A Short Proof of the Hajnal–Szemerédi Theorem on Equitable Colouring

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A proper vertex colouring of a graph is *equitable* if the sizes of colour classes differ by at most one. We present a new shorter proof of the celebrated Hajnal–Szemerédi theorem: for every positive integer r, every graph with maximum degree at most r has an equitable colouring with $r + 1$ colours. The proof yields a polynomial time algorithm for such colourings.

1. Introduction

An *equitable k-colouring* of a graph G is a proper k-colouring, for which any two colour classes differ in size by at most one. Equitable colourings naturally arise in some scheduling, partitioning, and load-balancing problems [1, 15, 16]. Pemmaraju [13] and Janson and Ruciński [6] used equitable colourings to derive deviation bounds for sums of random variables that exhibit limited dependence. In 1964 Erdős [3] conjectured that any graph with maximum degree $\Delta(G) \leqslant r$ has an equitable $(r + 1)$-colouring. This conjecture was proved in 1970 by Hajnal and Szemerédi [5] with a surprisingly long and complicated argument. Recently, Mydlarz and Szemerédi [11] found a polynomial time algorithm for such colourings.

In search of an easier proof, Seymour [14] strengthened Erdős's conjecture by asking whether every graph with minimum degree $\delta(G) \geqslant \frac{k}{k+1}|G|$ contains the kth power of a Hamiltonian cycle. (If $|G| = (r + 1)(s + 1)$ and $\Delta(G) \leqslant r$ then $\delta(\overline{G}) \geqslant \frac{s}{s+1}|G|$; each $(s + 1)$-interval of an sth power of a Hamiltonian cycle in $\overline{G}$ is an independent set in G.) The case $k = 1$ is Dirac's theorem and the case $k = 2$ is Pósa's conjecture. Fan and Kierstead [4] proved Pósa's conjecture with 'cycle'

replaced by 'path'. Komlós, Sárközy and Szemerédi [7] proved Seymour's conjecture for graphs with sufficiently many (in terms of k) vertices. Neither of these partial results has a simple proof. In fact, Komlós, Sárközy and Szemerédi [7] use the Regularity Lemma, the Blow-up Lemma and the Hajnal–Szemerédi theorem.

A different strengthening was suggested recently by Kostochka and Yu [9, 10]. In the spirit of Ore's theorem on Hamiltonian cycles [12], they conjectured that every graph in which $d(x) + d(y) \leqslant 2r$ for every edge xy has an equitable $(r + 1)$-colouring.

In this paper we present a short proof of the Hajnal–Szemerédi theorem and present another polynomial time algorithm that constructs an equitable $(r + 1)$-colouring of any graph G with maximum degree $\Delta(G) \leqslant r$. Our approach is similar to the original method, but a discharging argument leads to a simpler conclusion. Our techniques have paid further dividends. In another paper we will prove the above conjecture of Kostochka and Yu [9, 10] in a stronger form: with $2r + 1$ in place of $2r$. They also yield partial results towards the Chen–Lih–Wu conjecture [2] about equitable r-colourings of r-regular graphs and towards a list analogue of Hajnal–Szemerédi theorem (see [8] for definitions).

Most of our notation is standard; possible exceptions include the following. For a vertex y and set of vertices X, $N_X(y) := N(y) \cap X$ and $d_X(y) := |N_X(y)|$. If μ is a function on the edge set, then $\mu(A, B) := \sum_{xy \in E(A,B)} \mu(x, y)$, where $E(A, B)$ is the set of edges joining a vertex in A to a vertex in B. For a function $f : V \to Z$, the restriction of f to $W \subseteq V$ is denoted by $f|W$. Functions are viewed formally as sets of ordered pairs. Thus, if $u \notin V$ then $g := f \cup \{(u, \gamma)\}$ is the extension of f to $V \cup \{u\}$ such that $g(u) = \gamma$.

