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Discrete Applied Mathematics 156 (2008) 1542–1548
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Minimum degree conditions for H-linked graphs[☆]

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Received 31 March 2004; received in revised form 8 November 2005; accepted 2 November 2006
Available online 23 August 2007

Abstract

For a fixed multigraph H with vertices $w_1, \ldots, w_m$, a graph G is *H-linked* if for every choice of vertices $v_1, \ldots, v_m$ in G, there exists a subdivision of H in G such that v_i is the branch vertex representing w_i (for all i). This generalizes the notions of k-linked, k-connected, and k-ordered graphs.

Given a connected multigraph H with k edges and minimum degree at least two and $n \geqslant 7.5k$, we determine the least integer d such that every n-vertex simple graph with minimum degree at least d is H-linked. This value $D(H, n)$ appears to equal the least integer d' such that every n-vertex graph with minimum degree at least d' is $b(H)$-connected, where $b(H)$ is the maximum number of edges in a bipartite subgraph of H.

Keywords: Extremal graph problems; Degree conditions; H-linked graphs

1. Introduction

Let H be a multigraph. An *H-subdivision* in a graph G is a pair of mappings $f : V(H) \to V(G)$ and $g : E(H)$ into the set of paths in G such that:

(a) $f(u) \neq f(v)$ for all distinct $u, v \in V(H)$ and
(b) for every $uv \in E(H)$, $g(uv)$ is an $f(u)f(v)$-path in G, and distinct edges map into internally disjoint paths in G.

A graph G is *H-linked* if every injective mapping $f : V(H) \to V(G)$ can be extended to an H-subdivision in G. This is a natural generalization of k-linkage.

Recall that a graph is *k-linked* if for every list of $2k$ vertices $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$, there exist internally disjoint paths $P_1, \ldots, P_k$ such that each P_i is an s_i, t_i-path. By the definition, a graph G is k-linked if and only if G is H-linked for every graph H with $|E(H)| = k$ and $\delta(H) \geqslant 1$. It is known that a graph G on at least $2k$ vertices is k-linked if and only if G is M_k-linked, where M_k is the matching with k edges.

Let B_k denote the (multi)graph with two vertices and k parallel edges. By Menger's theorem, a simple graph G on at least $k + 1$ vertices is k-connected if and only if G is B_k-linked.

[☆] This work was supported by the NSF grants DMS-0099608 and DMS-0400498.
[1] Research was also partially supported by Grants 99-01-00581 and 00-01-00916 of the Russian Foundation for Basic Research.
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doi:10.1016/j.dam.2006.11.013

A graph is *k-ordered*, if for every ordered sequence of k vertices, there is a cycle that encounters the vertices of the sequence in the given order. Let C_k denote the cycle of length k. Clearly, a simple graph G is k-ordered if and only if G is C_k-linked.

Thus, the notion of H-linked graphs is a joint generalization of the notions of k-linked, k-ordered and k-connected graphs. Minimum degree conditions for graphs to be k-ordered or k-linked were considered by several authors (see [2,4–10]). Let $D(n, k)$ be the minimum positive integer d such that every n-vertex simple graph with minimum degree at least d is k-linked (i.e., G is H-linked for every H with k edges). It was proved in [5] that

$$D(n,k)=\begin{cases} n-1, & n\leqslant 3k-1,\\ \left\lfloor \dfrac{n+5k}{3}\right\rfloor -1, & 3k\leqslant n\leqslant 4k-2,\\ \left\lceil \dfrac{n-3}{2}\right\rceil +k, & n\geqslant 4k-1.\end{cases} \tag{1}$$

In fact, Egawa et al. [1] obtained a very similar result earlier in a bit different setting. In [8], we proved that the degree condition can be weakened if H has minimum degree at least two.

Theorem 1. *Let H be a loopless graph with k edges and $\delta(H)\geqslant 2$. Every simple graph G of order $n\geqslant 5k+6$ with $\delta(G)\geqslant \lceil (n+k)/2\rceil -1$ is H-linked.*

The minimum degree condition in Theorem 1 is sharp for all bipartite graphs H. The restriction $n\geqslant 5k+6$ probably can be weakened to about $n\geqslant 3k$, but not more. The main result of the present paper refines the bound of Theorem 1 for non-bipartite connected multigraphs H, but under stronger restrictions on n.

