_2}(u_j) \leqslant k_0' + \frac{(1-\alpha)^2}{20} n < 0.01(1-\alpha)n + \frac{(1-\alpha)^2 n}{20} < 0.04(1-\alpha)n \leqslant 0.02n, \qquad (4.1)$$

a contradiction. Therefore, $j \leqslant k_0$. Hence, $w_i \in V(T_1 \cup \cdots \cup T_{k_0'})$ and $\deg_{G_1}(w_i) \leqslant |V(T_{k_0'})| - 1$. By Claim 3.1(b), $|V(T_{k_0'})| \leqslant \frac{n}{n(1-\alpha) - k_0' + 1}$. In view of (3.4),

$$n(1-\alpha) - k_0' + 1 > n(1-\alpha) - 0.01n(1-\alpha) = 0.99n(1-\alpha).$$

It follows that $\deg_{G_1}(w_i) < \frac{1}{0.99(1-\alpha)}$ and the lemma holds.

Case 2: $\deg_{G_1}(w_i) > 0.5\alpha n + 2$. Since $e(G_1) = \alpha n$, there is only one vertex in G_1 with this property. Furthermore, with such a large degree, w_i is either in $V(G_1^*)$, or in $V(T_t)$. In either case, $u_j \notin U$, and by (4.1), $\deg_{G_2}(u_j) \leqslant 0.04(1-\alpha)n$. This proves the lemma. $\square$

Now we are ready to prove Theorem 1.8.

Proof. Recall that $m = \lceil 0.25\sqrt{n/\alpha^3} \rceil$. We will prove by induction on k, that for $k = 1, \dots, m$, there is a packing of $H_1, \dots, H_k$ such that the maximal degree, $\Delta(F_k)$, of the obtained graph $F_k = H_1 \cup \dots \cup H_k$ is at most $(1 - 0.96(1-\alpha))n + 2(k-2)/(1-\alpha)$.

For $k = 1$, the statement reduces to $\Delta(H_1) \leqslant \alpha n + 0.04n - 2/(1-\alpha)$. By (1.4), $0.04n - 2/(1-\alpha) \geqslant 0$ which proves the base case.

Suppose that the theorem is proved for some $k \leqslant m-1$. Let us check that Theorem 1.5 and Lemma 4.1 hold for our α and n, $c = e(F_k)/n^{3/2}$, $\varepsilon = 0.25(1-\alpha)$, $G_1 = H_{k+1}$, and $G_2 = F_k$. Indeed, since $k \leqslant m-1$, we have

$$e(F_k) \leqslant k\alpha n \leqslant (m-1)\alpha n < \frac{\alpha n\sqrt{n}}{4\alpha^{3/2}}$$

and hence $c \leqslant 0.25/\sqrt{\alpha}$. Therefore, $8c^2\alpha \leqslant 1/2$, which yields (1.1) and (1.2). Now, (1.3) follows from (1.4). By the inductive assumption,

$$\begin{aligned}\Delta(G_2) &\leqslant (1 - 0.96(1-\alpha))n + \frac{2(k-2)}{1-\alpha} \leqslant n - \frac{2}{1-\alpha} - \left(0.96(1-\alpha)n - \frac{2(m-2)}{1-\alpha}\right)\\ &\leqslant n - 2 - \left(0.96(1-\alpha)n - \frac{2\sqrt{n}}{4(1-\alpha)\alpha^{1.5}}\right) \leqslant n - 2 - \left(0.96(1-\alpha)n - \frac{\sqrt{2n}}{1-\alpha}\right).\end{aligned}$$

Observe that

$$0.96(1-\alpha)n > 0.96\sqrt{n}\frac{50^3}{(1-\alpha)^2} > \frac{100\sqrt{n}}{1-\alpha},$$

and hence

$$\Delta(G_2) \leqslant n - 2 - \frac{50\sqrt{n}}{1-\alpha}.$$

Thus, the conditions of Theorem 1.5 are satisfied, and by Lemma 4.1 we can pack H_{k+1} and F_k so that the maximum degree $\Delta(F_{k+1})$ of the resulting graph $F_{k+1} = F_k \cup H_{k+1}$ exceeds $(1 - 0.96(1-\alpha))n + 2(k-2)/(1-\alpha)$ by at most $2/(1-\alpha)$. This proves the theorem. □

5. Sparse graphs that do not pack

We will construct some series of pairs of sparse graphs that do not pack. We start from a simple series and then elaborate it.

Let $G_1 = G_1(n, 2)$ be a forest on n vertices whose components are stars S_1 and S_2 of degree at most $\lceil \frac{n}{2} \rceil$. By s_1 and s_2 we denote the centres of these stars.

Let $W = \{w_1, w_2, w_3\}$ and U be a set disjoint from W with $|U| = n-3$ partitioned into subsets U_1, U_2, and U_3 of about the same cardinality. We define $G_2 = G_2(n, 1, 2)$ as follows. Let $V_2 = V(G_2) = W \cup U$ and $E_2 = \{w_i w_j \mid 1 \leqslant i < j \leqslant 3\} \cup \bigcup_{i=1}^{3}\{uw_i, uw_{i+1} \mid u \in U_i\}$ (we sum the indices modulo 3). The graph G_2 possesses the property that every two vertices have a common neighbour and the maximum degree of G_2 is $\lceil 2n/3 \rceil$. Furthermore, G_2 is 2-degenerate, *i.e.*, very sparse.