2. Main result

Let G be a graph with $s(r + 1)$ vertices satisfying $\Delta(G) \leqslant r$. A *nearly equitable* $(r + 1)$-colouring of G is a proper colouring f, whose colour classes all have size s except for one *small* class $V^- = V^-(f)$ with size $s - 1$ and one *large* class $V^+ = V^+(f)$ with size $s + 1$. Given such a colouring f, define an auxiliary digraph $H = H(G, f)$ as follows. The vertices of H are the colour classes of f, and a directed edge VW belongs to $E(H)$ if and only if some vertex $y \in V$ has no neighbours in W. In this case we say that y is *movable* to W. Call $W \in V(H)$ *accessible* if V^- is reachable from W in H. Note that V^- is trivially accessible. Let $\mathscr{A} = \mathscr{A}(f)$ denote the family of accessible classes, $A := \bigcup \mathscr{A}$ and $B := V(G) \setminus A$. Let $m := |\mathscr{A}| - 1$ and $q := r - m$. Then $|A| = (m + 1)s - 1$ and $|B| = (r - m)s + 1$. Let $y \in B$. Since y cannot be moved to A,

$$d_A(y) \geqslant m + 1 \text{ and hence } d_B(y) \leqslant q - 1. \tag{2.1}$$

Lemma 2.1. *If G has a nearly equitable $(r + 1)$-colouring f whose large class V^+ is accessible, then G has an equitable $(r + 1)$-colouring.*

Proof. Let $\mathscr{P} := V_1 \ldots V_k$ be a path in $H(G, f)$ from $V_1 := V^+$ to $V_k := V^-$. This means that for each $j = 1, \ldots, k - 1$, V_j contains a vertex y_j that has no neighbours in V_{j+1}. So, if we move y_j to V_{j+1} for $j = 1, \ldots, k - 1$, then we obtain an equitable $(r + 1)$-colouring of G. $\square$

Suppose $V^+ \subseteq B$. If $A = V^-$ then $|E(A, B)| \leqslant r|V^-| = r(s - 1) < 1 + rs = |B|$, a contradiction to (2.1). Thus $m + 1 = |\mathscr{A}| \geqslant 2$. Call a class $V \in \mathscr{A}$ *terminal* if V^- is reachable from every

class $W \in \mathscr{A} \setminus \{V\}$ in the digraph $H - V$. Trivially, V^- is non-terminal. Every non-terminal class W partitions $\mathscr{A} \setminus \{W\}$ into two parts $\mathscr{S}_W$ and $\mathscr{T}_W \neq \emptyset$, where $\mathscr{S}_W$ is the set of classes that can reach V^- in $H - W$. Choose a non-terminal class U so that $\mathscr{A}' := \mathscr{T}_U \neq \emptyset$ is minimal. Then every class in $\mathscr{A}'$ is terminal and no class in $\mathscr{A}'$ has a vertex movable to any class in $(\mathscr{A} \setminus \mathscr{A}') \setminus \{U\}$. Set $t := |\mathscr{A}'|$ and $A' := \bigcup \mathscr{A}'$. Thus every $x \in A'$ satisfies

$$d_A(x) \geqslant m - t. \tag{2.2}$$

Call an edge zy with $z \in W \in \mathscr{A}'$ and $y \in B$ a *solo edge* if $N_W(y) = \{z\}$. The ends of the solo edges are called *solo vertices* and the vertices joined by the solo edges are called *special neighbours* of each other. Let S_z denote the set of special neighbours of z and let S^y denote the set of special neighbours of y in A'. Let $y \in B$. Since at most $r - (m + 1 + d_B(y))$ colour classes in A have more than one neighbour of y,

$$|S^y| \geqslant t - q + 1 + d_B(y). \tag{2.3}$$

Lemma 2.2. *If there exists $W \in \mathscr{A}'$ such that no solo vertex in W is movable to a class in $\mathscr{A} \setminus \{W\}$, then $q + 1 \leqslant t$. Furthermore, every vertex $y \in B$ is solo.*

Proof. Let S be the set of solo vertices in W and $D := W \setminus S$. Then every vertex in $N_B(S)$ has at least one neighbour in W and every vertex in $B \setminus N_B(S)$ has at least two neighbours in W. It follows that $|E(W, B)| \geqslant |N_B(S)| + 2(|B| - |N_B(S)|)|$. Since no vertex in S is movable, every $z \in S$ satisfies $d_B(z) \leqslant q$. By (2.2), every vertex $x \in W$ satisfies $d_B(x) \leqslant t + q$. Thus, using $s = |W| = |S| + |D|$,

$$qs + q|D| + 2 = 2(qs + 1) - q|S| \leqslant |E(W, B)| \leqslant q|S| + (t + q)|D| \leqslant qs + t|D|.$$

It follows that $q + 1 \leqslant t$. Moreover, by (2.3) every $y \in B$ satisfies $|S^y| \geqslant t - q + d_B(y) \geqslant 1$. Thus y is solo. $\qquad\square$

