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On Sufficient Degree Conditions for a Graph to be *k*-linked

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A graph is k-linked if for every list of $2k$ vertices $\{s_1,\ldots,s_k,t_1,\ldots,t_k\}$, there exist internally disjoint paths $P_1,\ldots,P_k$ such that each P_i is an s_i,t_i-path. We consider degree conditions and connectivity conditions sufficient to force a graph to be k-linked.

Let $D(n,k)$ be the minimum positive integer d such that every n-vertex graph with minimum degree at least d is k-linked and let $R(n,k)$ be the minimum positive integer r such that every n-vertex graph in which the sum of degrees of each pair of non-adjacent vertices is at least r is k-linked. The main result of the paper is finding the exact values of $D(n,k)$ and $R(n,k)$ for every n and k.

Thomas and Wollan [14] used the bound $D(n,k) \leqslant (n+3k)/2-2$ to give sufficient conditions for a graph to be k-linked in terms of connectivity. Our bound allows us to modify the Thomas–Wollan proof slightly to show that every $2k$-connected graph with average degree at least $12k$ is k-linked.

1. Introduction

Dirac [2] proved that every n-vertex graph G with minimum degree at least $n/2$ is Hamiltonian, and Ore [12] observed that the condition $\delta(G) \geqslant n/2$ in Dirac's result can be replaced by '$\sigma_2(G) \geqslant n$', where $\sigma_2(G)$ is the minimum value of the sum $\deg(u)+\deg(v)$ over all pairs $\{u,v\}$ of non-adjacent vertices in G.

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After Chartrand introduced the notion of *k-ordered graphs*, that is, graphs in which for every ordered sequence of k vertices there is a cycle that encounters the vertices of the sequence in the given order, several authors (see, *e.g.*, [4, 11, 8, 6, 5]) studied the analogue of Dirac's and Ore's sufficient conditions for a graph to be k-ordered. Let $D_0(n,k)$ denote the minimum positive integer d such that every n-vertex graph with minimum degree at least d is k-ordered. Similarly, let $R_0(n,k)$ denote the minimum positive integer r such that every n-vertex graph G with $\sigma_2(G) \geqslant r$ is k-ordered. Improving on results in [4, 11], it was shown in [6] that $R_0(n,k) = n + \lceil (3k-9)/2 \rceil$ for every $3 \leqslant k \leqslant n/2$. Furthermore, Kierstead, Sárközy and Selkow [8] showed that $D_0(n,k) = \lceil n/4 \rceil + \lfloor k/2 \rfloor - 1$ for $3 \leqslant k \leqslant (n+3)/11$. These bounds demonstrate the interesting phenomenon that $R_0(n,k) > 2D_0(n,k)$ for k small with respect to n.

A graph is *k-linked* if, for every list of $2k$ vertices $\{s_1,\ldots,s_k,t_1,\ldots,t_k\}$, there exist internally disjoint paths $P_1,\ldots,P_k$ such that each P_i is an s_i,t_i-path. It is a folklore observation that if the number n of vertices of a graph G is at least $2k$, then in the definition of a k-linked graph it is enough to consider only the lists of *distinct* $s_1,\ldots,s_k,t_1,\ldots,t_k$. As in the previous paragraph, let $D(n,k)$ be the minimum positive integer d such that every n-vertex graph with minimum degree at least d is k-linked. Also, let $R(n,k)$ denote the minimum positive integer r such that every n-vertex graph G with $\sigma_2(G) \geqslant r$ is k-linked. Thomas and Wollan [14] used the bound $D(n,k) \leqslant (n+3k)/2 - 2$ to give sufficient conditions for a graph to be k-linked in terms of connectivity. In this paper we determine the exact values of $D(n,k)$ and $R(n,k)$ for all n and k.