Theorem 2. *Let H be a loopless connected graph with k edges and $\delta(H)\geqslant 2$. Let $b(H)$ denote the maximum number of edges over all bipartite subgraphs of H. Then every simple graph G of order $n\geqslant 7.5k$ with $\delta(G)\geqslant \lceil (n+b(H))/2\rceil -1$ is H-linked.*

In the next section we present examples illustrating the theorem and start the proof of the upper bound. We assume that there is no appropriate H-subdivision for some choice of branching vertices in G and consider an optimal in some sense subgraph with a vertex set X. In Section 3, we estimate $|X|$. In Section 4 we finish the proof.

2. Preliminaries

First, we observe that the restriction $\delta(G)\geqslant \lceil (n+b(H))/2\rceil -1$ in Theorem 2 cannot be weakened for any $n\geqslant 3k$ and any H. Indeed, let G be the n-vertex graph with $V(G)=V_0\cup V_1\cup V_2$ such that $G[V_1]=K_{\lceil (n-b(H)+1)/2\rceil}$, $G[V_2]=K_{\lfloor (n-b(H)+1)/2\rfloor}$, and each vertex in V_0 (with $|V_0|=b(H)-1$) is adjacent to all other vertices in G. Then $\delta(G)=\lfloor (n+b(H)-1)/2\rfloor -1$.

Suppose that $b(H)$ edges in H connect disjoint $X\subset V(H)$ and $Y=V(H)-X$. We claim that G does not contain a subdivision of H such that X is mapped into V_1 and Y is mapped into V_2. This is because $b(H)$ edges of H should be mapped into $b(H)$ internally disjoint V_1, V_2-paths passing through V_0, but $|V_0|=b(H)-1$.

Now we start the proof of the upper bound. Let $f: V(H)\to V(G)$ be an injective mapping and $W=f(V(H))$. Let $E(H)=\{e_j=u_j^0v_j^0 : 1\leqslant j\leqslant k\}$. Let $u_j=f(u_j^0)$ and $v_j=f(v_j^0)$. Since $\delta(H)\geqslant 2$, we have $|W|=|V(H)|\leqslant k$.

Say that a family $\mathscr{C}$ of the form $\{P_1,\ldots,P_k\}$ is a *partial H-linkage* if each P_j is either the set $\{u_j, v_j\}$ or a u_j, v_j-path and the following properties hold:

(1) $|X|\leqslant |W|+2(k-b(H)+\alpha)+3$, where $X=\bigcup_{j=1}^{k}V(P_j)$ and α is the number of P_j's that are paths and
(2) the internal vertices of the paths P_j's are pairwise disjoint and disjoint from W.

Consider $\mathscr{C}_0=\{\{u_1,v_1\},\ldots,\{u_k,v_k\}\}$. This family satisfies properties (1) and (2) above with $X=\bigcup_{j=1}^{k}\{u_j,v_j\}=W$ and α=0. Therefore, $\mathscr{C}_0$ is a partial H-linkage. If all the P_j's in a partial H-linkage $\mathscr{C}$ are paths, then $\mathscr{C}$ is an H-subdivision in G.

A partial H-linkage $\mathscr{C}=\{P_1,\ldots,P_k\}$ is *optimal*, if as many as possible of the P_j's are paths and subject to this the set $X=\bigcup_{j=1}^{k}V(P_j)$ is as small as possible. We will prove that each optimal H-linkage is an H-subdivision in G.

Suppose for a contradiction that $\mathscr{C}=\{P_1,\ldots,P_k\}$ is an optimal partial H-linkage but is not an H-subdivision. Let, for definiteness, $P_k=\{u_k,v_k\}$ and $u_kv_k\notin E(G)$. Let $X=\bigcup_{j=1}^{k}V(P_j)$, $x=u_k$, and $y=v_k$. Let $A=N(x)-X$, $B=N(y)-X$, and $R=V(G)-(X\cup A\cup B)$.

