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When is an Almost Monochromatic K_4 Guaranteed?

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Suppose that $n > (\log k)^{ck}$, where c is a fixed positive constant. We prove that, no matter how the edges of K_n are coloured with k colours, there is a copy of K_4 whose edges receive at most two colours. This improves the previous best bound of $k^{c'k}$, where c' is a fixed positive constant, which follows from results on classical Ramsey numbers.

1. Introduction

Let p, q be positive integers with $2 \leqslant q \leqslant \binom{p}{2}$. A (p,q)-colouring of K_n is an edge-colouring such that every copy of K_p receives at least q distinct colours on its edges. Let $f(n,p,q)$ denote the minimum number of colours in a (p,q)-colouring of K_n. This parameter, introduced in [2] and subsequently investigated by Erdős and Gyárfás [3], is a generalization of the classical Ramsey numbers. Indeed, if $R_k(p)$ denotes the minimum n such that every k-edge-colouring of K_n results in a monochromatic K_p, then determining all $R_k(p)$ is equivalent to determining all $f(n,p,2)$. Many special cases of $f(n,p,q)$ lead to non-trivial problems (see, *e.g.*, [1, 6, 7, 8]). One particularly interesting case is $f(n,4,3)$. In [2] it was observed that an easy application of the probabilistic method yields $f(n,4,3) = o(n)$. This was subsequently improved in [3] to $f(n,4,3) = O(\sqrt{n})$ via the Local Lemma. The second author [5] then improved the upper bound further to $e^{O(\sqrt{\log n})} = n^{o(1)}$, and this is the current best-known upper bound. The lower

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bound follows from the well-known fact $R_k(4) < k^{O(k)}$, which implies that there is a constant c such that

$$f(n,4,3) \geqslant f(n,4,2) > \frac{c\log n}{\log\log n}.$$

Here we give the first improvement of this lower bound.

Theorem 1.1. *Let $a \geqslant 1$ be fixed. There is a constant c depending on a such that, for all $n \geqslant 2a$,*

$$f(n,2a,a+1) > \frac{c\log n}{\log\log\log n}.$$

Let $R_k(p,q)$ be the minimum n such that every k-edge-colouring of K_n yields a copy of K_p with at most $q-1$ colours. Then $R_k(p,q) \leqslant n$ implies that every k-edge-colouring of K_n yields a copy of K_p with at most $q-1$ colours. Therefore, in order to edge-colour K_n with every copy of K_p receiving at least q colours, we need at least $k+1$ colours. This means that $f(n,p,q) > k$. Our main result is

$$R_k(2a,a+1) \leqslant c'(\log k)^{c'k}, \tag{1.1}$$

where c' is a positive constant depending only on a.

Let us argue that Theorem 1.1 follows from (1.1). First observe that (1.1) implies that

$$f(\lfloor c'(\log k)^{c'k}\rfloor, 2a, a+1) > k.$$

Now suppose that $a \geqslant 1$ is fixed and n is sufficiently large. Let k be the largest integer such that $n \geqslant \lfloor c'(\log k)^{c'k}\rfloor$. Then

$$f(n,2a,a+1) \geqslant f(\lfloor c'(\log k)^{c'k}\rfloor, 2a, a+1) > k.$$

Note that as $n\to\infty$, we also have $k\to\infty$. All asymptotic notation below is taken as both of these parameters approach infinity. It suffices to solve for k in terms of n. By definition of k, we clearly have $n = c'(\log k)^{c'k+O(1)}$. Taking logs, this yields $\log n = \Theta(k\log\log k)$ or

$$k = \Theta\left(\frac{\log n}{\log\log k}\right). \tag{1.2}$$

Taking logs of the previous expression yields $\log\log n = \Theta(\log k + \log\log\log k) = \Theta(\log k)$, and taking logs once again gives $\log\log\log n = \Theta(\log\log k)$ or

$$\log\log k = \Theta(\log\log\log n).$$

Plugging this into (1.2) gives us a constant c such that $k > c\log n/\log\log\log n$, and this proves Theorem 1.1.

2. The set-up of the proof

Let $a \geqslant 1$ be a positive integer throughout the rest of the paper.

Clearly, $f(n,2,2) = 0$ for $n \geqslant 2$. The idea of our proof is to run induction on something related to a, but not on a itself, since in this case the scale would be too rough. To facilitate the induction, we introduce some definitions.

