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Extremal Graphs for a Graph Packing Theorem of Sauer and Spencer

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Let G_1 and G_2 be graphs of order n with maximum degree Δ_1 and Δ_2, respectively. G_1 and G_2 are said to *pack* if there exist injective mappings of the vertex sets into $[n]$, such that the images of the edge sets do not intersect. Sauer and Spencer showed that if $\Delta_1\Delta_2 < \frac{n}{2}$, then G_1 and G_2 pack. We extend this result by showing that if $\Delta_1\Delta_2 \le \frac{n}{2}$, then G_1 and G_2 do not pack if and only if one of G_1 or G_2 is a perfect matching and the other either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

1. Introduction

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be n-vertex graphs with maximum degrees $\Delta(G_i) = \Delta_i$ for $i = 1, 2$. G_1 and G_2 are said to *pack* if there exist injective mappings of their vertex sets into $[n]$, such that the images of the edge sets do not intersect. In other words, G_1 is isomorphic to a subgraph of the complement of G_2. The study of packings of graphs was started in the 1970s by Bollobás and Eldridge [1, 2], Sauer and Spencer [4], and Catlin [3]. (See the surveys by Wozniak [5] and Yap [6] for later developments in this field.) In particular, Sauer and Spencer [4] proved the following result.

Theorem 1.1 (Sauer and Spencer). *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be n-vertex graphs with maximum degrees $\Delta(G_i) = \Delta_i$ for $i = 1, 2$. If $2\Delta_1\Delta_2 < n$, then G_1 and G_2 pack.*

This is sharp for even n: take G_1 to be a perfect matching on n vertices and G_2 to be either $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd, or containing $K_{\frac{n}{2}+1}$. We are interested in describing the pairs of graphs with $2\Delta_1\Delta_2 = n$ that do not pack. We show that these examples are the only possibilities, thus somewhat extending the Sauer–Spencer theorem.

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Theorem 1.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be n-vertex graphs with maximum degrees $\Delta(G_i) = \Delta_i$ for $i = 1, 2$. Let $2\Delta_1\Delta_2 \le n$. G_1 and G_2 do not pack if and only if one of G_1 and G_2 is a perfect matching and the other either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.*

One can view Theorem 1.2 as a very small step towards the well-known conjecture by Bollobás and Eldridge [2] that any two n-vertex graphs G_1 and G_2 with maximum degrees Δ_1 and Δ_2 pack provided that $(\Delta_1 + 1)(\Delta_2 + 1) \le n + 1$. In the next section we give the outline of the proof and state all the lemmas. In the subsequent sections we prove the lemmas.

2. Outline of the proof

Say that a pair (G_1, G_2) of graphs is a *critical pair* if G_1 and G_2 do not pack, but for each $e_1 \in E(G_1)$, $G_1 - e_1$ and G_2 pack, and for each $e_2 \in E(G_2)$, G_1 and $G_2 - e_2$ pack.

Consider 1–1 mappings of V_1 onto V_2. The result of each such mapping f will be considered as the (multi)graph $G = G_f$ with labelled edges (with labels 1 and 2) whose vertices are pairs $(u, f(u))$ for $u \in V_1$, and for which two vertices $(u_1, f(u_1))$ and $(u_2, f(u_2))$ are connected by an edge in E_1 (respectively, E_2) if $u_1u_2 \in E_1$ (respectively, $f(u_1)f(u_2) \in E_2$). We will use the expression *i-neighbours* of a vertex to denote its neighbours under E_i, $i = 1, 2$. For each such mapping f, define a $(u_1, u_2; i, j)$-*link* to be a path of length two from $(u_1, f(u_1))$ to $(u_2, f(u_2))$ whose first edge has label i and the second edge has label j, $i, j \in \{1, 2\}$. For notational simplicity, we will often use u instead of $(u, f(u))$. A (u_1, u_2)-*switch* means replacing the mapping f with a mapping f' that differs from f only in that $f'(u_1) = f(u_2)$ and $f'(u_2) = f(u_1)$. In a (u_1, u_2)-switch, all the 2-neighbours of u_1 become the 2-neighbours of u_2 and vice versa.

