PROD. TYPE: FLP
PP: 1-10 (col.fig.: nil)
TCS5723 MODC+
ED: Babu
PAGN: Vish – SCAN: nil

ARTICLE IN PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Theoretical Computer Science ■■■ (■■■■) ■■■–■■■

Theoretical Computer Science

www.elsevier.com/locate/tcs

On equitable Δ-coloring of graphs with low average degree[☆]

A.V. Kostochka[a, b, *], K. Nakprasit[a]

[a] Department of Mathematics, The University of Illinois, Urbana, IL 61801, USA
[b] Institute of Mathematics, Novosibirsk, Russia

Abstract

An *equitable coloring* of a graph is a proper vertex coloring such that the sizes of any two color classes differ by at most 1. Hajnal and Szemerédi proved that every graph with maximum degree Δ is equitably k-colorable for every $k \geqslant \Delta + 1$. Chen, Lih, and Wu conjectured that every connected graph with maximum degree $\Delta \geqslant 3$ distinct from $K_{\Delta+1}$ and $K_{\Delta,\Delta}$ is equitably Δ-colorable. This conjecture has been proved for graphs in some classes such as bipartite graphs, outerplanar graphs, graphs with maximum degree 3, interval graphs. We prove that this conjecture holds for graphs with average degree at most $\Delta/5$.

Keywords: ■; ■; ■

1. Introduction

In several applications of coloring as a partition problem there is an additional requirement that color classes be not so large or be of approximately the same size. Examples are the mutual exclusion scheduling problem [1,17], scheduling in communication systems [7], construction timetables [9], and round-a-clock scheduling [18]. For other applications in scheduling, partitioning, and load balancing problems, one can look into [2,12,17]. A model imposing such a requirement is *equitable coloring*—a proper coloring such that color classes differ in size by at most one. A good survey on equitable colorings of graphs is given in [13]. Recently, Pemmaraju [16] and Janson and Ruciński [8] used equitable colorings to derive deviation bounds for sums of dependent random variables that exhibit limited dependence.

Unlike in the case of ordinary coloring, a graph may have an equitable k-coloring (i.e., an equitable coloring with k colors) but have no equitable $(k+1)$-coloring. For example, the complete bipartite graph $K_{2n+1,2n+1}$ has the obvious equitable 2-coloring, but has no equitable $(2n+1)$-coloring. Thus, it is natural to look for the minimum number, $\mathrm{eq}(G)$, such that for every $k \geqslant \mathrm{eq}(G)$, G has an equitable k-coloring.

The difficulty of finding $\mathrm{eq}(G)$ is not less than that of finding the chromatic number. Thus, already in the class of planar graphs, finding $\mathrm{eq}(G)$ is an NP-hard problem. This situation prompted studying extremal problems on relations of $\mathrm{eq}(G)$ with other graph parameters. Hajnal and Szemerédi [6] settled a conjecture of Erdős by proving that $\mathrm{eq}(G) \leqslant \Delta+1$ for every graph G with maximum degree at most Δ. This bound is sharp, as shows the example of $K_{2n+1,2n+1}$ above. Other natural examples showing sharpness of Hajnal–Szemerédi Theorem are graphs with chromatic number greater

☆ This work was partially supported by the NSF Grants DMS-0099608 and DMS-0400498.
* Corresponding author. Department of Mathematics, The University of Illinois, Urbana, IL 61801, USA.
E-mail addresses: kostochk@math.uiuc.edu (A.V. Kostochka), nakprasi@math.uiuc.edu (K. Nakprasit).

0304-3975/$ - see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.tcs.2005.09.031

than maximum degree, i.e. complete graphs and odd cycles. Chen et al. [3] proposed the following analogue of Brooks' Theorem for equitable coloring:

Conjecture 1 (*Chen et al. [3]*). *Let G be a connected graph with maximum degree* Δ. *If G is distinct from* $K_{\Delta+1}$, $K_{\Delta,\Delta}$, *and is not an odd cycle*, *then G has an equitable coloring with* Δ *colors*.

They proved the conjecture for graphs with maximum degree at most three. Later, Yap and Zhang [20,21] proved that the conjecture holds for outerplanar graphs and planar graphs with maximum degree at least 13. Nakprasit (unpublished) extended the result of Yap and Zhang [21] to planar graphs with maximum degree at least 9. Lih and Wu [14] verified the conjecture for bipartite graphs, and Chen et al. [4] verified it for interval graphs. Kostochka et al. [11] studied a list analogue of equitable coloring and proved the validity of Conjecture 1 for equitable list coloring in classes of interval graphs and 2-degenerate graphs. It follows from [10] that the conjecture holds for *d*-degenerate graphs with maximum degree Δ if $d \leqslant (\Delta - 1)/14$.

Recall that a graph G is *d-degenerate*, if each subgraph G' of G has a vertex of degree (in G') at most d (see, e.g., [19, p. 269]). In other words, one can destroy any *d*-degenerate graph by successively deleting vertices of degree at most d. Forests are exactly 1-degenerate graphs. It is also well known that every outerplanar graph is 2-degenerate (see, e.g., [19, p. 240]), and every planar graph is 5-degenerate. To say that a graph has 'low degeneracy' is about the same as to say that every subgraph of G has a 'small average degree'. In this paper, we prove Conjecture 1 for graphs that have 'low average degree' themselves without restrictions on average degrees of subgraphs.