Definition. A k-edge-colouring χ of K_n is a $(\gamma_1, \ldots, \gamma_k)$-*colouring* if, for each $i \in [k]$, colour i does not appear in any subgraph $K_{2\gamma_i+2}$ whose edges are coloured with at most $\gamma_i + 1$ colours. In particular, if $\gamma_i = 0$, then colour i does not appear in any subgraph K_2 whose edges are coloured with 1 colour, that is, does not appear at all.

Note that a k-edge-colouring of K_N is a $(2a, a+1)$-colouring if and only if it is an $(a-1, \ldots, a-1)$-colouring. Consequently, (1.1) states that if K_N admits an $(a-1, \ldots, a-1)$-colouring with k colours, then $N \leqslant c'(\log k)^{c'k}$, where c' depends only on a.

Definition. For an edge-colouring χ of K_n and a colour i, the *weakness* $\gamma_i(\chi)$ of i is the minimum p such that colour i does not appear in a K_{2p+2} with at most $p+1$ colours. In particular, $\gamma_i(\chi) = 0$ if and only if colour i is not present in χ at all. Then $\gamma(\chi) = \sum_{i=1}^{k} \gamma_i(\chi)$ is called the *weakness of* χ.

Note that by definition, each edge-colouring χ of K_n is a $(\gamma_1(\chi), \ldots, \gamma_k(\chi))$-colouring. Also by definition, the weakness of any $(a-1, \ldots, a-1)$-colouring with k colours is at most $(a-1)k$. Then (1.1) will follow from the following fact.

Theorem 2.1. *There is a positive constant c_1 such that if χ is an edge-colouring of K_N, then*

$$N \leqslant c_1(\log \gamma(\chi))^{c_1\gamma(\chi)}.$$

In all that follows, let γ_0 be sufficiently large such that for $\gamma \geqslant \gamma_0$, we have $\log\log\gamma > 1$,

$$\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^{15} > \frac{\log\gamma}{4500\log\log\gamma}, \quad \text{and} \quad 10^4\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^{5}\log\log\gamma > \log 2\gamma.$$

Let

$$\epsilon = \epsilon_\gamma = \frac{1000\log\log\gamma}{\log\gamma} < \frac{1}{100}, \quad t = t_\gamma = \lceil \epsilon^{-10} \rceil, \quad s = s_\gamma = \left\lceil \frac{(t-1)^{1/4}}{\sqrt{20}} \right\rceil > \frac{40}{\epsilon}. \qquad (2.1)$$

Let $c = R_{\gamma_0}(2\gamma_0)$ and define $g(\gamma) = c(\log\gamma)^{1000\gamma} = c\gamma^{\epsilon\gamma}$.

We will prove Theorem 2.1 by showing the following:

$$\text{Suppose that } \chi \text{ is a } (\gamma_1, \ldots, \gamma_k)\text{-colouring of } K_N \text{ and } \gamma = \textstyle\sum_i \gamma_i. \text{ Then } N < g(\gamma). \qquad (*)$$

We will prove $(*)$ by induction on γ and k. If $0 \leqslant \gamma \leqslant \gamma_0$, then certainly $N < c \leqslant g(\gamma)$, so we may assume that $\gamma > \gamma_0$. If some $\gamma_i = 0$, then colour i cannot appear at all, so we apply induction on k since the bound does not depend on k. Thus, we may assume that each γ_i is positive; in particular, $k \leqslant \gamma$. We will also assume that $N \geqslant g(\gamma) = c(\log\gamma)^{1000\gamma} = c\gamma^{\epsilon\gamma}$ and proceed to get a contradiction.

For a vertex x in a coloured K_n and a colour i, let $d_i(x)$ denote the number of edges of colour i incident to x.

Claim 2.2. *For $\gamma > \gamma_0$ and ϵ, t, s defined as above, we have $2t^{2s} < \gamma^{0.1s\epsilon-2}$.*

Proof. Since $2 < t^s$ and $s > 400/\epsilon$, the result follows from $t^{3s} < \gamma^{s\epsilon/20}$, which is equivalent to $60 \log t < \epsilon \log \gamma$. Since $t < \epsilon^{-11}$, we have

$$\frac{60 \log t}{\epsilon} < \frac{660 \log \epsilon^{-1}}{\epsilon} < \frac{660 \log \gamma}{1000 \log \log \gamma} \log\left[\frac{\log \gamma}{1000 \log \log \gamma}\right] < \frac{\log \gamma}{\log \log \gamma} \log \log \gamma = \log \gamma.$$

□

In the next section we prove that every dense bipartite graph $F(V_1, V_2; E)$ contains a 'large' subset M of V_1 in which every t-element subset has 'many' common neighbours in V_2. In Section 4 we prove the main result.