For $e \in E_1$ (respectively, $e \in E_2$), an *e-packing* of G_1 and G_2 is a mapping f of V_1 onto V_2 such that e is the only edge in E_1 (respectively, E_2) that shares its incident vertices with an edge from E_2 (respectively, E_1). Such a packing exists for every edge e in a critical pair. In this case, we will say that e is a *conflicting edge*. Each e-packing will be also called a *quasi-packing* if e is not specified.

The proof of the following lemma essentially repeats the proof of Theorem 1.1 by Sauer and Spencer.

Lemma 2.1. *Let (G_1, G_2) be a critical pair and $2\Delta_1\Delta_2 \le n$. Given any $e = u_1u_1' \in E_1$, the following statements hold for any e-packing of G_1 and G_2.*

(i) *For every $u \ne u_1'$, there exists either a unique $(u_1, u; 1, 2)$-link or a unique $(u_1, u; 2, 1)$-link.*
(ii) *There is neither a $(u_1, u_1'; 1, 2)$-link nor a $(u_1, u_1'; 2, 1)$-link.*
(iii) $2\deg_{G_1}(u_1)\deg_{G_2}(u_1) = n$.
Since e was chosen arbitrarily and the roles of G_1 and G_2 are symmetric, we have, in particular, that $2\Delta_1\Delta_2 = n$ and $\deg(w_i) \in \{0, \Delta_i\}$ for all $w_i \in V_i$, $i = 1, 2$.

By this lemma, we have a packing of G_1 and G_2 if $2\Delta_1\Delta_2 < n$, and we need to consider below only the case when $2\Delta_1\Delta_2 = n$ and in particular, n is even. The main part of our proof will be the following characterization of possibilities for G_1 and G_2.

Lemma 2.2. *If $2\Delta_1\Delta_2 = n$ and (G_1, G_2) is a critical pair, then every component of G_i is either K_{Δ_i,Δ_i} with Δ_i odd, or an isolated vertex, or K_{Δ_i+1}, $i = 1, 2$.*

This lemma alone allows us to settle the case of $\min\{\Delta_1, \Delta_2\} = 1$. Indeed, suppose $\Delta_2 = 1$, *i.e.*, G_2 is a matching. Then $\Delta_1 = \frac{n}{2}$. If G_1 contains K_{Δ_1,Δ_1}, then simply $G_1 = K_{\frac{n}{2},\frac{n}{2}}$. And the only case when we cannot pack $K_{\frac{n}{2},\frac{n}{2}}$ with a matching is when the matching is perfect and $\frac{n}{2}$ is odd. If $G_1 \neq K_{\frac{n}{2},\frac{n}{2}}$, then by Lemma 2.2, it contains $K_{\frac{n}{2}+1}$, and in this case G_1 has no room for any other non-trivial components described by Lemma 2.2. Again, if the graph consisting of $K_{\frac{n}{2}+1}$ and $\frac{n}{2} - 1$ isolated vertices does not pack with a matching, then the matching has to be perfect. This confirms the theorem for the case when $\Delta_2 = 1$ or $\Delta_1 = 1$.

The next lemma forbids some of the possibilities allowed by Lemma 2.2.