Theorem 1. *Let* $\Delta, n \geqslant 46$. *Suppose that an n-vertex graph* $G = (V, E)$ *has maximum degree at most* Δ *and* $|E| \leqslant \Delta n/5$. *If* $K_{\Delta+1}$ *is not a subgraph of G, then G has an equitable coloring with* Δ *colors.*

An immediate consequence of Theorem 1 is that Conjecture 1 holds for *d*-degenerate graphs with maximum degree Δ if $d \leqslant \Delta/10$.

In order to prove Theorem 1, we will need the following statement.

Theorem 2. *Let* $\Delta \geqslant 3$ *and G be a* $K_{\Delta+1}$*-free graph with* $\Delta(G) \leqslant \Delta$. *Suppose that* $G - v$ *has a* Δ*-coloring with color classes* $M_1, M_2, \ldots, M_\Delta$. *Then G has a* Δ*-coloring with color classes* $M'_1, M'_2, \ldots, M'_\Delta$ *such that* $|M_i| = |M'_i|$ *for all i apart from one*.

We call this statement a theorem, since it seems that it has its own merit. It can be considered as a slight refinement of the Brooks' Theorem. For example, Theorem 2 has the following easy consequence.

Corollary 1. *Let* $\Delta \geqslant 3$ *and G be a* $K_{\Delta+1}$*-free graph with* $\Delta(G) \leqslant \Delta$. *Let* $1 \leqslant k \leqslant \Delta$ *and* m_k *be the order of a maximum k-colorable subgraph of G. Then there is a proper coloring of G with at most* Δ *colors in which the total number of vertices in some k color classes is* m_k. *In particular*, *G has a proper coloring with at most* Δ *colors in which one of the color classes has* $\alpha(G)$ *vertices.*

The first step of the proof of Theorem 1 uses the Hajnal–Szemerédi Theorem whose complexity we have not analyzed. The rest of the proof can be rewritten as a polynomial time algorithm for equitable coloring of a graph.

The structure of the paper is as follows. In the next section, we prove Theorem 2. In Section 3 we introduce some useful notions and verify Theorem 1 for graphs with few vertices. In Section 4, we prove the bounded-size version of Theorem 1: we demand each color class to be of size at most $\lceil n/\Delta \rceil$ but allow 'small' color classes. We finalize the proof of Theorem 1 in the last section.

Throughout the paper, we use standard graph-theoretic definitions and notation (see, e.g., [5,19]).

2. Proof of Theorem 2

The proof follows the steps of a proof of Brooks' Theorem by Mel'nikov and Vizing [15] (see also [5, p. 99]).

Let $\Delta \geqslant 3$ and G be a $K_{\Delta+1}$-free graph with $\Delta(G) \leqslant \Delta$. Let f be a proper coloring of $G - v$ with at most Δ colors. Suppose that $M_1, M_2, \ldots, M_\Delta$ are color classes (maybe empty) of f and that G has no Δ-coloring with color classes $M'_1, M'_2, \ldots, M'_\Delta$ such that $|M_i| = |M'_i|$ for all i apart from one. Let G_{ij} denote the subgraph of G induced by $M_i \cup M_j \cup \{v\}$, and G'_{ij} denote the component of G_{ij} containing v. We will deliver the proof in a series of claims.

Claim 2.1. *The vertex v has exactly one neighbor in each M_i.*

Proof. Otherwise, there is a color class M_l containing no neighbors of v. We simply add v to M_l. Since other color classes do not change, the conclusion of the theorem holds, a contradiction. □

In view of this claim, let w_i denote the only neighbor of v in M_i.

Claim 2.2. *For each $1 \leqslant i < j \leqslant \Delta$, the graph G'_{ij} is an odd cycle.*

Proof. Suppose that some G'_{ij} is not an odd cycle. By Claim 2.1, G'_{ij} cannot be an even cycle. Thus G'_{ij} has a vertex of degree (in G'_{ij}) distinct from 2. Let u be a vertex closest to v in G'_{ij} with degree in G'_{ij} not equal to 2. Let $P = (v, v_1, \ldots, v_k)$, where $v_k = u$, be the shortest v, u-path in G'_{ij} (since $G'_{ij} - v$ is bipartite, this path is unique). Note that each internal vertex of P has degree 2 in G'_{ij}. Denote $A_j = \{v_1, \ldots, v_k\} - M_i$, and $A_i = \{v_1, \ldots, v_k\} - M_j$. Let $B_i = M_i \cup A_j - A_i$, and $B_j = M_j \cup A_i - A_j$. We may assume that $v_1 = w_i$.

Case 1: $d_{G'_{ij}}(u) = 1$. Then $P - v$ is a component in $G_{ij} - v$. Therefore, the sets B_i and B_j are independent. Note that B_i does not contain neighbors of v, since the only neighbor, v_1, of v in M_i belongs to B_j and A_j does not contain any neighbor of v. Thus $M'_i = B_i \cup \{v\}$, $M'_j = B_j$ and $M'_m = M_m$, for $m \neq i, j$, are color classes of a proper coloring of G. If k is odd, then $|M'_i| = |M_i|$, and if k is even, then $|M'_j| = |M_j|$. This contradicts the choice of G and f.