3. A probabilistic lemma

One of our main tools is the following lemma, essentially Lemma 1 in [4]. The proof uses ideas of Sudakov [9]. By $N(A)$ we denote the set of common neighbours of all vertices in A.

Lemma 3.1. *Let positive integers m, n, h, d and reals α, β be such that*

$$m^{d/h} < \beta. \tag{3.1}$$

Let $F = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = m$, $|V_2| = n$ such that

$$\deg_F(v) \geqslant n/\alpha \quad \text{for each } v \in V_1.$$

Then there is a subset V_1'' of V_1 with $|V_1''| > m/\alpha^h - 1$ such that every d-tuple D of vertices in V_1'' has at least n/β common neighbours.

Proof. Let $x_1, \ldots, x_h$ be a sequence of h not necessarily distinct vertices of V_2, which we choose uniformly and independently at random and denote $S = \{x_1, \ldots, x_h\}$. Denote by V_1' the set $N(S)$ of common neighbours of vertices in S. Note that the size of V_1' is a random variable and that $S \subseteq N(v)$ for every $v \in V_1'$. Then, using (3.1), we can estimate the expected size of V_1' as follows:

$$\mathbf{E}\big(|V_1'|\big) = \sum_{v \in V_1} \mathbf{Pr}\big(v \in V_1'\big) = \sum_{v \in V_1} \left(\frac{|N(v)|}{n}\right)^h \geqslant m\,\alpha^{-h}. \tag{3.2}$$

On the other hand, by definition, the probability that a given set of vertices $W \subset V_1$ is contained in V_1' equals $\big(|N(W)|/n\big)^h$. Denote by Z the number of subsets W of V_1' of size d with $|N(W)| < n/\beta$. Then by (3.1) the expected value of Z is at most

$$\mathbf{E}(Z) = \sum_{W \subseteq V_1 \,:\, |W|=d,\, |N(W)|<n/\beta} \mathbf{Pr}\big(W \subset V_1'\big) \leqslant \binom{m}{d}\left(\frac{1}{\beta}\right)^h \leqslant m^d \left(\frac{1}{\beta}\right)^h < 1. \tag{3.3}$$

Hence, the expectation of $|V_1'| - Z$ is greater than $m\,\alpha^{-h} - 1$, and thus there is a choice S_0 of S such that the corresponding value of $|V_1'(S_0)| - Z(S_0)$ is greater than $m\,\alpha^{-h} - 1$. For every d-tuple D of vertices of $V_1'(S_0)$, delete a vertex $v_D \in D$ from $V_1'(S_0)$. The resulting set V_1'' satisfies the lemma. □

4. Proof of the theorem

Call a t-set of vertices *rainbow* if its edges are coloured with at least $10t^{3/2}$ colours.

Claim 4.1. *Suppose that $n \geqslant \gamma > \gamma_0$, the edges of K_n are coloured (with any number of colours) and $d_i(x) \leqslant 2n\gamma^{-\epsilon/10}$ for each $x \in V(K_n)$ and each colour i. Then the number of t-sets that are not rainbow is at most $\binom{n}{t}/\gamma$.*

Proof. First, let us estimate $v(i,t,n)$, the number of t-sets in K_n in which there is a vertex incident with at least s edges of colour i in this t-set. We can first choose the vertex, then choose s incident edges of colour i and include the other ends of these edges, and then add $n-s-1$ other vertices. This gives

$$v(i,t,n) \leqslant \sum_{x \in V(K_n)} \binom{d_i(x)}{s}\binom{n-1-s}{t-1-s} \leqslant n\binom{\frac{2n}{\gamma^{\epsilon/10}}}{s}\binom{n-1-s}{t-1-s} \leqslant \binom{n}{t}\gamma^{-s\epsilon/10}\, t^{2s}.$$