It is well known (see, e.g., [11]) that

$$b(H)\geqslant(k+1)/2 \tag{2}$$

for every H with $k>0$ edges. Therefore, each of A and B has size at least

$$\delta(G)-(|X|-2)\geqslant\frac{n+b(H)-2}{2}-(|W|+2(k-b(H)+(k-1))+3-2)$$
$$\geqslant\frac{7.5k+b(H)-2}{2}-5k+2b(H)+1=1.25(2b(H)-k)>1.25.$$

It follows that we may choose distinct $a_1,a_2\in A$ and $b_1,b_2\in B$.

For $v\in V(G)$, let $d_j(v)$ denote the number of neighbors of v 'inside' P_j plus $\beta_j=1/\deg_H(u_j^0)$ if $u_j\in N_G(v)$ and plus $\gamma_j=1/\deg_H(v_j^0)$ if $v_j\in N_G(v)$. For example, if $P_j=u_jw_1w_2v_j$, $\deg_H(u_j^0)=3$ and v is adjacent to u_j and w_2 in P_j, then $d_j(v)=\frac{4}{3}$. It is easy to check that

$$\sum_{j=1}^{k}d_j(v)=|N_G(v)\cap X|\quad\forall v\in V(G). \tag{3}$$

Let l_p be the number of P_j's of length p for $p\geqslant 1$, and l_0 be the number of P_j that are not paths. Then

$$|X|=|W|+\sum_{p\geqslant 1}(p-1)l_p=\sum_{j=1}^{k}(\beta_j+\gamma_j)+\sum_{p\geqslant 1}(p-1)l_p \tag{4}$$

and

$$k=\sum_{p\geqslant 0}l_p=\alpha+l_0. \tag{5}$$

3. A bound on the size of X

We will assume that every path P_j is of the form $P_j=u_j,w_{1,j},\ldots,w_{p_j-1,j},v_j$. Sometimes, for simplicity we will write p instead of p_j and w_i instead of $w_{i,j}$ if j is clear from the context. In the rest of the paper, for every $j=1,\ldots,k$, we denote $\beta_j=1/\deg_H(u_j^0)$, $\gamma_j=1/\deg_H(v_j^0)$, $M_j=d_j(x)+d_j(y)$, and $L_j=d_j(a_1)+d_j(a_2)+d_j(b_1)+d_j(b_2)$.

The following lemma (which is Lemma 5 in [8]) will be very helpful.

Lemma 3. *For a $P_j=u_j,w_1,\ldots,w_{p-1},v_j$, let $s_j=M_j+0.5L_j$, $\beta=\beta_j$, and $\gamma=\gamma_j$. Define*

$$D_1(p,\beta,\gamma)=\begin{cases}p+1+2\beta+2\gamma, & \text{for } p\leqslant 1,\\ p+3+2\beta+2\gamma, & \text{for } p\geqslant 2.\end{cases}$$

Then

(a) $s_j\leqslant D_1(p,\beta,\gamma)$ *and*
(b) $s_k\leqslant 2(\beta_k+\gamma_k)$.

Furthermore, if $xy\notin E(G)$, then $s_k=\beta_k+\gamma_k$.

Based on Lemma 3, we prove the following.

Lemma 4. *Let* $Z=\{a_1,a_2,b_1,b_2\}$ *and* $V_0=(A\cup B)-Z-N_G(Z)$. *Then* $|X|\leqslant|W|+2(\alpha+k-b(H))-|R|-|V_0|$.