Case 2: $d_{G'_{ij}}(u) \geqslant 3$. Since $d_G(u) \leqslant \Delta$ and $d_{G_{ij}}(u) \geqslant 3$, there is a color class M_l, $l \neq i, j$, with no neighbors of u. Hence $M_l \cup \{u\}$ is independent. Similarly to Case 1, the sets $B_i - v_k$ and $B_j - v_k$ are independent. Thus, $M'_i = B_i \cup \{v\}$, $M'_j = B_j$, $M'_l = M_l \cup \{u\}$ and $M'_m = M_m$, for $m \neq i, j, l$, are color classes of a proper coloring of G. Note that independently of the parity of k, $|M'_i| = |M_i|$, and $|M'_j| = |M_j|$. This contradicts the choice of G and f. □

Claim 2.3. *For any distinct i, j and s, the components G'_{ij} and G'_{js} share exactly two vertices, namely, the vertex v and the neighbor, w_j, of v in M_j.*

Proof. By the definition, $\{v, w_j\} \subseteq G'_{ij} \cap G'_{js}$. Suppose $u \in G'_{ij} \cap G'_{js} - \{v, w_j\}$. By Claim 2.2, u has four neighbors in $G'_{ij} \cup G'_{js}$. Hence there is a color class M_l, $l \neq j$, with no neighbors of u. Let $v, v_1, \ldots, v_k, u$ be the v, u-path in G'_{ij} with $v_1 = w_i$. Note that k is odd, since $u \in M_j$ and $v_1 = w_i \in M_i$. Denote $A_j = \{v_1, \ldots, v_k\} - M_i$, and $A_i = \{v_1, \ldots, v_k\} - M_j$. Let $B_i = M_i \cup A_j - A_i$, and $B_j = M_j \cup A_i - A_j - \{u\}$. Similarly to the proof in Claim 2.2, the sets B_i and B_j are independent. Also similarly to the proof of Claim 2.2, the sets $M'_i = B_i \cup \{v\}$, $M'_j = B_j$, $M'_l = M_l \cup \{u\}$ and $M'_m = M_m$, for $m \neq i, j, l$, are color classes of a proper coloring of G. Moreover, $|M_m| = |M'_m|$ for all m apart from $m = l$. This proves the claim. □

Now, we are ready to finish the proof of Theorem 2. Among all colorings of $G - v$ with color class sizes $|M_1|, \ldots, |M_\Delta|$, choose a coloring $f = (M_1, \ldots, M_\Delta)$ and indices i and j so that $G'_{ij} - v = G'_{ij}(f) - v$ has the largest order. Recall that by the claims above, $G'_{ij} - v$ is a path with end-vertices w_i and w_j. We may assume that $i = 1$, $j = 2$, and $G'_{12} - v$ is a path $P_1 = (v_1, \ldots, v_q)$, where $v_1 = w_1$ and $v_q = w_2$. If $q = 2$, then by the maximality of G'_{ij}, all w_is are adjacent to each other and together with v form a $K_{\Delta+1}$, a contradiction. Thus, $q \geqslant 4$ and hence $v_{q-1} \neq w_1$.

Similarly, $G'_{23} - v$ is a path, P_2, with end-vertices w_2 and w_3. Let coloring f_1 be obtained from f by swapping the colors 2 and 3 on vertices in $G'_{23} - v$. Since $G'_{23} - v$ is a w_2, w_3-path, the color class sizes in f_1 are the same as in f. By Claim 2.2, the new bicolored subgraph $H = G'_{12}(f_1) - v$ has to be a w_1, w_3-path (because w_3 is the only neighbor of v in the second color class of f_1). Note that H contains $P_1 - v_q$. By the maximality of q, the vertices v_{q-1} and w_3 must be adjacent. But then v_{q-1} belongs to $G'_{13}(f)$, a contradiction to Claim 2.3. This proves the theorem. □

Proof of Corollary 1. Suppose that H is a k-colorable subgraph of G of the maximum order. Let $V(G) - V(H) = \{v_1, \ldots, v_t\}$. Let $G_0 = H$ and $G_i = G[V(H) \cup \{v_1, \ldots, v_i\}]$ for $i = 1, \ldots, t$. Let $f_0 = (M_1, \ldots, M_\Delta)$ be a coloring of H with $M_{k+1} = M_{k+2} = \cdots = M_\Delta = \emptyset$. Now, for $i = 1, \ldots, t$, apply Theorem 2 to G_i and f_{i-1} to produce a coloring f_i of G_i. Then by this theorem, the total number of vertices of colors $1, \ldots, k$ in $G_t = G$ is at least $|V(H)|$, but it cannot be greater by the maximality of H. This proves the corollary. As an additional feature, we have that none of the sizes of the first k color classes changed. □

Corollary 2. *Let $\Delta \geqslant 3$ and $G = (V, E)$ be a $K_{\Delta+1}$-free graph with $\Delta(G) \leqslant \Delta$. Let $|V| = n = k(\Delta + 1) + r$, where $0 \leqslant r \leqslant \Delta$. Then G has a Δ-coloring f with color classes $M_1, M_2, \ldots, M_\Delta$ such that*

(i) *$|M_i| \geqslant k$ for every i;*
(ii) *for every set Z of color classes, $|\bigcup_{M \in Z} M| \leqslant k + |Z|\lceil n/\Delta + 1\rceil$; in particular, $|M_i| \leqslant k + \lceil n/\Delta + 1\rceil$ for every i;*
(iii) *if $|M_i| = k + 1 + p$ for some $p \geqslant 1$, then the degree of every $v \in M_i$ in G is at least $\Delta - (k + r - 1)/p$.*

Proof. By the Hajnal–Szemerédi Theorem, G has an equitable $(\Delta + 1)$-coloring f'. Under the conditions of the corollary, exactly r color classes of f' have size $k + 1$. Let M' be a color class of f' with $|M'| = k$. Adding the vertices of M' one by one to $G - M'$ and applying Theorem 2 on every step, we get a Δ-coloring f'' of G satisfying (i) and (ii).