Theorem 1.1. *If $k \geqslant 2$, then*

$$R(n,k) = \begin{cases} 2n-3, & n \leqslant 3k-1, \\ \lfloor \frac{2(n+5k)}{3} \rfloor - 3, & 3k \leqslant n \leqslant 4k-2, \\ n+2k-3, & n \geqslant 4k-1, \end{cases} \tag{1.1}$$

and

$$D(n,k) = \left\lceil \frac{R(n,k)}{2} \right\rceil = \begin{cases} n-1, & n \leqslant 3k-1, \\ \lfloor \frac{n+5k}{3} \rfloor - 1, & 3k \leqslant n \leqslant 4k-2, \\ \lceil \frac{n-3}{2} \rceil + k, & n \geqslant 4k-1. \end{cases} \tag{1.2}$$

Note that $R(3k,k) < R(3k-1,k)$. This is the only place for a fixed k where $R(n,k)$ decreases with growing n.

Egawa, Faudree, Győri, Ishigami, Schelp and Wang [3] proved the following very closely related result.

Theorem 1.2. ([3]) *Let $k \geqslant 2$ and $n \geqslant 3k$. Let $D_1(n,k)$ be the minimum positive integer d such that, for every n-vertex graph G with minimum degree at least d and every matching $M = \{s_it_i \mid i = 1,\ldots,m\}$ of size $m \leqslant k$ in G, there exist vertex-disjoint cycles $C_1,\ldots,C_m$ such that C_i contains s_it_i for each $i = 1,\ldots,m$. Similarly, let $R_1(n,k)$ be the minimum positive* *integer r such that, for every n-vertex graph G with $\sigma_2(G) \geqslant r$ and every matching $M = \{s_i t_i \mid i = 1, \ldots, m\}$ of size $m \leqslant k$ in G, there exist vertex-disjoint cycles $C_1, \ldots, C_m$ such that C_i contains $s_i t_i$ for each $i = 1, \ldots, m$. Then*

$$R_1(n,k) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 4k - 2, & 3k \leqslant n \leqslant 4k-2, \\ n + 2k - 2, & n \geqslant 4k-1, \end{cases} \tag{1.3}$$

and

$$D_1(n,k) = \begin{cases} \lceil \frac{n+5k}{3} \rceil - 1, & 3k \leqslant n \leqslant 4k-2, \\ \lceil \frac{n}{2} \rceil + k - 1, & n \geqslant 4k-1. \end{cases} \tag{1.4}$$

This is closely related because, for a graph G and a matching $M = \{s_i t_i \mid i = 1, \ldots, m\}$, the existence of cycles provided by Theorem 1.2 is equivalent to the existence in $G' = G - M$ of vertex-disjoint paths linking s_i with t_i provided by Theorem 1.1. Although the graphs G and G' differ only by a matching, the values of $R(n,k)$ and $R_1(n,k)$ for $3k < n < 4k$ differ significantly. On the other hand, the ideas of the proofs are similar. But neither of the bounds of Theorem 1.2 and Theorem 1.1 can be derived from the other. Also, in terms of linkages, Theorem 1.1 gives slightly better bounds for some parities of n and k, which could perhaps be used for extremal problems on linkages.

A very interesting problem is estimating $f(k)$ – the minimum positive integer f such that every f-connected graph is k-linked. After a series of papers by Jung [7], Larman and Mani [9], Mader [10], and Robertson and Seymour [13], the first linear upper bound for f, namely $f(k) \leqslant 22k$, was proved by Bollobás and Thomason [1]. Very recently, Thomas and Wollan [14] improved this bound to $f(k) \leqslant 16k$. If one were to use Theorem 1.1 in the Thomas–Wollan proof [14], then their sufficient condition for a graph to be k-linked could be relaxed.

Theorem 1.3. *Every $2k$-connected graph $G = (V, E)$ with $|E| \geqslant 6k|V|$ is k-linked. In particular, every $12k$-connected graph is k-linked.*

We note that applying Theorem 1.2 also would yield Theorem 1.3. In the next section we prove lower bounds for $D(n,k)$ and $R(n,k)$. Then, in Section 3, the upper bounds are established. In the final section, we show how to modify the Thomas–Wollan proof [14] in order to derive Theorem 1.3.

Using new ideas (in particular, ideas of this paper), Thomas and Wollan improved the upper bound on $f(k)$ further to $f(k) \leqslant 10k$.