Lemma 2.3. *There exists a solo vertex $z \in W \in \mathscr{A}'$ such that either z is movable to a class in $\mathscr{A} \setminus \{W\}$ or z has two nonadjacent special neighbours in B.*

Proof. Suppose not. Then by Lemma 2.2 every vertex in B is solo. Moreover, S_z is a clique for every solo vertex $z \in A'$. Consider a weight function μ on $E(A', B)$ defined by

$$\mu(xy) := \begin{cases} \frac{q}{|S_x|} & \text{if } xy \text{ is solo,} \\ 0 & \text{if } xy \text{ is not solo.} \end{cases}$$

For $z \in A'$ we have $\mu(z, B) = |S_z|\frac{q}{|S_z|} = q$ if z is solo; otherwise $\mu(z, B) = 0$. Thus $\mu(A', B) \leqslant q|A'| = qst$. On the other hand, consider $y \in B$. Let $c_y := \max\{|S_z| : z \in S^y\}$, say $c_y = |S_z|, z \in S^y$. Using that S_z is a clique and (2.1), $c_y - 1 \leqslant d_B(y) \leqslant q - 1$. So $c_y \leqslant q$. Together with (2.3) this yields

$$\mu(A', y) = \sum_{z \in S^y} \frac{q}{|S_z|} \geqslant |S^y|\frac{q}{c_y} \geqslant (t - q + c_y)\frac{q}{c_y} = (t - q)\frac{q}{c_y} + q \geqslant t.$$

Thus $\mu(A', B) \geqslant t|B| = t(qs + 1) > qst \geqslant \mu(A', B)$, a contradiction. $\qquad\square$

We are now ready to prove the Hajnal–Szemerédi theorem.

Theorem 2.4. *If G is a graph satisfying $\Delta(G) \leqslant r$, then G has an equitable $(r + 1)$-colouring.*

Proof.   We may assume that $|G|$ is divisible by $r + 1$. To see this, suppose that $|G| = s(r + 1) - p$, where $p \in [r]$. Let $G' := G + K^p$. Then $|G'|$ is divisible by $r + 1$ and $\Delta(G') \leqslant r$. Moreover, the restriction of any equitable $(r + 1)$-colouring of G' to G is an equitable $(r + 1)$-colouring of G.

Argue by induction on $\|G\|$. The base step $\|G\| = 0$ is trivial, so consider the induction step $\|G\| \geqslant 1$. Let $e = xy$ be an edge of G. By the induction hypothesis there exists an equitable $(r + 1)$-colouring f_0 of $G - e$. Hence we are done unless some colour class V contains both x and y. Since $d(x) \leqslant r$, there exists another class W such that x is movable to W. Doing so yields a nearly equitable $(r + 1)$-colouring f of G with $V^-(f) = V \setminus \{x\}$ and $V^+(f) = W \cup \{x\}$. We now show by a secondary induction on $q(f)$ that G has an equitable $(r + 1)$-colouring.

If $V^+ \in \mathscr{A}$ then we are done by Lemma 2.1; in particular, the base step $q = 0$ holds. Otherwise, by Lemma 2.3 there exists a class $W \in \mathscr{A}'$, a solo vertex $z \in W$ and a vertex $y_1 \in S_z$ such that either z is movable to a class $X \in \mathscr{A} \setminus \{W\}$ or z is not movable in $\mathscr{A}$ and there exists another vertex $y_2 \in S_z$, which is not adjacent to y_1. By (2.1) and the primary induction hypothesis, there exists an equitable q-colouring g of $B^- := B \setminus \{y_1\}$. Let $A^+ := A \cup \{y_1\}$.

Case 1: z is movable to $X \in \mathscr{A}$. Move z to X and y_1 to $W \setminus \{z\}$ to obtain a nearly equitable $(m + 1)$-colouring φ of A^+. Since $W \in \mathscr{A}'(f)$, $V^+(\varphi) = X \cup \{z\} \in \mathscr{A}(\varphi)$. By Lemma 2.1, A^+ has an equitable $(m + 1)$-colouring φ'. Then $\varphi' \cup g$ is an equitable $(r + 1)$-colouring of G.

