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PROBLEM SECTION
Ore-type graph packing problems

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We say that n-vertex graphs $G_1, G_2, \ldots, G_k$ *pack* if there exist injective mappings of their vertex sets onto $[n] = \{1, \ldots, n\}$ such that the images of the edge sets do not intersect. The notion of packing allows one to make some problems on graphs more natural or more general. Clearly, two n-vertex graphs G_1 and G_2 pack if and only if G_1 is a subgraph of the complement $\overline{G}_2$ of G_2.

In terms of packing, Dirac's [2] and Ore's [3] theorems on hamiltonian cycles in graphs can be stated as follows.

Theorem 1. (Dirac) *Let $n \geq 3$. If G is an n-vertex graph and its maximum degree, $\Delta(G)$, is at most $\frac{1}{2}n - 1$, then G packs with the cycle C_n of length n.*

Let the *maximum edge-average degree of G*, $\overline{\sigma}_2(G)$, denote the maximum of $\frac{1}{2}(\deg_G(v) + \deg_G(u))$ over all edges vu of G.

Theorem 2. (Ore) *If $n \geq 3$ and G is an n-vertex graph with $\overline{\sigma}_2(G) \leq \frac{1}{2}n - 1$, then G packs with the cycle C_n.*

The study of extremal problems on packings of graphs was started in the 1970s by Sauer and Spencer [4] and Bollobás and Eldridge [1]. The following result of Sauer and Spencer [4] can be viewed as a generalization of Dirac's Theorem to packing of general graphs.

Theorem 3. (Sauer and Spencer) *Suppose that G_1 and G_2 are graphs of order n such that $2\Delta(G_1)\Delta(G_2) < n$. Then G_1 and G_2 pack.*

One of the main conjectures in the area is the following.

Conjecture 4. (Bollobás–Eldridge-Catlin) *If G_1 and G_2 are n-vertex graphs and* $(\Delta(G_1)+1)(\Delta(G_2)+1) \leq n+1$, *then G_1 and G_2 pack.*

This conjecture also has the spirit of Dirac's Theorem. If true, it would be a considerable extension of the Hajnal-Szemerédi Theorem on equitable colourings, since an equitable k-colouring of an n-vertex graph G is a packing of G with the n-vertex graph whose components are cliques on $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ vertices. The conjecture has been proved for some narrow classes of graphs, but is wide open in general.

The aim of this note is to suggest the consideration of packing of graphs where some restrictions on maximum degrees of graphs are replaced by restrictions on their maximum edge-average degrees. The maximum edge-average degree of a graph G does not look too exotic when we observe that it is closely related to the maximum degree of the line graph $L(G)$:

$$\overline{\sigma}_2(G) = \frac{1}{2}\Delta(L(G)) + 1,$$

and that any bound on $\overline{\sigma}_2(G)$ is in fact a bound on $\Delta(L(G))$.

We can prove the following "one-sided" Ore-type version of the Sauer-Spencer Theorem.

Theorem 5. *Suppose that G_1 and G_2 are graphs of order n such that $2\overline{\sigma}_2(G_1)\Delta(G_2) < n$. Then G_1 and G_2 pack.*

Note that the set of the "extremal" graphs for this result, that is, graphs G_1 and G_2 such that $2\overline{\sigma}_2(G_1)\Delta(G_2) = n$ and the graphs G_1 and G_2 do not pack, is somewhat wider than the corresponding set for the Sauer-Spencer Theorem. We think that the following stronger statement holds (but cannot prove it).

Conjecture 6. *If G_1 and G_2 are n-vertex graphs and $\overline{\sigma}_2(G_1)\overline{\sigma}_2(G_2) < n$, then G_1 and G_2 pack.*

In all extremal pairs (G_1, G_2) for the BEC-conjecture that we know of, G_1 contains a $\Delta(G_1)$-regular component and G_2 contains a $\Delta(G_2)$-regular component. This prompts us to put forward the following Ore-type analogue of the BEC-conjecture.

Conjecture 7. *If G_1 and G_2 are n-vertex graphs and $(\overline{\sigma}_2(G_1)+1)(\Delta(G_2)+1) \leq n+1$, then G_1 and G_2 pack.*

Conjecture 3 would imply the following Ore-type version of the Hajnal-Szemerédi Theorem.

Conjecture 8. *Every graph G has an equitable coloring with k colors for any $k \geq \overline{\sigma}_2(G)+1$.*

Some other Ore-type generalizations of packing and colouring results are also possible.

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