2. Constructions

In this section we present examples giving lower bounds for $D(n,k)$ and $R(n,k)$. Consider several cases.

Case 1: $n \leqslant 3k - 1$. Let G be K_n with a deleted matching $\{(v_1, v_2), \ldots, (v_{2\lfloor n/2 \rfloor - 1}, v_{2\lfloor n/2 \rfloor})\}$. Clearly, $\delta(G) = n - 2$ and $\sigma_2(G) = 2n - 4$. Let $m = \min\{k, \lfloor n/2 \rfloor\}$. For $i = 1, \ldots, m$, let $s_i = v_{2i-1}$ and $t_i = v_{2i}$. Assume that there exist internally disjoint paths $P_1, \ldots, P_k$ such

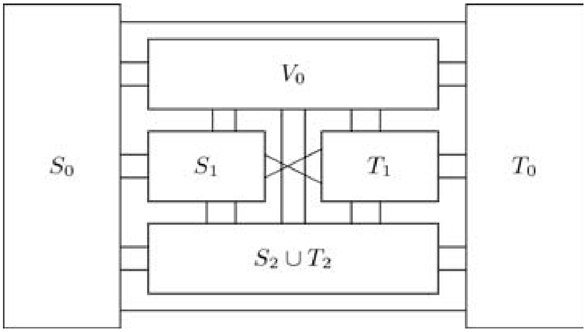

Figure 1. Graph G for Case 2.

that each P_i is an s_i, t_i-path. Denote $S = \{s_1, \ldots, s_m, t_1, \ldots, t_m\}$. For every i, the path P_i contains a vertex $x_i \notin S$, since $s_i t_i \notin E(G)$. Therefore, $n \geqslant |S| + m = 3m$. This is impossible if $n \leqslant 3k - 1$, $k \geqslant 2$, and $m = \min\{k, \lfloor n/2 \rfloor\}$.

Case 2: $3k \leqslant n \leqslant 4k - 2$. Let $x = \lceil \frac{2n-5k}{3} \rceil + 1$. It is easier to describe the complement $\overline{G} = (V, \overline{E})$ of G. Fix six disjoint subsets $S_0, S_1, S_2, T_0, T_1, T_2$ of V with $|S_0| = |T_0| = x - 1$, $|S_1| = |T_1| = x$, and $|S_2| = |T_2| = k - x$. Let $V_0 = V - S_0 - S_1 - S_2 - T_0 - T_1 - T_2$.

Looking ahead, we may assume that

$$S_1 = \{s_1, \ldots, s_x\}, \qquad S_2 = \{s_{x+1}, \ldots, s_k\},$$
$$T_1 = \{t_1, \ldots, t_x\}, \qquad T_2 = \{t_{x+1}, \ldots, t_k\}.$$

The set $\overline{E}$ of edges of $\overline{G}$ is $E_1 \cup E_2 \cup E_3$, where

$$E_1 = \{s_i t_i \ : \ i = 1, \ldots, k\},$$
$$E_2 = \{vw \ : \ v \in S_0, w \in T_1\},$$
$$\text{and} \quad E_3 = \{vw \ : \ v \in T_0, w \in S_1\}.$$

The graph G itself is drawn in Figure 1.

Assume that there exist internally disjoint paths $P_1, \ldots, P_k$ such that each P_i is an s_i, t_i-path. As in Case 1, each P_i contains a vertex in $S_0 \cup T_0 \cup V_0$. Moreover, if $i \leqslant x$ (*i.e.*, $s_i \in S_1$ and $t_i \in T_1$), then either P_i contains a vertex in V_0, or it has at least two internal vertices, since no vertex in $S_0 \cup T_0$ is adjacent to both s_i and t_i. Therefore, $n \geqslant 2k + k + (x - |V_0|) = 5k - n - 2 + 3x$. By the definition of x, the last expression exceeds n, a contradiction.