Now, consider the following procedure: If a vertex v in a color class M_i of size z has no neighbors in a color class M_j of size at most $z - 2$, then move v from M_i to M_j. Clearly, we will stop after a finite number of steps. We claim that the final Δ-coloring f is what we need. Indeed, once a coloring satisfies (i) and (ii), such moves do not destroy these properties. Since we have stopped our procedure, if for some i, we have $|M_i| = k + 1 + p$ and $v \in M_i$, then v has neighbors in every color class of size at most $k + p - 1$. By (i), the number of color classes of size at least $k + p$ (including M_i) is at most $(n - k\Delta - 1)/p = (k + r - 1)/p$ (here -1 arises, because $|M_i| = k + 1 + p$). It follows that v is adjacent to vertices in at least $\Delta - (k + r - 1)/p$ color classes. □

3. Background for the proof of Theorem 1

In this section, we do preparatory work for the proof of Theorem 1: introduce some notions and prove Theorem 1 for $n \leqslant 8.8\Delta$.

Let $G = (V, E)$ be a graph with maximum degree Δ and f be a vertex coloring of G with color classes $M_1, \ldots, M_\Delta$ (some color classes can be empty). For a set Y_0 of color classes of f and a subset V' of V, we define the *(V', f)-expansion of Y_0 in G* as follows.

Say that a vertex $w \in V' - \bigcup_{M \in Y_0} M$ is a *Y_0-candidate* if w has no neighbors in some color class $M(w) \in Y_0$. Let Y_1 be the set of color classes of f containing a Y_0-candidate. Similarly, for $h \geqslant 1$, a vertex $w \in V' - \bigcup_{M \in Y_0 \cup \cdots \cup Y_h} M$ is a *Y_h-candidate* if w has no neighbors in some color class $M(w) \in Y_h$. Let Y_{h+1} be the set of color classes containing a Y_h-candidate. Finally, the set $Y = \bigcup_{j=0}^{\infty} Y_j$ will be called the *(V', f)-expansion of Y_0 in G*.

By the construction, we have the following.

Claim 3.1. *If Y is the (V', f)-expansion of Y_0 in G, then every vertex $u \in V' - \bigcup_{M \in Y} M$ has a neighbor in every $M \in Y$.*

For each $M \in Y - Y_0$, we define an *(M, Y)-recoloring* as follows. Suppose that $M \in Y_{h+1}$ for some $h \geqslant 0$. By the definition of Y_{h+1}, M contains a Y_h-candidate $x_{h+1} \in V'$. Furthermore, for $j = h, h - 1, \ldots, 1$, the color class $M_j = M(x_{j+1})$ contains a Y_{j-1}-candidate $x_j \in V'$. Then an *(M, Y)-recoloring* of f is the coloring f' that differs from f only at $x_1, \ldots, x_{h+1}$: for every x_j, $j = h + 1, h, \ldots, 1$, we let $f'(x_j) = M_{j-1} = M(x_j)$. For different choices of $x_1, \ldots, x_{h+1}$, we get different recolorings, but the following claim holds by the definition.

Claim 3.2. *If $h \geqslant 0$ and $M \in Y_{h+1}$ for some (V', f)-expansion Y of Y_0 in G and f' is an (M, Y)-recoloring of f, then*

(a) *the sizes of almost all color classes in f and f' are the same, only the size of M decreases by one and the size of one color class in Y_0 increases by one;*
(b) *the colors of vertices in $V(G) - V'$ are the same in f and f'.* □

Lemma 1. *Let $\Delta \geqslant 3$ and G be a $K_{\Delta+1}$-free n-vertex graph with maximum degree Δ and*

$$|E(G)| \leqslant n\Delta/10. \tag{1}$$

If $n/\Delta \leqslant 8.8$, then G has an equitable Δ-coloring.

Proof. Let G be an inclusion minimal counterexample to the lemma, and e be an edge adjacent to a vertex v of the lowest positive degree α. If $\alpha > 0.4\Delta$, then by (1) and the choice of v, the cardinality of the set V' of non-isolated vertices in G is at most $2|E(G)|/\alpha < 2 \cdot 0.1n\Delta/0.4\Delta = n/2$. Applying Lemma 2 to $G[V']$, we obtain a Δ-coloring of $G[V']$ in which every color class has at most $\lceil n/\Delta \rceil$ vertices and each but one color class has less than $\lceil n/\Delta \rceil$ vertices. Now we can add the remaining isolated vertices to these color classes in order to get the size of every class equal to $\lceil n/\Delta \rceil$ or $\lfloor n/\Delta \rfloor$. This contradicts the choice of G. Therefore,

$$\deg(v) \leqslant 0.4\Delta. \tag{2}$$

By the minimality of G, $G-e$ has an equitable Δ-coloring f. We may assume that the color classes of f are $M_1, \ldots, M_\Delta$ and that $v \in M_\Delta$. By the definition, the size of every M_i is either t or $t-1$, where $t = \lceil n/\Delta \rceil$.

Let $M_0 = M_\Delta - v$. Then $f_0 = (M_0, \ldots, M_{\Delta-1})$ is a proper coloring of $G_0 = G - v$. Let Y_0 denote the set of color classes in f_0 of size $|M_0|$. If some $M_j \in Y_0$ contains no neighbors of v, then$|Y_0| > 1$ and hence $|M_0| = t-1$. In this case, we color v_i with M_j and get an equitable Δ-coloring of G, a contradiction. Thus, every $M_j \in Y_0$ contains a neighbor of v. Let $Y = \bigcup_{j=0}^{\infty} Y_j$ be the $(V(G_0), f_0)$-expansion of Y_0 in G_0 (defined at the beginning of the section) and $y = |Y|$.