Case 2: z is not movable to any class in $\mathscr{A}$. Then $d_{A^+}(z) \geqslant d_A(z) + 1 \geqslant m + 1$. Thus $d_{B^-}(z) \leqslant q - 1$. So we can move z to a colour class $Y \subseteq B$ of g to obtain a new colouring g' of $B^* := B^- \cup \{z\}$. Also, move y_1 to W to obtain an $(m + 1)$-colouring ψ of $A^* := V(G) \setminus B^*$. Set $\psi' := \psi \cup g'$. Then ψ' is a nearly equitable colouring of G with $A^* \subseteq A(\psi')$. Moreover, y_2 is movable to $W^* := W \cup \{y_1\} \setminus \{z\}$. Thus $q(\psi') < q(f)$, and so by the secondary induction hypothesis, G has an equitable $(r + 1)$-colouring ψ''.   □

3. A polynomial algorithm

Our proof clearly yields an algorithm. However it may not be immediately clear that its running time is polynomial. The problem lies in the secondary induction, where we may apply Case 2 $O(r)$ times, each time calling the algorithm recursively. Lemma 2.2 is crucial here; it allows us to claim that when we are in Case 2 (doing much work) we make much progress. As above, G is a graph satisfying $\Delta(G) \leqslant r$ and $|G| = s(r + 1) =: n$. Let f be a nearly equitable $(r + 1)$-colouring of G.

Theorem 3.1. *There exists an algorithm $\mathscr{P}'$ that from input (G, f) constructs an equitable $(r + 1)$-colouring of G in $c(q + 1)n^3$ steps.*

Proof.   We shall show that the construction in the proof of Theorem 2.4 can be accomplished in the stated number of steps. Argue by induction on q. The base step $q = 0$ follows immediately from Lemma 2.1 and the observation that the construction of H and the recolouring can be carried

out in $\frac{1}{4}cn^3$ steps. Now consider the induction step. In $\frac{1}{4}cn^3$ steps construct $\mathscr{A}, \mathscr{A}', B, W, z, y_1$. Using the induction hypothesis on the input $(G[B^-], f|B^-)$, construct the colouring g of B^- in $c(q(f|B^-) + 1)(qs)^3 \leqslant cqn^3$ steps. In $\frac{1}{4}cn^3$ steps determine whether Case 1 or Case 2 holds.

If Case 1 holds, construct the recolouring φ' in $\frac{1}{4}cn^3$ steps. This yields an equitable $(r + 1)$-colouring $g \cup \varphi'$ in a total of $\frac{3}{4}cn^3 + cqn^3 \leqslant c(q + 1)n^3$.

If Case 2 holds then, by Lemma 2.2, $q + 1 \leqslant t$. Thus we used only $\frac{1}{8}cqn^3$ steps to construct g. Use an additional $\frac{1}{4}cn^3$ steps to extend g to ψ'. Notice that W^* is non-terminal in ψ'. Thus we can choose $\mathscr{A}'(\psi')$ so that $A'(\psi') \subseteq B$. If Case 1 holds for ψ' then, as above, we can construct an equitable colouring in an additional $\frac{1}{4}cn^3 + \frac{1}{8}cqn^3$ steps. So the total number of steps is at most $c(q + 1)n^3$. Otherwise, by Lemma 2.2, $q(\psi') < \frac{1}{2}q$. Thus, by the induction hypothesis, we can finish the construction in $c\frac{qn^3}{16}$ additional steps. Therefore, the total number of steps is less than $c(q + 1)n^3$. $\qquad\square$

Theorem 3.2. *There is an algorithm $\mathscr{P}$ of complexity $O(n^5)$ that constructs an equitable $(r + 1)$-colouring of any graph G satisfying $\Delta(G) \leqslant r$ and $|G| = n$.*

Proof. As above, we may assume that n is divisible by $r + 1$. Let $V(G) := \{v_1, \dots, v_n\}$. Delete all the edges from G to form G_0 and let f_0 be an equitable colouring of G_0. Now, for $i = 1, \dots, n - 1$, do the following.

(i) Add back all the edges of G incident with v_i to form G_i.

(ii) If v_i has no neighbours in its colour class in f_{i-1}, then set $f_i := f_{i-1}$.

(iii) Otherwise, move v_i to a colour class that has no neighbours of v_i to form a nearly equitable colouring f'_{i-1} of G_i. Then apply $\mathscr{P}'$ to (G_i, f'_{i-1}) to get an equitable $(r + 1)$-colouring f_i of G_i.

Then f_{n-1} is an equitable $(r + 1)$-colouring of $G_{n-1} = G$. Since we have only $n - 1$ stages and each stage runs in $O(n^4)$ steps, the total complexity is $O(n^5)$. $\qquad\square$

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