The vertices in $S_0 \cup S_1 \cup T_0 \cup T_1$ have degree x in $\overline{G}$ and all other vertices have degree at most 1. It follows that $\delta(G) = n - 1 - x = \lfloor \frac{n+5k}{3} \rfloor - 2$ and $\sigma_2(G) = 2n - 2 - 2x = 2\lfloor \frac{n+5k}{3} \rfloor - 4$. This proves the lower bound on $D(n,k)$ for every $3k \leqslant n \leqslant 4k - 2$ and on $R(n,k)$ for $3k \leqslant n \leqslant 4k - 2$ such that $n + 5k \not\equiv 2 \pmod 3$. For $n + 5k \equiv 2 \pmod 3$, we slightly modify the construction: we change x to $x = \lceil \frac{2n-5k+2}{3} \rceil$ and move one vertex from V_0 to T_0, so that $|T_0| = x$. Then, as in the previous paragraph, we obtain $n \geqslant 2k + k + (x - |V_0|) = 5k - n - 2 + 3x + 1$, which contradicts the new definition of x. On the other

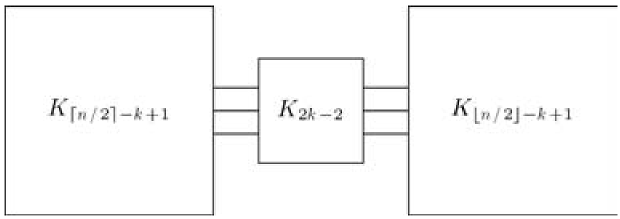

Figure 2. Graph G for Case 3.

hand, $\sigma_2(G) = 2n-3-2x = 2\lfloor\frac{n+5k-2}{3}\rfloor - 3$. Since we have $2\lfloor\frac{n+5k-2}{3}\rfloor - 3 = \lfloor\frac{2(n+5k)}{3}\rfloor - 4$, for $n+5k \equiv 2 \pmod 3$, this completes Case 2.

Case 3: $n \geqslant 4k-1$. This is a standard example of a graph with connectivity $2k-2$ (see Figure 2). Clearly, $\delta(G) = \lfloor n/2\rfloor + k - 2$ and $\sigma_2(G) = n+2k-4$. If $s_1,\dots,s_{k-1},t_1,\dots,t_{k-1}$ are the vertices of the central K_{2k-2}, s_k belongs to the left $K_{\lceil n/2\rceil-k+1}$, and t_k belongs to the right $K_{\lfloor n/2\rfloor-k+1}$, then there is no corresponding linkage: there is simply no room for an s_k,t_k-path.

3. Upper bounds in Theorem 1.1

Observe that it is enough to prove the upper bound for $R(n,k)$, since $D(n,k) \geqslant R(n,k)/2$. The case of $n \leqslant 3k-1$ is obvious, because $\sigma_2(G) \geqslant 2n-3$ means that $G = K_n$.

Remark 1. If $n \leqslant 4k-1$, then $n+2k-4 \leqslant \frac{2n+10k-1}{3} - 4$.

Let $n \geqslant 3k$ and $G = (V,E)$ be a graph on n vertices satisfying the conditions of the theorem. Let $M_i = \{s_i,t_i\}$, $i = 1,\dots,k$, be arbitrary disjoint vertex pairs to be linked, and let $M = \cup_{i=1}^{k} M_i$. If, say, $s_kt_k \in E(G)$, then, for the graph $G' = G - s_k - t_k$ with $n' = n-2$ vertices and for $k' = k-1$, we have $\sigma_2(G') \geqslant R(n,k) - 4 \geqslant R(n',k')$. Therefore, if the theorem holds for G', it also holds for G. Thus, we may assume that none of the pairs s_it_i is an edge in G.

We will find the linkage in 3 steps (resembling the steps of the main proof in [6]). On each of the steps, if we cannot perform this step, then $\sigma_2(G) < R(n,k)$.

Construct the auxiliary bipartite graph H with partite sets W_1 and W_2 as follows. Let $W_1 = \{M_1,\dots,M_k\}$, $W_2 = V - M$, and a pair (M_i, v) be an edge in H if $v \in N_G(s_i) \cap N_G(t_i)$. If H has a matching saturating W_1, then this matching gives the required linkage. Otherwise, let m be the size of a maximum matching in H.