Suppose that for some $h \geqslant 0$, a color class M_{h+1} in Y_{h+1} does not contain a neighbor of v. Let f' be an (M, Y)-recoloring of f_0. By Claim 3.2, if we additionally color v with M_{h+1}, then we obtain an equitable Δ-coloring of G, a contradiction. Thus, every color class in Y contains a neighbor of v and therefore $y \leqslant \deg_G(v)$.

Let $V^+ = V(G) - \bigcup_{M \in Y} M$.

Case 1: $|M_0| = t-1$. By the definition of Y_0, every color class outside of Y has size t. Therefore, $|V^+| = t(\Delta - y) \geqslant n(\Delta - y)/\Delta$. By (1) and Claim 3.1, we have

$$\frac{n\Delta}{10} \geqslant |E(G)| \geqslant y(\Delta - y)\,\frac{n}{\Delta}. \tag{3}$$

For $\lambda = y/\Delta$, (3) gives $\lambda^2 - \lambda + 0.1 \geqslant 0$. It follows that either $\lambda > 0.88$ or $\lambda < 0.12$. The former contradicts (2), so $y < 0.12\Delta$. Since every vertex in V^+ is adjacent to M_0, we get $\Delta(t-1) \geqslant t(\Delta - y)$, i.e., $ty \geqslant \Delta$. This yields $t > \frac{1}{0.12} > 8$, which means $n > 8\Delta$ and $t \geqslant 9$. Furthermore, since every color class outside of Y has size t and $y < 0.12\Delta$, we get

$$n \geqslant t(\Delta - y) + (t-1)y \geqslant 9\Delta - y \geqslant 8.88\Delta.$$

Case 2: $|M_0| = t-2$. Recall that no other color class has size less than $t-1$. Suppose that some $M \in Y$, say, $M \in Y_h$ has t vertices. Then any (M, Y)-recoloring f' of f_0 satisfies the conditions of Case 1 with $M_h - x_h$ in place of M_0. Since Case 1 is proved, we can assume that every $M \in Y$ has at most $t-1$ vertices. It follows that the average size of color classes outside of Y is higher than in Y, and therefore higher than n/Δ. Thus, (3) holds, and we get $y < 0.12\Delta$ exactly as in Case 1. Similarly to Case 1, we obtain $\Delta(t-2) \geqslant (t-1)(\Delta - y)$, i.e., $(t-1)y \geqslant \Delta$. It follows that $t - 1 > \frac{1}{0.12} > 8$. Since $t-1$ is an integer, we have $n/\Delta > t - 1 \geqslant 9$. □

4. The bounded-size version of Theorem 1

An *l-bounded* coloring of a graph G is a proper vertex coloring of G in which the size of each color class is at most l. Clearly, every equitable k-coloring of an n-vertex graph is $\lceil n/k \rceil$-bounded, but not every $\lceil n/k \rceil$-bounded k-coloring of an n-vertex graph is equitable. Thus, the theorem below is a weaker statement than Theorem 1.

Theorem 3. *Let $\Delta, n \geqslant 46$. Suppose that an n-vertex graph $G = (V, E)$ has maximum degree at most Δ and $|E| \leqslant \Delta n/5$. If G does not contain $K_{\Delta+1}$, then G has an $\lceil n/\Delta \rceil$-bounded coloring with Δ colors.*

Proof. Let G be a $K_{\Delta+1}$-free graph with maximum degree $\Delta \geqslant 46$ and average degree at most $\Delta/5$. Let $t = \lceil n/\Delta \rceil$. Let $V_l = \{v_1, \dots, v_l\}$ be the set of vertices of degree at least $4\Delta/5$. For $i = l+1, \dots, n$, consider the following procedure:
(1) let Δ_i be the maximum degree in $G - V_{i-1}$;
(2) if $\Delta_i \geqslant 2\Delta/5$, then let v_i be a vertex of degree Δ_i in $G - V_{i-1}$;
(3) if $\Delta_i < 2\Delta/5$, then let v_i be a vertex of maximum degree in G among vertices in $G - V_{i-1}$;
(4) let $V_i = V_{i-1} \cup \{v_i\}$ and $G_i = G[V_i]$.

Note that, in general, the resulting ordering is not unique.

Claim 4.1. *Let m be the largest index i such that $\Delta_i \geqslant 2\Delta/5$. Then $m \leqslant \lfloor n/4 \rfloor$.*

Proof. By the definition, every v_i for $l+1 \leqslant i \leqslant m$ has at least $2\Delta/5$ adjacent vertices v_j with $j > i$. Therefore, G contains at least $(m-l)2\Delta/5$ edges not incident with $v_1, \dots, v_l$. Thus, if $\Delta/4 < m$, then

$$|E(G)| \geqslant l \cdot \frac{4\Delta}{5 \cdot 2} + (m-l) \cdot \frac{2\Delta}{5} > \frac{n}{4}\frac{2\Delta}{5} = \frac{\Delta n}{10},$$

a contradiction. □

Suppose that $m = k(\Delta+1) + r$ with $0 \leqslant r \leqslant \Delta$. Then$G_m$ has a proper Δ-coloring $f_m = (M_1^m, \dots, M_\Delta^m)$ satisfying Corollary 2. Note that by Claim 4.1, $k + \lceil m/\Delta+1 \rceil = \lfloor m/\Delta+1 \rfloor + \lceil m/\Delta+1 \rceil < t$. Therefore, f_m is a t-bounded coloring of G_m.