Proof. Let

$$\Sigma'=\deg_G(x)+\deg_G(y)+\tfrac{1}{2}(\deg_G(a_1)+\deg_G(a_2)+\deg_G(b_1)+\deg_G(b_2)). \tag{6}$$

Every vertex $w\in A\cup B$ contributes to Σ' at most 2: if $w\in A$ (respectively, $w\in B$), then it is not adjacent to y, b_1, and b_2 (respectively, to x, a_1, and a_2). By the definition, every vertex in V_0 is not adjacent to any vertex in Z and to at least one of x and y. Therefore, every vertex in V_0 contributes to Σ' at most 1. Furthermore, every $z\in Z$ contributes to Σ' at most 1.5, since it is not adjacent to itself. Thus, in total $A\cup B$ contributes to Σ' at most $2|A\cup B|-|V_0|-0.5|Z|$. Every $r\in R$ contributes to Σ' at most 2. By the definition, for every j, the vertices of P_j contribute to Σ' exactly s_j. Therefore,

$$\Sigma'\leqslant 2|A\cup B|-2+2|R|+\sum_{j=1}^{k}s_j-|V_0|. \tag{7}$$

By Lemma 3,

$$\begin{aligned}\sum_{j=1}^{k}s_j&\leqslant l_0+2l_1+\sum_{p\geqslant 2}(p+3)l_p+2\sum_{j=1}^{k}(\beta_j+\gamma_j)-1\\&=l_0+2l_1+\sum_{p\geqslant 2}(p+3)l_p+2|W|-1.\end{aligned} \tag{8}$$

Therefore,

$$\Sigma'\leqslant 2(|A|+|B|+|W|)+2|R|-|V_0|-3+l_0+2l_1+\sum_{p\geqslant 2}(p+3)l_p. \tag{9}$$

Combining with (4) and (5), we get

$$|X|+\Sigma'\leqslant 2n+|W|+2k+2\alpha-3-l_0-2l_1-|V_0|.$$

By (2), $\delta(G)\geqslant((n+b(H))/2)-1$ and hence $\Sigma'\geqslant 2n+2b(H)-4$. Thus,

$$|X|\leqslant|W|+2(k-b(H)+\alpha)-l_0-2l_1-|V_0|+1\leqslant|W|+2(k-b(H)+\alpha)-|V_0|. \tag{10}$$

If an $r\in R$ has a neighbor $a_0\in A$ and a neighbor $b_0\in B$, then one can add to $\mathscr{C}$ the path $P_k=x,a_0,r,b_0,y$. The new set of paths will be a better partial linkage, since the new X would have size at most $|W|+2(k-b(H)+\alpha)+3=|W|+2(k-b(H)+\alpha+1)+1$. Since this contradicts the choice of $\mathscr{C}$, no $r\in R$ has both a neighbor in A and a neighbor in B. Thus, every $r\in R$ contributes to Σ' at most 1, and (7) becomes

$$\Sigma'\leqslant 4\cdot 1.5+2(|A\cup B|-4)+|R|+\sum_{j=1}^{k}s_j-|V_0|.$$

Correspondingly, (10) transforms into

$$|X|\leqslant|W|+2(k-b(H)+\alpha)-|V_0|-|R|. \qquad\square \tag{11}$$

This lemma has the following two immediate consequences.

Lemma 5. $|A|+|B|>3k$.

Proof. By Lemma 4, $|A|+|B|=n-(|X|+|R|)\geqslant n-(|W|+2(k-b(H)+\alpha))\geqslant 7.5k-(k+2(k-((k+1)/2)+k-1))>3k$. $\square$

Lemma 6. *Each $v \in V(G)$ is adjacent to at least three vertices in $A \cup B - V_0$. In particular, either v has two neighbors in A that belong or are adjacent to the set $\{a_1, a_2\}$, or two neighbors in B that belong or are adjacent to the set $\{b_1, b_2\}$.*

Proof. By Lemma 4, $\delta(G)-(|X|+|R|+|V_0|)\geqslant 0.5(7.5k+b(H)-2)-|W|-2(k-b(H)+\alpha)\geqslant 3.75k+0.5b(H)-1-k-2(k-b(H)+k-1)=1.25(2b(H)-k)+1>2$. Thus each vertex has at least three neighbors in $V(G)-X-R-V_0$. □