Lemma 2.3. *Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1\Delta_2 = n$. If (G_1, G_2) is a critical pair and the conflicted edge in a quasi-packing belongs to a component H of G_2 isomorphic to K_{Δ_2,Δ_2}, then every component of G_1 sharing vertices with H is K_{Δ_1,Δ_1}.*

Since for every edge $e \in E(G_1) \cup E(G_2)$, there exists an e-packing, and each quasi-packing of G_1 and G_2 is a connected graph, Lemma 2.3 yields that if at least one component of G_2 is K_{Δ_2,Δ_2} or at least one component of G_1 is K_{Δ_1,Δ_1}, then *every* component of G_2 is K_{Δ_2,Δ_2} and every component of G_1 is K_{Δ_1,Δ_1}. Furthermore, in this case both Δ_1 and Δ_2 are odd, G_1 consists of Δ_2 copies of K_{Δ_1,Δ_1} and G_1 consists of Δ_1 copies of K_{Δ_2,Δ_2}. Thus the following two packing results will complete the proof of the theorem.

Lemma 2.4. *Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1\Delta_2 = n$. If every non-trivial component of G_i is K_{Δ_i+1}, $i = 1, 2$, then G_1 and G_2 pack.*

Proof. Let the components of G_1 be $H_1, \ldots, H_k, H_{k+1}, \ldots, H_{k+m}$, where each of $H_1, \ldots, H_k$ is a K_{Δ_1+1}, and each of $H_{k+1}, \ldots, H_{k+m}$ is an isolated vertex. We may assume that $m \leq \Delta_1$ (otherwise, we may form additional copies of K_{Δ_1+1} from the isolated vertices without violating the conditions of the lemma). Thus

$$k \geq \frac{n - \Delta_1}{\Delta_1 + 1} = \frac{2\Delta_1\Delta_2 - \Delta_1}{\Delta_1 + 1} = \Delta_2 + \frac{\Delta_1\Delta_2 - \Delta_1 - \Delta_2}{\Delta_1 + 1} = \Delta_2 + \frac{(\Delta_1 - 1)(\Delta_2 - 1) - 1}{(\Delta_1 + 1)}.$$

If $\Delta_1 \geq 3$, or $\Delta_2 \geq 3$, or $m < \Delta_1$, this implies that $k \geq \Delta_2 + 1$. In this case, order $H_1, \ldots, H_k$ cyclically and map each vertex of a copy of K_{Δ_2+1} in G_2 to a vertex in H_i in this order. After all the vertices in H_i, $1 \leq i \leq k$, are exhausted, continue on to the isolated vertices. Since $k \geq \Delta_2 + 1$, no two vertices in a component of G_2 will be assigned to the same component of G_1, giving a valid packing.

If $\Delta_1 = \Delta_2 = 2$ and $m = \Delta_1 = 2$, then we need to pack two 8-vertex graphs each consisting of two triangles and two isolated vertices. Let $u_1u_2u_3$ and $v_1v_2v_3$ be the two triangles of G_1 and x and y be the isolated vertices in G_1. Map the vertices of the two triangles of G_2 to u_2v_2x and u_3v_3y, respectively, and the isolated vertices to u_1 and v_1. □

Lemma 2.5. *Suppose that $\Delta_1, \Delta_2 \geq 3$ and odd, and $2\Delta_1\Delta_2 = n$. If G_1 consists of Δ_2 copies of K_{Δ_1,Δ_1} and G_2 consists of Δ_1 copies of K_{Δ_2,Δ_2}, then G_1 and G_2 pack.*

Proof. We may assume that $\Delta_1 \geq \Delta_2$. Suppose that the components of G_1 are $H_1, \dots, H_{\Delta_2}$ and the partite sets of H_i are

$$\{u_{(i-1)\Delta_1+1}, u_{(i-1)\Delta_1+2}, \dots, u_{i\Delta_1}\} \quad \text{and} \quad \{u_{\frac{n}{2}+(i-1)\Delta_1+1}, u_{\frac{n}{2}+(i-1)\Delta_1+2}, \dots, u_{\frac{n}{2}+i\Delta_1}\}$$