We will now complete f_m to a t-bounded Δ-coloring of G by constructing consecutively colorings f_i of G_i for $i = 1+m, 2+m, \dots, n$, in such a way that

$$f_i(v) = f_m(v) \quad \text{for every } v \in V_m. \tag{4}$$

Observe that f_m satisfies (4). Now, suppose that $m+1 \leqslant i \leqslant n$ and G_{i-1} has a t-bounded coloring f_{i-1} satisfying (4). We will construct f_i for G_i.

Let $M_1, \dots, M_\Delta$ be the color classes of f_{i-1}. Let Y_0 denote the set of color classes of cardinality less than t. If some $M_j \in Y_0$ contains no neighbors of v_i, then we color v_i with M_j and have a t-bounded coloring f_i satisfying (4). Otherwise, let $Y = \bigcup_{j=0}^{\infty} Y_j$ be the $(V_{i-1} - V_m, f_{i-1})$-expansion of Y_0 in G_{i-1} (defined in Section 3) and $y = |Y|$.

If for some $h \geqslant 0$, a color class M_{h+1} in Y_{h+1} does not contain a neighbor of v, then consider an (M, Y)-recoloring f' of f_{i-1}. By Claim 3.2(a), if we additionally color v_i with M_{h+1}, then we obtain a t-bounded Δ-coloring of G_i. Moreover, by Claim 3.2(b), this new coloring also satisfies (4), as required. Thus, we may assume that every color class in Y contains a neighbor of v. This together with the definition of Y_0 and Claim 3.1 yields the following.

Claim 4.2. *If G_i has no t-bounded coloring f_i satisfying* (4), *then the $(V_{i-1} - V_m, f_{i-1})$-expansion Y of Y_0 in G_{i-1} possesses the following properties*:
(a) *every color class in Y contains a neighbor of v_i and thus $y \leqslant \deg_G(v_i)$*,
(b) *every vertex $u \in V_{i-1} - V_m - \bigcup_{M \in Y} M$ has a neighbor in every $M \in Y$*,
(c) *every color class outside of Y has t vertices.*

Let

$$V^- = V_m - \bigcup_{M \in Y} M$$

and

$$V^+ = V_{i-1} - V_m - \bigcup_{M \in Y} M = V_{i-1} - \bigcup_{M \in Y} M - V^-.$$

Claim 4.3. *Using the notation above, $|V^-| \leqslant 3n/8\Delta(\Delta - y)$.*

Proof. Recall that $m = k(\Delta+1)+r$ with $0 \leqslant r \leqslant \Delta$. By Corollary 2(ii),

$$|V^-| \leqslant k + \left\lceil \frac{m}{\Delta+1} \right\rceil (\Delta - y) \leqslant \frac{n}{4\Delta} + \left\lceil \frac{n}{4\Delta} \right\rceil (\Delta - y).$$

Since $\Delta \geqslant 46$ and $i > l$, we have $\Delta - y \geqslant \Delta/5 > 9$. By Lemma 1, $n/4\Delta \geqslant 2.2$ and therefore, $\lceil n/4\Delta \rceil \leqslant \frac{3}{2.2} \cdot n/4\Delta$. It follows that

$$|V^-| \leqslant (\Delta - y)\left(\frac{n}{4\Delta(\Delta - y)} + \frac{3}{2.2}\cdot\frac{n}{4\Delta}\right) \leqslant \frac{n}{4\Delta}(\Delta - y)\left(\frac{1}{10} + \frac{3}{2.2}\right) < \frac{3n}{8\Delta}(\Delta - y).$$

This proves the claim. □

Claim 4.4. *The size y of Y is less than* 0.15Δ.

Proof. By Claim 4.2(b), at least $y|V^+|$ edges connect V^+ with $\bigcup_{M\in Y} M$. Recall that every v_q for $l+1 \leqslant q \leqslant m$ has at least $2\Delta/5$ adjacent vertices v_j with $j > q$. Thus, at least $0.4\Delta|V^-|$ edges of G are incident with V^- and hence

$$|E(G)| \geqslant y|V^+| + \frac{y}{2}|V^-|.$$

Since $|V(G) - \bigcup_{M\in Y} M| \geqslant t(\Delta - y) \geqslant n/\Delta(\Delta - y)$, we have

$$|E(G)| \geqslant y\left((\Delta - y)\frac{n}{\Delta} - |V^-|\right) + \frac{y}{2}|V^-| \geqslant y\left((\Delta - y)\frac{n}{\Delta} - \frac{y}{2}|V^-|\right).$$

This and Claim 4.3 yield

$$\frac{n\Delta}{10} \geqslant y\left(\frac{n}{\Delta}(\Delta - y) - (\Delta - y)\frac{3n}{16\Delta}\right) = \frac{13}{16} yn\left(1 - \frac{y}{\Delta}\right).$$

Denoting $\lambda = y/\Delta$ and dividing both parts by $n\Delta$, we obtain $\frac{1}{10} \geqslant \frac{13}{16}\lambda(1-\lambda)$. Solving this inequality, we get $\lambda > 0.85$ or $\lambda < 0.15$. Since $i > m \geqslant l$, we have $y \leqslant \deg(v_i) < 0.8\Delta$. We conclude that $\lambda = y/\Delta < 0.15$. □

Claim 4.5. *The size of V^+ is greater than* $\frac{2n}{3}$.