By the König–Egerváry theorem, there is a $Q \subseteq W_1$ with $k-m = |Q| - |N_H(Q)|$. Denote $R = N_H(Q)$ and $S = V(G) - M - R$. We may assume that $Q = \{M_i : i = 1,\dots,q\}$. Let $Q' = \bigcup_{i=1}^{q}\{s_i,t_i\}$ (the elements of Q are pairs, and the elements of Q' are all the elements of these pairs). Note that $|Q| = q$, $|Q'| = 2q$, $|R| = q-k+m$, and $|S| = n-k-q-m$.

Let P be a maximum matching in H. We may assume that only vertices in $D = \{M_1,\dots,M_{k-m}\}$ are not covered by this matching. Let $D' = \{s_i,t_i : 1 \leqslant i \leqslant k-m\}$.

Consider the linkage $\mathscr{P}$ of $M - D$ corresponding to P. Let Z be the set of vertices not participating in the linkage.

Lemma 3.1. $n \geqslant 4k - m$.

Proof. We need to prove this lemma only for $n \leqslant 4k - 1$.

Assume $n \leqslant 4k - m - 1$. Let $M_i \in Q$ and $x \in S$. Then either s_i or t_i is not a neighbour of x. Therefore,

$$\deg(s_i) + \deg(t_i) \leqslant 2(n-2) - |S| = n + k - 4 + q + m. \tag{3.1}$$

For the same reasons, x is not adjacent to at least q vertices in Q'. We may assume that $xs_1 \notin E(G)$. If $s_1 y \notin E(G)$ for every $y \in S$, then

$$\deg(s_1) + \deg(x) \leqslant (n - 2 - |S|) + (n - 1 - q) = n - 3 + k + m \leqslant n - 4 + 2k.$$

By Remark 1, this contradicts (1.1). Otherwise, there is a $y \in S$ with $ys_1 \in E(G)$ and, therefore, $yt_1 \notin E(G)$. Thus, by (3.1),

$$3\sigma_2(G) \leqslant (\deg(x) + \deg(s_1)) + (\deg(y) + \deg(t_1)) + (\deg(s_1) + \deg(t_1))$$

$$\leqslant 2(n + k - 4 + q + m) + 2(n - q - 1) \leqslant 2(n + 5k - 6).$$

It follows that $\sigma_2(G) < R(n, k)$. □

By Lemma 3.1, for every $i = 1, \ldots, k - m$, we can assign a vertex $z_i \in Z$ to s_i and a vertex $z_i' \in Z$ to t_i so that we assign distinct members of Z to distinct vertices. Also, for every $k - m + 1 \leqslant i \leqslant k$, let y_i be the common neighbour of s_i and t_i corresponding to the matching P above. Among such assignments, choose an assignment $\mathscr{A}$ with as many edges $z_i s_i$ and $z_i' t_i$ as possible. Let $Z' = \bigcup_{i=1}^{k-m} \{z_i, z_i'\}$.

Lemma 3.2. *In $\mathscr{A}$, every z_i is adjacent to s_i and every z_i' is adjacent to t_i.*

Proof. Assume that $s_1 z_1 \notin E(G)$. We will prove that

$$|N(s_1) - M| \leqslant |M - N(z_1)|. \tag{3.2}$$

To do this, for every neighbour w of s_1 outside M we show a non-neighbour $f(w)$ of z_1 in M. First, observe that either $w \in Z'$ or w was used in the linkage $\mathscr{P}$, since otherwise we can assign w as z_1.

Case 1. If w is used in a path s_i, w, t_i in the linkage $\mathscr{P}$ and z_1 is adjacent to both s_i and t_i, then by swapping w with z_1, we will get an assignment with new z_1 (former w) adjacent to s_1. Thus, either s_i or t_i can be chosen as $f(w)$.