Similarly, let $\psi(i,t,n)$ be the number of t-sets in K_n in which there is a matching of colour i of size at least s. Let e_i be the number of edges of colour i. Since

$$e_i \leqslant \frac{n}{2}\max_{x \in V(K_n)} d_i(x) \leqslant n^2\gamma^{-\epsilon/10},$$

we have

$$\psi(i,t,n) \leqslant \binom{e_i}{s}\binom{n-2s}{t-2s} \leqslant \binom{\frac{n^2}{\gamma^{\epsilon/10}}}{s}\binom{n-2s}{t-2s} \leqslant \binom{n}{t}t^{2s}\gamma^{-s\epsilon/10}.$$

Now Claim 2.2 implies that

$$v(i,t,n) + \psi(i,t,n) \leqslant 2\binom{n}{t}t^{2s}\gamma^{-s\epsilon/10} < \frac{1}{\gamma^2}\binom{n}{t}.$$

Suppose that a t-set T contains more than s^2 edges of colour i, and let G_i be the graph of these edges. Either G_i has a vertex incident with at least s edges, or Vizing's theorem implies that G_i has a proper edge-colouring with at most s colours. In the latter case, G_i has a matching of size at least $s^2/s = s$. We have already shown that the number of t-sets that contain a monochromatic matching of size s or a vertex with s edges of the same colour is at most $\binom{n}{t}/\gamma^2$. Consequently, the number of t-sets that contain more than s^2 edges of some colour is at most

$$k\binom{n}{t}/\gamma^2 \leqslant \binom{n}{t}/\gamma.$$

Each t-set not included above has at most s^2 edges in each colour and therefore at least $\binom{t}{2}/s^2$ colours. By the choice of s, this is at least $10t^{3/2}$. Hence the number of rainbow t-sets is at least $(1-1/\gamma)\binom{n}{t}$. $\square$

Claim 4.2. *Let $u \in V(K_N)$ and $S = S(u) = \{j \in [k] \;:\; d_j(u) \leqslant N/\gamma^{1+\epsilon/2}\}$. Then, for every $i \in [k] - S$ and $j \in [k]$, the number of vertices $x \in N_i(u)$ for which*

$$|N_j(x) \cap N_i(u)| \geqslant 2d_i(u)/\gamma^{\epsilon/10} \tag{4.1}$$

is at most $\gamma^{\epsilon\gamma-3}$.

Proof. Suppose the contrary. Then there are colours $i \in [k] - S(u)$ and $j \in [k]$ such that $N_i(u)$ contains a set M of $\lceil \gamma^{\epsilon\gamma-3} \rceil$ vertices x such that (4.1) holds. Consider the bipartite graph $F(V_1, V_2; E)$ with partite sets $V_1 = M$ and $V_2 = N_i(u) - M$ whose edges are all edges of colour j in our K_N connecting V_1 with V_2. By (4.1) and since $|M| = \lceil \gamma^{\epsilon\gamma-3} \rceil < \lceil N/\gamma^3 \rceil < d_i(u)/\gamma^{\epsilon/10}$, we have, for every $v \in V_1$,

$$\deg_F(v) > \frac{2d_i(u)}{\gamma^{\epsilon/10}} - |M| > \frac{d_i(u)}{\gamma^{\epsilon/10}} > \frac{|V_2|}{\gamma^{\epsilon/10}}.$$

Observe that graph F satisfies the conditions of Lemma 3.1 with

$$m = |M|, \quad n = |V_2|, \quad h = \gamma/\sqrt{t}, \quad d = t, \quad \alpha = \gamma^{\epsilon/10}, \quad \beta = 2m^{t/h}.$$

Hence, there is a subset M' of V_1 with

$$|M'| > m/\alpha^h - 1 \geqslant \gamma^{\epsilon\gamma-3}\alpha^{-h} - 1 > \gamma^{\epsilon\gamma-3}\gamma^{-(\gamma/\sqrt{t})\epsilon/10} - 1 > \gamma^{0.9\epsilon\gamma} \tag{4.2}$$

such that every d-tuple D of vertices in M' has at least n/β common neighbours.