For given $a_1, a_2 \in A$, $b_1, b_2 \in B$, let $A''=A''(a_1,a_2)$ (respectively, $B''=B''(b_1,b_2)$) denote the set of vertices in X having at least two neighbors in A (respectively, in B) that belong or are adjacent to the set $\{a_1,a_2\}$ (respectively, $\{b_1,b_2\}$). The above lemma yields that for every choice of a_1, a_2, b_1, and b_2,

$$A'' \cup B'' = X. \tag{12}$$

4. Proof of Theorem 2

Lemma 7. *For every non-adjacent $s, t \in A$ (or B), $|N(s)\cap N(t)-X|\geqslant 3$.*

Proof. Suppose to the contrary that $a_1, a_2 \in A$, $a_1a_2 \notin E(G)$ and the cardinality of the set T of common neighbors of a_1 and a_2 outside of X is at most two. Consider arbitrary $b_1, b_2 \in B$ and let $Z=\{a_1,a_2,b_1,b_2\}$. Then the contribution of every $a \in A-Z-T$ to the sum Σ' defined in (6) is at most 1.5. Thus, repeating the proof of Lemma 4, instead of (11), we will get $|X|\leqslant|W|-|R|+2(k-b(H)+\alpha)-|V_0|-0.5(|A-V_0|-4)$. In other words,

$$|X|+0.5|A|+|R|\leqslant|W|+2(k-b(H)+\alpha)+2\leqslant 5k-2b(H). \tag{13}$$

On the other hand, $\deg_{G-X}(a_1)+\deg_{G-X}(a_2)\leqslant|A|+|T|+|R|-2$ (the -2 arises because neither of a_1 and a_2 is adjacent to a_1 or a_2). It follows that

$$2\frac{n+b(H)}{2}-2\leqslant 2\delta(G)\leqslant 2|X|+|A|+|R|,$$

which together with (13) yields $n+b(H)-2\leqslant 2(5k-2b(H))$. Thus, $n\leqslant 10k-5b(H)+2\leqslant 10k-5((k+1)/2)+2=7.5k-0.5$, a contradiction. □

For the rest of the section, we fix some distinct $a_1, a_2 \in A$ and $b_1, b_2 \in B$, and let $A''=A''(a_1,a_2)$ and $B''=B''(b_1,b_2)$. The next fact from [6] was used in [8].

Lemma 8. *Let X be optimal, $1\leqslant j\leqslant k-1$, and either $\{u_j,v_j\}\subset A''$ or $\{u_j,v_j\}\subset B''$. Then for each $a\in A$ and $b\in B$,*

$$(N(a)\cap N(b)\cap P_j)\backslash\{u_j,v_j\}=\emptyset.$$

Proof. Assume to the contrary that $r\in N(a)\cap N(b)\cap P_j\backslash\{u_j,v_j\}$. Let $P'_k=(x,a,r,b,y)$. Without loss of generality, assume that $\{u_j,v_j\}\subset A''$. Then there exist $s\in N(u_j)\cap A\backslash\{a\}$ and $t\in N(v_j)\cap A\backslash\{a\}$. If $s=t$ or s is adjacent to t, then let $P'_j=(u_j,s,t,v_j)$.

If s and t are non-adjacent, then by Lemma 7, we have $|(N(s)\cap N(t))\backslash X|\geqslant 3$, and therefore there exists $q\in N(s)\cap N(t)\backslash(X\cup\{a,b\})$. In this case, let $P'_j=(u_j,s,q,t,v_j)$. In both cases, P'_j is a path disjoint from P'_k. Thus, in both cases we increase the number of P_j's that are paths by one and, by (11), maintain $|X|\leqslant|W|+2(k-b(H)+\alpha+1)+2$. This is a contradiction. □

Lemma 9. *Let X be optimal, $1\leqslant j\leqslant k-1$, $P_j=(w_0,w_1,\ldots,w_p)$, where $w_0=u_j\in A''$ and $w_p=v_j\in B''$. If some w_i, $1\leqslant i\leqslant p-1$ has a neighbor $a_0\in A\cup\{x\}$ and a neighbor $b_0\in B\cup\{y\}$, then each $w_{i'}$ for $i<i'\leqslant p$ has no neighbors in $A-a_0$ and each $w_{i''}$ for $0\leqslant i''<i$ has no neighbors in $B-b_0$.*