Proof. Assume that $|V^+| \leqslant \frac{2n}{3}$. As in the proof of Claim 4.4, at least $y|V^+|$ edges connect V^+ with $\bigcup_{M\in Y} M$ and at least $0.4\Delta|V^-|$ edges are incident with V^-. Hence

$$|E(G)| \geqslant y|V^+| + 0.4\Delta\left((\Delta - y)\frac{n}{\Delta} - |V^+|\right) \geqslant |V^+|(y - 0.4\Delta) + 0.4\frac{n}{\Delta}\Delta(\Delta - y).$$

Recall that $y < 0.15\Delta$ by Claim 4.4 and $|V^+| \leqslant \frac{2n}{3}$. Thus the last inequality yields

$$\frac{n\Delta}{10} > \frac{2n}{3}(y - 0.4\Delta) + 0.4n(\Delta - y).$$

Dividing both parts by $n\Delta$, we obtain

$$\frac{1}{10} > \frac{2}{3}\frac{y}{\Delta} - \frac{8}{30} + \frac{2}{5} - \frac{2y}{5\Delta},$$

which is false for $y/\Delta \geqslant 0$. This contradiction proves the claim. □

Claim 4.6. *Let M_1 be a color class of the smallest size in Y. Then*

$$|M_1| \leqslant \frac{n+y}{\Delta} - 1. \tag{5}$$

Proof. Since v_i is not colored, $|M_1| < n/\Delta$. If $|M_1| \leqslant t-2$, the conclusion is obvious. Suppose that $|M_1| = t-1$. Since every color class not in Y has size t, by the minimality of $|M_1|$, $n + y \geqslant t\Delta$. This proves the claim. □

Now we are ready to finish the proof. Define $z = 4n/9\Delta + \frac{17}{30}$.

Case 1: $|M_1 \cap V_m| \leqslant z$. Then the number of neighbors of M_1 is at most $z\Delta + (|M_1| - z)(2\Delta/5) = z(3\Delta/5) + |M_1|2\Delta/5$. By Claims 4.6 and 4.4, this is less than

$$\left(\frac{4n}{9\Delta} + \frac{17}{30}\right)\frac{3\Delta}{5} + \left(\frac{n}{\Delta} - 0.85\right)\cdot\frac{2\Delta}{5} = \frac{2n}{3}. \tag{6}$$

Since every vertex in V^+ is a neighbor of M_1, (6) contradicts Claim 4.5.

Case 2: $|M_1 \cap V_\mu| = z + x$, where $x > 0$. We will prove that M_1 has at most $z(3\Delta/5) + |M_1|(2\Delta/5)$ neighbors in V^+, which by (6) would give the same contradiction as in Case 1.

By Corollary 2(iii), every vertex of $M_1 \cap V_m$ has at most $(k + r - 1)/(z + x - k - 1)$ neighbors in V^+. Thus, it is enough to prove that

$$\frac{k + r - 1}{z + x - k - 1}\,(z + x) \leqslant z\Delta + x\,\frac{2\Delta}{5}, \tag{7}$$

since the RHS of (7) is the maximum amount contributed by $z + x$ vertices to $z\Delta + (|M_1| - z)(2\Delta/5)$ in Case 1. Note that (7) is equivalent to

$$\left(z\Delta + x\,\frac{2\Delta}{5}\right)(z + x - k - 1) - (k + r - 1)(z + x) \geqslant 0. \tag{8}$$

For $x = 0$, (8) becomes

$$z\Delta(z + k - 1) - z(k + r - 1) \geqslant 0,$$

which reduces to

$$z \geqslant \frac{k + r - 1}{\Delta} + k + 1. \tag{9}$$

Recall that $m = k(\Delta + 1) + r \leqslant \lfloor n/4 \rfloor$, i.e., $n \geqslant 4(k\Delta + k + r)$, and that $z = 4n/9\Delta + \frac{17}{30}$. It follows that

$$z \geqslant \frac{16k}{9} + \frac{16}{9}\,\frac{k + r}{\Delta} + \frac{17}{30}, \tag{10}$$

which yields (9) for $k \geqslant 1$. This proves (8) for $x = 0$.

Now consider the LHS of (8) as a function $g(x)$. Then

$$g'(x) = \frac{2\Delta}{5}\,(z + x - k - 1) + \left(z\Delta + x\,\frac{2\Delta}{5}\right) - (k + r - 1) = \frac{4\Delta}{5}\,x + \frac{7\Delta}{5}\,z - \frac{2\Delta}{5}\,(k + 1) - (k + r - 1).$$

We want to show that $g'(x) \geqslant 0$ for every $x > 0$. This would prove (8) and thus the theorem. By the last equality, the condition $g'(x) \geqslant 0$ is equivalent to

$$\frac{7}{5}\,z \geqslant \frac{2}{5}\,(k + 1) - \frac{k + r - 1}{\Delta}.$$

This inequality is implied by (10) for $k \geqslant 1$. □

5. Proof of Theorem 1

The algorithm in the previous section produces a t-bounded Δ-coloring of G, but this coloring might have 'small' color classes. In order to 'correct' the coloring, we use a slight variation of the technique used above.

Consider t-bounded colorings of G obtained in the course of proof of Theorem 3. In particular, each vertex $v \in V(G) - V_m$ has at most 0.8Δ neighbors in G and at most 0.4Δ neighbors in $G - V_m$. Among such colorings with a fixed coloring f_m of $G[V_m]$ satisfying Corollary 2, choose a coloring f_0 with fewest color classes of size t. We will prove that f_0 has no color classes of size $t - 2$ or less.

Let Y_0 be the set of color classes of size at most $t-2$ and assume that Y_0 is non-empty. Let $Y = \bigcup_{j=0}^{\infty} Y_j$ be the $(V(G) - V_m, f_0)$-expansion of Y_0 in G and $y = |Y|$.