We will construct a sequence $M_0 \subset M_1 \subset \cdots$ of subsets of M' as follows. Let $M_0 = M'$. Suppose that $M_0, M_1, \ldots, M_l$ are constructed. If there is a vertex $x_{l+1} \in M_l$ and a colour j_{l+1} such that $|N_{j_{l+1}}(x_{l+1}) \cap M_l| \geqslant |M_l|\gamma^{-\epsilon/10}$, then we let $M_{l+1} = N_{j_{l+1}}(x_{l+1}) \cap M_l$; otherwise we stop. Suppose that we stop at step q. Each colour i appears at most $2\gamma_i + 1$ times in $\{j_1, \ldots, j_q\}$, since otherwise we have a monochromatic $K_{2\gamma_i+2}$, which is forbidden. Consequently, $q \leqslant \sum_i (2\gamma_i + 1) = 2\gamma + k \leqslant 3\gamma$. From this and (4.2),

$$|M_q| > |M_0|(\gamma^{-\epsilon/10})^{3\gamma} = |M_0|\gamma^{-3\gamma\epsilon/10} > \gamma^{0.9\epsilon\gamma}\gamma^{-3\gamma\epsilon/10} = \gamma^{0.6\gamma\epsilon} > \gamma.$$

Hence, by Claim 4.1, M_q contains a rainbow t-tuple D (in fact it contains many). Let $N_F(D) = U$. By Lemma 3.1, $|U| \geqslant n/\beta$. Now suppose ℓ is a colour that appears in D. Then the weakness of ℓ within U is strictly smaller than γ_ℓ, since if ℓ appears in a K_{2p} within U that receives at most p colours, then this copy together with an edge of colour ℓ from D yields a $K_{2(p+1)}$ with at most $p + 1$ colours (the only new colour is possibly j). Therefore, the weakness of χ when restricted to U is at most $\gamma' = \gamma - 10t^{3/2}$. Hence, by the induction hypothesis, $|U| < g(\gamma') = c(\log \gamma')^{1000\gamma'}$. Since $|U| \geqslant n/\beta$,

$$n \leqslant \beta c(\log \gamma')^{1000\gamma'}.$$

On the other hand, since $|M| < d_i(u)/2$,

$$n = |V_2| = d_i(u) - |M| > \frac{d_i(u)}{2} > \frac{N}{2\gamma^{1+\epsilon/2}}.$$

This gives

$$N < 2\gamma^{1+\epsilon/2}(2m^{t/h})c(\log \gamma')^{1000\gamma'} = 4\gamma^{1+\epsilon/2}m^{t\sqrt{t}/\gamma}c(\log \gamma')^{1000\gamma'} < \gamma^{2+\epsilon t^{3/2}}c(\log \gamma')^{1000\gamma'},$$

where the last inequality holds because $m = |M| < \gamma^{\epsilon\gamma}$. As $N \geqslant g(\gamma) = c(\log \gamma)^{1000\gamma}$, we get

$$(\log \gamma)^{1000\gamma} < \gamma^{2+\epsilon t^{3/2}}(\log \gamma')^{1000\gamma'} < \gamma^{2+\epsilon t^{3/2}}(\log \gamma)^{1000\gamma'}.$$

Taking logs, this reduces to

$$1000\gamma \log\log \gamma < (2 + \epsilon t^{3/2}) \log \gamma + 1000\gamma' \log\log \gamma.$$

Consequently,

$$(1000 \log\log\gamma)10t^{3/2} < (2 + \epsilon t^{3/2})\log\gamma = 2\log\gamma + 1000t^{3/2}\log\log\gamma.$$

Simplifying, we obtain $9000t^{3/2}\log\log\gamma < 2\log\gamma$. Finally, this yields

$$\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^{15} = \epsilon^{-15} \leqslant t^{3/2} < \frac{\log\gamma}{4500\log\log\gamma},$$

which contradicts our choice of γ. □

Claim 4.3. *For every $u \in V(K_N)$, the number of rainbow t-sets on $V(K_N) - \{u\}$ all of whose vertices are connected with u by edges of the same colour is at least* $0.3\sum_{i=1}^{k}\binom{d_i(u)}{t}$.