Proof. Suppose some $w_{i'}$ for $i<i'\leqslant p$ has a neighbor $a'\in A-a_0$. By the definition of A'', u_j has a neighbor $a''\in A-a_0$. By Lemma 7, the length of a shortest path P' from a'' to a' in $G[A-a_0]$ is at most two. Thus, we

can replace P_j by the path $(u_j, a'', P', a', w_{i'}, P'_j, v_j)$(where P'_j is the part of P_j connecting $w_{i'}$ with v_j) and add the path $P_k = (x, a_0, w_i, b_0, y)$. The new set of $\alpha + 1$ paths has at most $|X| + 5$ vertices, which by (11) is at most $|W| + 2(k - b(H) + \alpha + 1) + 3$, a contradiction to the choice of $\mathscr{C}$. □

Similarly to $d_j(v)$, let $d_j(u, v)$ denote the number of common neighbors of u and v 'inside' P_j plus $\beta_j \cdot |N(u) \cap N(v) \cap \{u_j\}|$ plus $\gamma_j \cdot |N(u) \cap N(v) \cap \{v_j\}|$.

Lemma 10. *Let $\mathscr{C}$ be optimal, $a \in A, b \in B$. Then there exists some $j = j(a, b)$ such that $d_j(a, b) > 1$.*

Proof. Since $N(a) \cap N(b) \cap (V(G) - X + x + y) = \emptyset$, we have

$$\sum_{j=1}^{k-1} d_j(a, b) = |N(a) \cap N(b)| \geqslant 2\delta(G) - (n - 2) \geqslant b(H). \tag{14}$$

Suppose that $d_j(a, b) \leqslant 1$ for each $1 \leqslant j \leqslant k - 1$. We will find an edge cut in H with more than $\sum_{j=1}^{k-1} d_j(a, b)$ edges, a contradiction to (14). Let E' be the set of edges e_j in H such that an internal vertex of P_j is in $N(a) \cap N(b)$. Let V' be the set of vertices u^0 in H such that the vertex $f(u^0)$ (i.e., the branching vertex in G corresponding to u^0) is in $N(a) \cap N(b)$. By our assumption, no vertex in V' is incident to an edge in E', and for each $e_j \in E'$, the path P_j contains exactly one vertex of $N(a) \cap N(b)$. Thus, it is enough to find in H an edge cut of size greater than $|E'| + |V'|$.

By Lemma 8, for each $e_j \in E'$, either $u_j \in A'' - B''$ and $v_j \in B'' - A''$ or $v_j \in A'' - B''$ and $u_j \in B'' - A''$. Recall that $x = f(u_k^0)$, $y = f(v_k^0)$,$x \in A'' - B''$ and $y \in B'' - A''$. It follows that the set $E' \cup \{e_k\}$ is contained in an edge-cut in H. Let V_1 and V_2 be the disjoint subsets of $V(H)$ such that:

(a) each edge in $E' \cup \{e_k\}$ is incident to a vertex in V_1 and a vertex in V_2 and
(b) each vertex in $V_1 \cup V_2$ is incident to an edge in $E' \cup \{e_k\}$.

By the above, $V' \cap (V_1 \cup V_2) = \emptyset$ and hence $|V(H) - (V_1 \cup V_2)| \geqslant |V'|$. Since H is connected, there is a vertex u^0 adjacent to $V_1 \cup V_2$. If u^0 is adjacent to V_1, then we add u^0 to V_2, otherwise add it to V_1. In any case the number of edges between the new V_1 and V_2 is greater than between the old ones. We continue adding vertices to $V_1 \cup V_2$ so that with each added vertex, the number of edges between V_1 and V_2 grows by at least one. When we add the last vertex of H, we get a partition (V_1, V_2) of $V(H)$ such that the number of edges between V_1 and V_2 is at least

$$|E' \cup \{e_k\}| + |V(H) - (V_1 \cup V_2)| \geqslant |E'| + 1 + |V'|,$$

a contradiction to (14). □

Lemma 11. *Let X be optimal, $1 \leqslant j \leqslant k - 1$. Then there is at most one $a \in A$, such that there is more than one $b \in B$ with $j = j(a, b)$.*