for $i = 1, \dots, \Delta_2$. We map the first $2\Delta_2$ vertices in this numeration to a copy of K_{Δ_2,Δ_2} in G_2, then the second $2\Delta_2$ vertices in this numeration to another copy of K_{Δ_2,Δ_2} in G_2, and so on. In order to see that this is a packing, we will check that any $2\Delta_2$ consecutive vertices in our numeration of $V(G_1)$ form an independent set. Indeed, consider the set $U_j = \{u_j, u_{j+1}, \dots, u_{j+2\Delta_2-1}\}$ for some j. Note that u_i is adjacent in G_1 to u_j only if $|i-j| > \frac{1}{2}n - \Delta_1$. But $\Delta_2 \geq 3$, hence $\frac{1}{2}n - \Delta_1 \geq 3\Delta_1 - \Delta_1 \geq 2\Delta_2$. □

Thus, to complete the proof of Theorem 1.2 it is enough to prove Lemmas 2.1, 2.2, and 2.3.

3. Proof of Lemmas 2.1 and 2.2

Proof of Lemma 2.1. Let (G_1, G_2) be a critical pair with $\Delta_1\Delta_2 \leq \frac{n}{2}$. Given an $e_1 \in E_1$, consider an e_1-packing of (G_1, G_2) with $e_1 = u_1u_1'$.

Suppose that for some $u \neq u_1'$ there exists neither a $(u_1, u; 1, 2)$-link nor a $(u_1, u; 2, 1)$-link. Then make a (u_1, u)-switch. Any new conflicting edge must be incident to either u_1 or u. If there is a conflicting edge incident to u_1 then, under the original quasi-packing, there is either a $(u, u_1; 1, 2)$-link or u_1u is a conflicting edge. The latter does not happen, since $u \neq u_1'$. Similarly, if there is a conflicting edge incident to u then there is a $(u, u_1; 2, 1)$-link in the original quasi-packing. This contradicts our assumption, and hence there exists at least one $(u_1, u; 1, 2)$- or $(u_1, u; 2, 1)$-link.

Since each of $d_{G_1}(u_1)$ 1-neighbours of u_1 (respectively, each of $d_{G_2}(f(u_1))$ 2-neighbours of u_1) has at most Δ_2 2-neighbours (respectively, at most Δ_1 1-neighbours), at most $d_{G_1}(u_1)\Delta_2 + d_{G_2}(f(u_1))\Delta_1$ such links start at u_1. Two of these links also finish at u_1. We have already seen that each $u \neq u_1', u_1$, is connected by such a link to u_1. This gives

$$n - 2 \leq d_{G_1}(u_1)\Delta_2 + d_{G_2}(f(u_1))\Delta_1 - 2.$$

Under the conditions of the lemma, this is possible only if $d_{G_1}(u_1) = \Delta_1$, $d_{G_2}(f(u_1)) = \Delta_2$, $2\Delta_1\Delta_2 = n$, and no link was wasted, that is, no $u \neq u_1$ is connected with u_1 by two or more links, and u_1' is not connected by a link with u_1. This proves the lemma. □

Now we will prove Lemma 2.2 in a series of claims. Let (G_1, G_2) be a critical pair with $\Delta_1\Delta_2 = \frac{n}{2}$. Suppose that G_1 is not the union of copies of K_{Δ_1+1} and isolated vertices. Let H_1 be a component of G_1 which is neither K_{Δ_1+1} nor an isolated vertex. By Lemma 2.1, H_1 is Δ_1-regular. Thus, H_1 contains an induced $P_3 = (u_1', u_1, u_2)$. Let $e_1 = u_1u_1'$. Consider an e_1-packing f of G_1 and G_2.

Claim 3.1. *If $u_1u_2 \in E_1$ and $u_2u_1' \notin E_1$, then there exists a $(u_1, u_2; 1, 2)$-link.*

Proof. By Lemma 2.1, there exists either a unique $(u_1, u_2; 1, 2)$-link or a unique $(u_1, u_2; 2, 1)$-link. If the claim does not hold, then there exists a $(u_1, u_2; 2, 1)$-link. Make a (u_1', u_2)-switch.