Case 2. If $w \in Z'$, say, $w = z_i'$ (possibly, $i = 1$), and z_1 is adjacent to t_i, then swapping z_1 with w produces a better assignment. Thus, $z_1 t_i \notin E$, and we let $f(w) = t_i$.

Since all s_i and t_i are disjoint, (3.2) holds, and therefore $\deg(s_1) + \deg(z_1) \leqslant 2(n-2) - |V - M| = n - 4 + 2k$. By Remark 1, this yields $\sigma_2(G) < R(n, k)$ for each $n \geqslant 3k$. □

The last step in the proof is given by the next lemma.

Lemma 3.3. *The assignment $\mathscr{A}$ in Lemma 3.2 can be chosen in such a way that every z_i is adjacent to every z_i'.*

Proof. Consider an assignment $\mathscr{A}$ satisfying Lemma 3.2. For $i = 1, \ldots, k-m$, let $X_i = \{s_i, t_i, z_i, z_i'\}$, and for $i = k-m+1, \ldots, k$, let $X_i = \{s_i, t_i, y_i\}$. Let $X = \bigcup_{i=1}^{k} X_i$. Choose $\mathscr{A}$ in Lemma 3.2 so that as many as possible z_i are adjacent to corresponding z_i'. Suppose that the lemma does not hold. Then we may renumber (s_i, t_i) so that $z_1 z_1' \notin E(G)$. Let $A = N(s_1) \cap (V(G) - X)$ and $B = N(t_1) \cap (V(G) - X)$.

Note that $N(A + z_1) \cap (B + z_1') = \emptyset$. For $i = 1, \ldots, k$, let k_i denote the number of neighbours of X_1 (with multiplicities) in X_i. Since each member of X_1 has exactly one neighbour in X_1, $k_1 = 4$.

Claim 3.4. *If $2 \leqslant i \leqslant k-m$, then $k_j \leqslant 12$. If $k-m+1 \leqslant i \leqslant k$, then $k_j \leqslant 10$.*

Proof.
Case 1: $2 \leqslant i \leqslant k-m$. By the maximality of m, neither of z_1, z_1' is a common neighbour of s_i and t_i and neither of z_i, z_i' is a common neighbour of s_1 and t_1. Therefore, $k_i \leqslant |X_1| \cdot |X_i| - 4 = 12$.

Case 2: $k-m+1 \leqslant i \leqslant k$.

Subcase 2.1: $s_1 y_i, t_1 y_i \in E(G)$. If some of z_1, z_1' (say, z_1) is a common neighbour of s_i and t_i, then assigning z_1 as the new y_i and assigning the old y_i as the new y_1 will contradict the maximality of m. Otherwise, $k_i \leqslant |X_1| \cdot |X_i| - 2 = 3 \cdot 4 - 2 = 10$.

Subcase 2.2: $s_1 y_i \notin E(G)$. If all the four edges $z_1' s_i$, $z_1' t_i$, $z_1 y_i$, $t_1 y_i$ are in $E(G)$, then we can swap y_i with z_1' and get a better assignment, since the new z_1' is adjacent to z_1. Otherwise, we again have $k_i \leqslant 10$. □

Claim 3.5. *For each $v \notin X$, $|N(v) \cap \{s_1, t_1, z_1, z_1'\}| \leqslant 2$.*

Proof. Otherwise, we can swap v with either z_1 or z_1' so that the new assignment is better. □

Let $F = \deg(s_1) + \deg(t_1) + \deg(z_1) + \deg(z_1')$. In view of the claims above, $F \leqslant 2(n - |X|) + 4 + 12(k-m-1) + 10m$. Since $|X| = 4(k-m) + 3m$, we obtain

$$F \leqslant 2(n - 4k + m) + 4 + 12(k-m-1) + 10m = 2n + 4k - 8.$$

Since $F \geqslant 2\sigma_2(G) \geqslant 4\delta(G)$, we obtain a contradiction to both (1.1) and (1.2). This proves the lemma, and thus the theorem as well. □

4. Connectivity conditions

Thomas and Wollan [14] showed that $f(k) \leqslant 16k$ by proving the following (stronger) result.