Suppose that $G_1(n, 2)$ and $G_2(n, 1, 2)$ pack, *i.e.*, that there is an edge-disjoint placement f of the vertex set V_1 of G_1 onto V_2. Let $t_1 = f(s_1)$ and $t_2 = f(s_2)$. By the previous paragraphs, t_1 and t_2 have a common neighbour, say, t_0, in G_2. Then the vertex s_0 in G_1 with $f(s_0) = t_0$ cannot be adjacent to any of s_1 and s_2. This contradicts the definition of G_1. Thus G_1 and G_2 do not pack.

Note that this example disproves Conjecture 1.6 and shows that to extend the statement of Theorem 1.5 even to $\alpha = 1$, one needs to impose sufficiently stricter conditions on the maximum degree of G_2. The maximum of maximum degrees of G_1 and G_2 is $\lceil 2n/3 \rceil$. Below, we elaborate the above example to make this maximum less by making greater the average degree of G_2.

Let $G_1 = G_1(n,k)$ be a forest on n vertices whose k components are stars $S_1, \dots, S_k$ of degree at most $\lceil \frac{n}{k} \rceil$. By $s_1, \dots, s_k$ we denote the centres of these stars.

Let q be a prime power. For a nonnegative integer d, let $q_d = \frac{q^{d+1}-1}{q-1}$. In particular, $q_0 = 1$ and $q_1 = q + 1$. Suppose that $n > q^{k+1}$. To construct $G_2 = G_2(n,q,k)$, consider a k-dimensional projective space W over the field GF_q. It has q_k points and q_k hyperplanes. Let U be a set of $n - q_k$ vertices partitioned into q_k sets $U_1, \dots, U_{q_k}$ with $|U_i| \leqslant \lceil \frac{n}{q_k} \rceil - 1$ for all i. Let $\{H_1, \dots, H_{q_k}\}$ be a list of all hyperplanes in W. The graph $G_2 = G_2(n,q,k)$ has the vertex set $V_2 = W \cup U$ and the edge set

$$E_2 = \{w_1 w_2 \mid w_1 \in H_1,\ w_2 \in W,\ w_1 \neq w_2\} \cup \bigcup_{i=1}^{q_k} \{wu \mid w \in H_i,\ u \in U_i\}.$$

Claim 5.1. *If $n > q^{k+1}$, then*

(a) *$G_2(n,q,k)$ is q_{k-1}-degenerate,*

(b) *$|E_2| < q_{k-1} n$,*

(c) *the maximum degree of $G_2(n,q,k)$ is at most $\frac{n}{q} + q_k$.*

Proof. Order the vertices of G_2 so that first we list the vertices in U, then the vertices in $W - H_1$, and finally the points of H_1. Then every vertex v has at most q_{k-1} neighbours following v in this order. This proves (a). Note that (a) yields (b).

To check (c), observe that every vertex in U has degree q_{k-1}. Every point of a k-dimensional projective space over GF_q is contained in q_{k-1} hyperplanes. Therefore, every $w \in W$ is adjacent to at most $q_{k-1}(\lceil \frac{n}{q_k} \rceil - 1) < \frac{n}{q}$ vertices in U. Since $|W| = q_k$, this proves (c). □

Claim 5.1 implies that for fixed q and k, $G_2(n,q,k)$ has linear in n number of edges. Furthermore, if $n > q \cdot q_k$, then the maximum degree of G_2 is less than $\frac{2n}{q}$. Thus, for every k and any prime power $q \geqslant 2k$, if $n > q \cdot q_k$, then both $G_1(n,k)$ and $G_2(n,q,k)$ have maximum degree at most n/k.

Claim 5.2. *If $n > q \cdot q_k$, then $G_1(n,k)$ and $G_2(n,q,k)$ do not pack.*

Proof. Suppose that there exists a packing of $G_1(n,q)$ and $G_2(n,q,k)$, *i.e.*, that there is an edge-disjoint placement f of the vertex set V_1 of G_1 onto V_2. Let $t_j = f(s_j)$ for $j = 1, \dots, k$. By the definition of G_2, the neighbourhood of every of t_j contains some $H_{i(j)}$ (if $t_j \in H_1$, then it contains many H_i). Suppose that the set $T = \{t_1, \dots, t_k\}$ contains exactly r vertices of H_1. Since any $k-r$ hyperplanes of W have a common r-dimensional subspace, the neighbourhoods in G_2 of the remaining $k-r$ elements of T have at least q_r vertices in common. Since $q_r > r$ and vertices of H_1 are adjacent to every vertex in W, there exists a common neighbour $t_0 \in W$ of all vertices in T. But then the vertex $s_0 = f^{-1}(t_0)$ cannot be adjacent in G_1 to any of $s_1, \ldots, s_k$. This contradicts the definition of G_1. □

These two claims prove Theorem 1.7.

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