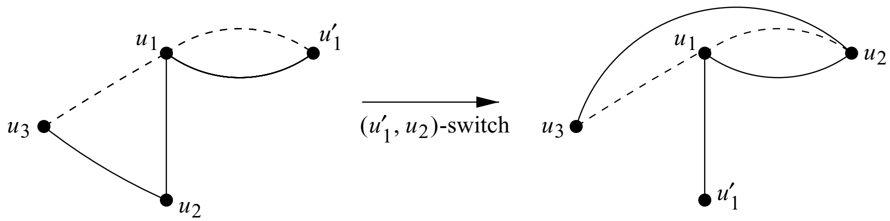

Figure 1. Claim 3.1 with edges in E_1 denoted by — and edges in E_2 denoted by - - - .

Then there exists a $(u_1, u_2; 2, 1)$-link under the new mapping f' which is a u_1u_2-packing. This contradicts Lemma 2.1(ii). See Figure 1. □

Claim 3.2. *Under the same conditions, let u_3 be the intermediate vertex on the $(u_1, u_2; 1, 2)$-link. Then $u_3u'_1 \notin E_1$ and $u_3u'_1 \notin E_2$.*

Proof. Otherwise, there exist two $(u_1, u_3; i, j)$-links, $i \neq j$, $i, j \in \{1, 2\}$, $u_1 \to u_2 \to u_3$ and $u_1 \to u'_1 \to u_3$, contradicting Lemma 2.1(i). □

Claim 3.3. *Under the conditions of Claim 3.2,*

(i) *u_2 is the only common neighbour of u_1 and u_3,*
(ii) *there is no common 1-neighbour of u_1 and u'_1,*
(iii) *all 1-neighbours of u_2 are also 1-neighbours of u'_1.*

Proof. (i) Let u_4 be another common neighbour of u_1 and u_3. By Claim 3.2, $u_4 \neq u'_1$. By Lemma 2.1, there is either a $(u_1, u_3; 1, 1)$-link or a $(u_1, u_3; 2, 2)$-link through u_4.

If there is a $(u_1, u_3; 1, 1)$-link through u_4, then make a (u'_1, u_4)-switch. Then there exist two $(u_1, u_3; i, j)$-links, $u_1 \to u_2 \to u_3$ and $u_1 \to u_4 \to u_3$, under the new mapping f', which is a u_1u_4-packing, $i \neq j$, $i, j \in \{1, 2\}$. This contradicts Lemma 2.1(i). See Figure 2(a).

If there is a $(u_1, u_3; 2, 2)$-link through u_4, then make a (u_1, u'_1)-switch. Then there exist two $(u_1, u_4; 1, 2)$-links under the new mapping f', which is a $u_1u'_1$-packing. This contradicts Lemma 2.1(i). See Figure 2(b).

(ii) Let u_5 be a common 1-neighbour of u_1 and u'_1. Make a (u'_1, u_3)-switch. By (i), there is no edge between u'_1 and u_3, and hence none exists after the switch as well. In the new mapping f', a u_1u_3-packing with u'_1 as the intermediate vertex on the $(u_1, u_2; 1, 2)$-link, u_5 is a second common neighbour of u_1 and u'_1, contradicting (i). See Figure 3.

(iii) Suppose u'_2 is a 1-neighbour of u_2 but not of u'_1. By Lemma 2.1(i), there exists a unique $(u_1, u'_2; i, j)$-link whose intermediate vertex u is not u'_1, since u'_2 is not a 1-neighbour of u'_1 and u'_2 is also not a 2-neighbour of u'_1, otherwise there would be two $(u'_1, u_2; 2, 1)$-links (through u_1 and u'_2).

If $u = u_3$, then u_3 is a 2-neighbour of u'_2. The (u_1, u_2)-switch leads to two $(u_3, u_2; 2, 1)$-links under the new mapping f', which is a u_1u_3-packing. This contradicts Lemma 2.1(i).