Theorem 4.1. ([14]) *Every $2k$-connected graph G with $|E(G)| \geqslant 8k|V(G)|$ is k-linked.*

Most of the proof works under weaker restrictions on the average degree. The bottleneck for the bounds in Theorem 4.1 is the following claim.

Claim 4.2. ([14]) *For $\alpha = 8$, every graph H on at most $2\alpha k$ vertices with minimum degree at least αk contains a k-linked subgraph.*

If one proves Claim 4.2 for any $3 \leqslant \alpha < 8$, this would imply the strengthening of Theorem 4.1 with α in place of 8. Using Theorem 1.1, we prove below an analog of Claim 4.2 with $\alpha = 6$. This will make Theorem 4.1 work with 6 in place of 8, *i.e.*, will yield Theorem 1.3. The beginning of the proof is reminiscent of that for Claim 4.2, but for completeness, we present the full proof.

Lemma 4.3. *For $\alpha = 6$, every graph H on at most $2\alpha k$ vertices with minimum degree at least αk contains a k-linked subgraph.*

Proof. Consider an H satisfying the conditions of the lemma. If H itself is not k-linked, then there is a set $X = \{s_i, t_i : 1 \leqslant i \leqslant k\} \subseteq V(H)$ such that there are no disjoint paths $P_1, \ldots, P_k$ such that each P_i is an s_i, t_i-path and all s_i and t_i are distinct. Link as many as possible pairs (s_i, t_i) by paths of length at most 6 and, subject to this, minimize the sum of the lengths of these paths. Suppose that l_1 pairs are not linked and the number of paths of length i is l_{i+1}, $1 \leqslant i \leqslant 6$. We may assume that s_1 and t_1 are not linked.

Let S be the union of X with vertex sets of all the paths of the linkage. Let $A = N_H(s_1) - S$ and $B = N_H(t_1) - S$. By the choice, A and B are disjoint and are at distance at least 5 in $H - S$. Since the paths P_i are chosen to be of the minimum total length, we have

$$|N_H(v) \cap V(P_i)| \leqslant 3 \quad \forall v \in V(G) - S \quad \forall P_i, \tag{4.1}$$

$$|N_H(v) \cap \{s_1, t_1\}| \leqslant 1 \quad \forall v \in V(G) - S, \tag{4.2}$$

and

$$|N_H(s_1) \cap N_H(t_1) \cap V(P_i)| \leqslant 3 \quad \forall i, \tag{4.3}$$

Claim 4.4. *For each $v \in V(H) - S - A - B$, $|N(v) \cap (A \cup B)| \geqslant 2$ and either $N(v) \cap A = \emptyset$ or $N(v) \cap B = \emptyset$.*

Proof. Suppose that $|N(v) \cap (A \cup B)| \leqslant 1$ for some $v \in V(H) - A - B$. By (4.1), (4.2), and (4.3), $|N_H(v) - S| \geqslant \delta(H) - 3(k-1) - 1$ and $|A| + |B| \geqslant 2\delta(H) - |S| - 3k$, we have

$$|V(H)| \geqslant |A| + |B| + |S| + (1 + \deg_{H-S}(v)) - 1 \geqslant 2 + 3\delta(H) - 6k > 12k \geqslant |V(H)|,$$

a contradiction. If both $N(v) \cap A \neq \emptyset$ and $N(v) \cap B \neq \emptyset$, then there is an s_1, t_1-path of length 4 outside S, a contradiction. $\square$

By Claim 4.4, every vertex in $V(H)-S$ has distance at most two from either s_1 or t_1, but not both. Therefore, $H-S$ is the union of $H_1=H[A\cup N(A)-S]$ and $H_2=H[B\cup N(B)-S]$, and there are no edges connecting $V(H_1)$ with $V(H_2)$. Assume that $|V(H_1)|\leqslant|V(H_2)|$.