Proof. Let $P_j = (w_0, w_1, \ldots, w_p)$, where $w_0 = u_j$ and $w_p = v_j$. Assume to the contrary that there are $a_1, a_2 \in A$ and $b_1, b_2, b_3, b_4 \in B$ such that $j(a_1, b_1) = j(a_1, b_2) = j(a_2, b_3) = j(a_2, b_4) = j$, where $a_1 \neq a_2, b_1 \neq b_2, b_3 \neq b_4$. By Lemma 8, we may assume that $u_j \in A'' \backslash B''$ and $v_j \in B'' \backslash A''$.

Since $\beta_j + \gamma_j \leqslant 1$, there exists i, $1 \leqslant i \leqslant p - 1$, such that $w_i \in N(a_1) \cap N(b_1)$. Since $b_3 \neq b_4$, we may assume that $b_3 \neq b_1$. By Lemma 9, no vertex in $V(P_j) - w_i$ can belong to $N(a_2) \cap N(b_3)$. This contradicts the fact that $d_j(a_2, b_3) > 1$. □

By Lemma 5, $|A| + |B| > 2k$. We may assume that $|A| \leqslant |B|$. Thus, $|B| \geqslant k$. If $|A| \geqslant k$, then since $|B| \geqslant k$, by Lemma 10, for each $a \in A$ there is some $j(a)$ and $b_1(a)$ and $b_2(a)$ such that $j(a) = j(a, b_1(a)) = j(a, b_2(a))$. Furthermore, since $|A| \geqslant k$, for some $a_1, a_2 \in A$, the indices $j(a_1)$ and $j(a_2)$ are the same. This contradicts Lemma 11.

Thus, we may assume that $|A| < k$. Since $|B| \geqslant k$, for each $a \in A$ there is some $j(a)$ and $b_1(a)$ and $b_2(a)$ such that $j(a) = j(a, b_1(a)) = j(a, b_2(a))$. Let $J = \{j(a) \mid a \in A\}$. By Lemma 11, the indices $j(a)$ are distinct for distinct $a \in A$ and hence $|J| = |A|$.

Lemma 12. *Suppose that $j \in J$. Then x is not adjacent to some interior vertex of P_j.*

Proof. Let $P_j = (w_0, w_1, \dots, w_p)$, where $w_0 = u_j$ and $w_p = v_j$. By the definition of J, there exists $a \in A$ and $b_1, b_2 \in B$ such that $d_j(a, b_1)$ and $d_j(a, b_2) > 1$. Since $\beta_j + \gamma_j \leqslant 1$, this implies that $p \geqslant 2$. Assume that $u_j \in A'' - B''$ and $v_j \in B'' - A''$.

Since $u_j \notin B''$, we may assume that $u_j b_1 \notin E(G)$. Let $w_{i'}, w_{i''} \in N(a) \cap N(b_1)$and $i' < i''$. By the choice, $1 \leqslant i' \leqslant p-1$. If $xw_{i'} \in E(G)$, then we get a contradiction to Lemma 9 with $a_0 = x$, since $w_{i''}a \in E(G)$. Thus, $xw_{i''} \notin E(G)$. □

Proof (*End of the proof*). By Lemma 12, x is not adjacent to at least $|J|$ vertices in $X - W$. It also is not adjacent to itself. Thus, $|N(x) \cap X| \leqslant |X| - |J| - 1 \leqslant |W| + 2(k - b(H) + k - 1) - |J| - 1 \leqslant 5k - 2b(H) - 3 - |J|$. Since $|J| = |A| = |N(x) - X|$, we get

$$\frac{n + b(H)}{2} - 1 \leqslant \deg(x) \leqslant 5k - 2b(H) - 3,$$

which yields $n \leqslant 10k - 5b(H) - 1 < 7.5k - 2$, a contradiction. □

Acknowledgment

We thank the referees for their helpful comments.

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