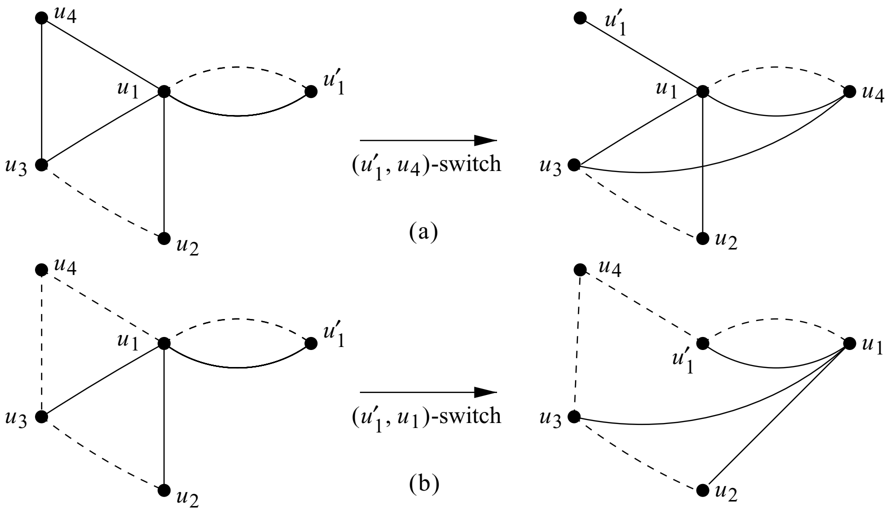

Figure 2. Claim 3.3(i).

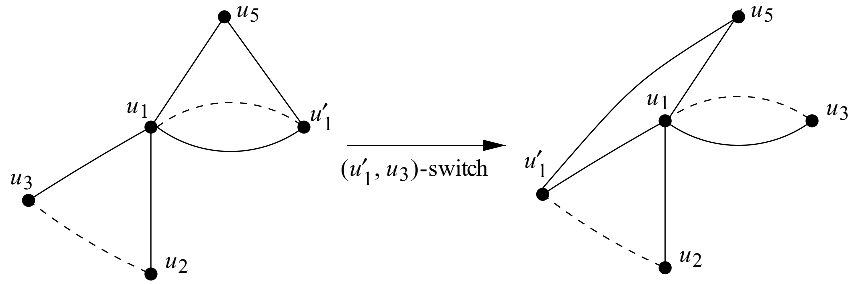

Figure 3. Claim 3.3(ii).

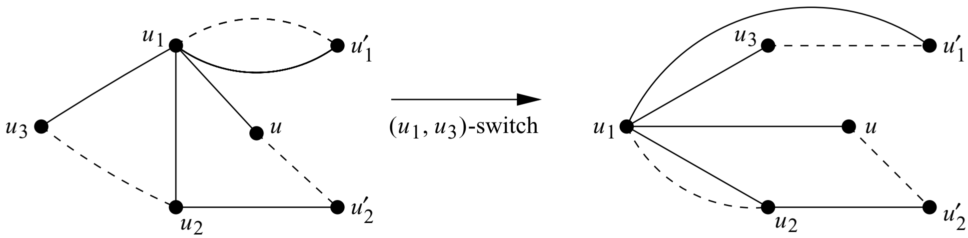

Figure 4. Claim 3.3(iii).