Observe that $\delta(H_1)\geqslant\delta(H)-3(k-l_1)-l_1\geqslant 6k-3k+2l_1=3k+2l_1$ and $|V(H_1)|\geqslant 1+\delta(H_1)>3k$. If $|V(H_1)|\leqslant 4k$, then $3k+2l_1\geqslant\frac{|V(H_1)|+5k}{3}$, and by Theorem 1.1, H_1 is k-linked. This proves the lemma when $|V(H_1)|\leqslant 4k$.

Now consider the case $|V(H_1)|,|V(H_2)|\geqslant 4k+1$. If $\delta(H_1)\geqslant(|V(H_1)|-3)/2+k$ or $\delta(H_2)\geqslant(|V(H_2)|-3)/2+k$, then by Theorem 1.1, either H_1 or H_2 is k-linked. Thus we may assume that there exist $v_1\in H_1$ and $v_2\in H_2$ such that $\deg_{H_1}(v_1)\leqslant|V(H_1)|/2+k-2$ and $\deg_{H_2}(v_2)\leqslant|V(H_2)|+k-2$. Hence,

$$\deg_{H_1}(v_1)+\deg_{H_2}(v_2)\leqslant(|V(H_1)|+|V(H_2)|)/2+2k-4,$$

that is,

$$\deg_H(v_1)+\deg_H(v_2)-(|N_H(v_1)\cap S|+|N_H(v_2)\cap S|)\leqslant(|V(H)|-|S|)/2+2k-4.$$

Therefore

$$\begin{aligned}|N_H(v_1)\cap S|+|N_H(v_2)\cap S| &\geqslant 2\delta(H)-|V(H)|/2+|S|/2-2k+4\\ &\geqslant 12k-6k+|S|/2-2k+4\geqslant 4k+4+|S|/2. \qquad (4.4)\end{aligned}$$

Claim 4.5. $|N(v_1)\cap S|+|N(v_2)\cap S|\leqslant 4l_1+4l_2+5l_3+6(l_4+l_5+l_6+l_7)$.

Proof. It is enough to prove that v_1 and v_2 together have at most:

(a) 4 neighbours (counted with multiplicities) on each P_i of length 1 and each unlinked pair in S,

(b) 5 neighbours on each P_i of length 2,

(c) 6 neighbours on each P_i of length at least 3.

Statement (a) is evident and (c) follows from (4.1). To prove (b), suppose that each of v_1 and v_2 has exactly 3 neighbours in a path $P_i=(s_i,w,t_i)$. Since $|S|\leqslant|V(H)|-|V(H_1)|-|V(H_2)|\leqslant 12k-2(4k+1)=4k-2$, every $v\in V(H)$ has at least $6k-|S|\geqslant 2k+2$ neighbours outside S. Thus we may assume that $wu\in E(G)$ for some $u\in V(H_1)$, $u\neq v_1$.

Since no $u\in V(H_1)+s_1$ is adjacent to any $w\in V(H_2)+t_1$, each distinct pair $u_1,u_2\in V(H_1)+s_1$ has at least $6k+6k-(8k-2)=4k+2\geqslant|S|+4$ common neighbours. Similarly, each distinct pair $w_1,w_2\in V(H_2)+t_1$ has at least $|S|+4$ common neighbours. Thus, the graph $H-(S-s_1-t_1)$ contains an s_1,u-path Q_1 of length at most 2 avoiding v_1 and a t_1,v_2-path Q_2 of length at most 2. Then we replace P_i by $s_iv_1t_i$ and add the path $s_1Q_1uwv_2Q_2t_1$ of length at most 6. This contradicts the maximality of the linkage. □

Now by Claim 4.5 we have

$$\begin{aligned}|N(v_1)\cap S|+|N(v_2)\cap S| &\leqslant 4l_1+4l_2+5l_3+6(l_4+l_5+l_6+l_7)\\ &=4\sum_{i=1}^{7} l_i+(l_3+2l_4+2l_5+2l_6+2l_7)\\ &=4k+(l_3+2l_4+2l_5+2l_6+2l_7)\leqslant 4k+|S|/2,\end{aligned}$$

which contradicts (4.4). □

As we have mentioned above, proving Lemma 4.3 for smaller α would yield the corresponding improvement for $f(k)$.

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