Proof. Fix some $u \in V(K_N)$. Let $S = \{i \in [k] \ : \ d_i(u) \leqslant N/\gamma^{1+\epsilon/2}\}$. Then

$$\sum_{i\in S}\binom{d_i(u)}{t} \leqslant k\binom{\lfloor\frac{N}{\gamma^{1+\epsilon/2}}\rfloor}{t} \leqslant k\binom{\lfloor\frac{N}{k}\rfloor}{t}\gamma^{-\epsilon t/2} \leqslant 2\gamma^{-\epsilon t/2}\sum_{i=1}^{k}\binom{d_i(u)}{t}.$$

We added the factor 2 since $d_1(u) + \cdots + d_k(u) = N - 1$ and not N. Since $t = \lceil\epsilon^{-10}\rceil$, we have $t\epsilon > 20$ and hence

$$\sum_{i\in S}\binom{d_i(u)}{t} \leqslant \gamma^{-10}\sum_{i=1}^{k}\binom{d_i(u)}{t}. \tag{4.3}$$

Now, let $i \notin S$. Let M be the set of vertices $x \in N_i(u)$ such that, for some colour j, (4.1) holds. Let $\overline{M} = N_i(u) - M$. By Claim 4.2,

$$|M| < \gamma^{\epsilon\gamma-2} < \frac{N}{\gamma^2} < \frac{|N_i(u)|}{t}.$$

Hence, for the subgraph F of our K_N on $\overline{M}$, the conditions of Claim 4.1 are satisfied since $|\overline{M}| > (1 - 1/t)d_i(u) > 0.9d_i(u) > \gamma$. Thus, by Claim 4.1, at least $(1 - 1/\gamma)\binom{|\overline{M}|}{t}$ t-sets in $\overline{M}$ are rainbow. Now

$$\frac{\gamma-1}{\gamma}\binom{|\overline{M}|}{t} \geqslant \frac{\gamma-1}{\gamma}\binom{d_i(u)(1-1/t)}{t}.$$

For large γ, the last expression is at least

$$0.9\left(\frac{t-1}{t}\right)^t\binom{d_i(u)}{t} \geqslant \frac{1}{3}\binom{d_i(u)}{t}.$$

Combining this with (4.3), we finish the proof. □

By Claim 4.3, the total number of $(t+1)$-sets $\{u_0, u_1, \ldots, u_t\}$ of vertices of $V(K_N)$ such that the t-set $\{u_1, \ldots, u_t\}$ is rainbow and all edges from u_0 to $u_1, \ldots, u_t$ are of the same colour is at least

$$0.3\sum_{u\in V(K_N)}\sum_{i=1}^{k}\binom{d_i(u)}{t} \geqslant 0.3N\cdot k\binom{(N-1)/k}{t} \geqslant N\cdot(2k)^{1-t}\binom{N}{t}.$$

It follows that some rainbow t-set $\{u_1, \dots, u_t\}$ is contained in at least $N \cdot (2k)^{1-t}$ such $(t+1)$-sets. Let U be the set of all vertices u_0 in these $(t+1)$-sets containing our chosen $\{u_1, \dots, u_t\}$. Then, for some $1 \leqslant i \leqslant k$, the size of the subset U_i of U that is connected with each of $u_1, \dots, u_t$ by an edge of colour i is at least $2N \cdot (2k)^{-t}$. Since $\{u_1, \dots, u_t\}$ is rainbow, it contains edges of at least $10t^{3/2}$ colours. For every colour ℓ that appears within $\{u_1, \dots, u_t\}$, the weakness of ℓ when restricted to U_i is at most $\gamma_\ell - 1$. Hence, by the induction hypothesis, $|U_i| \leqslant g(\gamma') = c(\log \gamma')^{1000\gamma'}$, where $\gamma' = \gamma - 10t^{3/2}$. Since $|U_i| \geqslant 2N/(2k)^t$ and $N \geqslant g(\gamma)$, we obtain

$$c(\log \gamma)^{1000\gamma} \leqslant N < (2k)^t c(\log \gamma')^{1000\gamma'} < (2k)^t c(\log \gamma)^{1000\gamma'}.$$

Dividing by c and taking logs,

$$1000\gamma \log \log \gamma < t \log 2\gamma + 1000\gamma' \log \log \gamma.$$

Consequently,

$$(1000 \log \log \gamma) 10t^{3/2} < t \log 2\gamma.$$

Plugging in the values of t and ϵ, we obtain

$$10^4 \left(\frac{\log \gamma}{1000 \log \log \gamma} \right)^5 \log \log \gamma = 10^4 \epsilon^{-5} \log \log \gamma < 10^4 \sqrt{t} \log \log \gamma < \log 2\gamma.$$

This contradicts our choice of γ and completes the proof. □

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