If $u \neq u_3$ then there exists a $(u_1, u_2'; i, j)$-link through intermediate vertex u, $i \neq j$, $i, j \in \{1, 2\}$. The (u_1, u_3)-switch leads to two $(u_1, u_2'; i, j)$-links under the new mapping f', a u_1u_2-packing. This contradicts Lemma 2.1(i). See Figure 4. □

As noted above, each vertex in the non-trivial component H_1 of G_1 has Δ_1 1-neighbours. If u_2' ($\neq u_2$), a 1-neighbour of u_1, has a 1-neighbour that is not a 1-neighbour of u_2 then by Claim 3.3 with u_2' playing the role of u_2, u_1' will have more than Δ_1 1-neighbours, a contradiction. Hence, all the 1-neighbours of u_1 have exactly the same Δ_1 1-neighbours, that is, the component of G_1 containing u_1 is K_{Δ_1,Δ_1}. Also, by Claim 3.3(i), there is a pairing of 1-neighbours of u_1 (respectively, u_1') other than u_1' (respectively, u_1), with a 2-edge between each such pair and no other 2-edges induced by the vertices of H_1. This means that these 2-edges induce a matching. By counting the 1-neighbours of u_1, we see that Δ_1 is odd, as u_1' is also a 1-neighbour of u_1. This completes the proof of Lemma 2.2.

4. Proof of Lemma 2.3

Suppose that $e_0 = u_{1,1}u_{2,1}$ is the only conflicting edge in a quasi-packing of G_1 with G_2 and that the component H_2 of G_2 containing e_0 is a K_{Δ_2,Δ_2} with partite sets $U_1 = \{u_{1,1}, u_{1,2}, \ldots, u_{1,\Delta_2}\}$ and $U_2 = \{u_{2,1}, u_{2,2}, \ldots, u_{2,\Delta_2}\}$. Let H_1 be the component of G_1 containing e_0.

Claim 4.1. $V(H_1) \cap V(H_2) = \{u_{1,1}, u_{2,1}\}$.

Proof. If a vertex $u \in V(H_2) - \{u_{1,1}, u_{2,1}\}$ belongs to $V(H_1)$, then it is adjacent in G_1 to $\{u_{1,1}, u_{2,1}\}$. Suppose, for definiteness, that it is adjacent to $u_{1,1}$. Since $uu_{1,1}$ is not a conflicting edge, $u \in U_1$. But then there is a $(u_{1,1}, u_{2,1}; 1, 2)$-link, a contradiction to Lemma 2.1. □

Claim 4.2. *Every component of G_1 shares with H_2 either 0 or 2 vertices.*

Proof. Let $u \in U_1 - u_{1,1}$. By Lemma 2.1, there is either a $(u_{2,1}, u; 1, 2)$-link or a $(u_{2,1}, u; 2, 1)$-link. The former cannot happen by Claim 4.1, so u should be adjacent in G_1 to another vertex in U_1. Moreover, if it were adjacent in G_1 to more than one vertex in U_1, then there would be more than one $(u_{2,1}, u; 2, 1)$-link, a contradiction to Lemma 2.1. Thus, each vertex of H_2 is adjacent in G_1 to exactly one other vertex of H_1. □

Consider now any component H_0 of G_1 sharing a vertex with H_2 and distinct from H_1. By Claim 4.2, it shares with H_2 exactly two vertices (from the same partite set). Suppose for definiteness that $V(H_0) \cap V(H_2) = \{u_{1,2}, u_{1,3}\}$. If $H_0 = K_{\Delta_1+1}$, then, since $\Delta_1 \geq 2$, the graph H_0 contains a vertex u_0 adjacent to both $u_{1,2}$ and $u_{1,3}$. But then there are at least two $(u_{2,1}, u_0; 2, 1)$-links, a contradiction to Lemma 2.1. Thus, each component of G_1 sharing a vertex with H_2 and distinct from H_1 is a K_{Δ_1,Δ_1}.

Consider a $(u_{2,1}, u_{1,2})$-switch. By Lemma 2.1, the new mapping f' is also a quasi-mapping. Furthermore, now H_1 does not have conflicting edges with H_2. Thus, by the previous paragraph, H_1 is a K_{Δ_1,Δ_1}. This finishes the proof of Lemma 2.3. □

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