Claim 5.1. *Y possesses the following properties*:
(a) *every vertex* $u \in V(G) - V_m - \bigcup_{M \in Y} M$ *has a neighbor in every* $M \in Y$,
(b) *every color class in* Y *has at most* $t-1$ *vertices*,
(c) *every color class outside of* Y *has at least* $t-1$ *vertices*.

Proof. Claim 3.1 implies (a), and the definition of Y_0 yields (c). To prove (b), assume by contradiction that for some $h \geqslant 0$, a color class M_{h+1} in Y_{h+1} has cardinality t. Consider an (M, Y)-recoloring f' of f_0. By Claim 3.2(a), f' is a t-bounded Δ-coloring of G with fewer color classes of size t. Moreover, by Claim 3.2(b), f' satisfies (4). This contradicts the choice of f_0. □

Since there is a color class M' of size t, $y < \Delta$. Since every vertex in $M' - V_m$ has neighbors in each color class of Y,

$$y \leqslant 0.8\Delta. \tag{11}$$

Let

$$V^- = V_m - \bigcup_{M \in Y} M$$

and

$$V^+ = V - V_m - \bigcup_{M \in Y} M = V - \bigcup_{M \in Y} M - V^-.$$

Now, Claim 4.3 holds: the proof simply repeats that in the previous section. By Claim 5.1(b) and (c),

$$\left| V(G) - \bigcup_{M \in Y} M \right| \geqslant (\Delta - y)\,\frac{n}{\Delta}. \tag{12}$$

Thus we can essentially repeat the proofs of Claims 4.4 and 4.5, and conclude that they hold for our Y.

Let $M_1 \in Y_0$. By the definition of Y_0, $|M_1| \leqslant t-2 < n/\Delta - 1$, which is stronger than Claim 4.6. Therefore, all the calculations in the previous section following Claim 4.6 go through and we get a contradiction to our assumption.

References

[1] B. Baker, E. Coffman, Mutual exclusion scheduling, Theorot. Comput. Sci. 162 (1996) 225–243.
[2] J. Blazewicz, K. Ecker, E. Pesch, G. Schmidt, J. Weglarz, Scheduling Computer and Manufacturing Processes, second ed., Springer, Berlin, 2001, p. 485.
[3] B.-L. Chen, K.-W. Lih, P.-L. Wu, Equitable coloring and the maximum degree, European J. Combin. 15 (1994) 443–447.
[4] B.-L. Chen, K.-W. Lih, J.-H. Yan, A note on equitable coloring of interval graphs, Manuscript, 1998.
[5] R. Diestel, Graph Theory, Springer, New York, 1997.
[6] A. Hajnal, E. Szemerédi, Proof of conjecture of Erdős, in: P. Erdős, A. Rényi, V.T. Sós (Eds.), Combinatorial Theory and its Applications, Vol. II, North-Holland, Amsterdam, 1970, pp. 601–603.
[7] S. Irani, V. Leung, Scheduling with conflicts and applications to traffic signal control, in: Proc. of Seventh Annu. ACM-SIAM Symp. on Discrete Algorithms, Atlanta, GA, SIAM, Philadelphia, PA, 1996, pp. 85–94.
[8] S. Janson, A. Ruciński, The infamous upper tail, Random Structures and Algorithms 20 (2002) 317–342.
[9] F. Kitagawa, H. Ikeda, An existential problem of a weight-controlled subset and its application to school timetable construction, Discrete Math. 72 (1988) 195–211.
[10] A.V. Kostochka, K. Nakprasit, Equitable colorings of k-degenerate graphs, Combinatorics, Probab. Comput. 12 (2003) 53–60.
[11] A.V. Kostochka, M.J. Pelsmajer, D.B. West, A list analogue of equitable coloring, J. Graph Theory 44 (2003) 166–177.
[12] J. Krarup, D. de Werra, Chromatic optimisation: limitations, objectives, uses, references, European J. Oper. Res. 11 (1982) 1–19.
[13] K.-W. Lih, The equitable coloring of graphs, in: D.-Z. Du, P. Pardalos (Eds.), Handbook of Combinatorial Optimization, Vol. 3, Kluwer, Dordrecht, 1998, pp. 543–566.
[14] K.-W. Lih, P.-L. Wu, On equitable coloring of bipartite graphs, Discrete Math. 151 (1996) 155–160.
[15] L.S. Mel'nikov, V.G. Vizing, New proof of Brooks' theorem, J. Combin. Theory 7 (1969) 289–290.
[16] S.V. Pemmaraju, Equitable colorings extend Chernoff–Hoeffding bounds, in: Proc. of Fifth Internat. Workshop on Randomization and Approximation Techniques in Computer Science, APPROX-RANDOM 2001, 2001, pp. 285–296.

[17] B.F. Smith, P.E. Bjorstad, W.D. Gropp, Domain decomposition, Parallel Multilevel Methods for Elliptic Partial Differential Equations, Cambridge University Press, Cambridge, 1996, p. 224.
[18] A. Tucker, Perfect graphs and an application to optimizing municipal services, SIAM Rev. 15 (1973) 585–590.
[19] D.B. West, Introduction to Graph Theory, secon ed., Prentice Hall, Upper Saddle River, 2001.
[20] H.-P. Yap, Y. Zhang, The equitable Δ-colouring conjecture holds for outerplanar graphs, Bull. Inst. Math. Acad. Sinca 25 (1997) 143–149.
[21] H.-P. Yap, Y. Zhang, Equitable colourings of planar graphs, J. Combin. Math. Combin. Comput. 27 (1998) 97–105.