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Communication

Smaller planar triangle-free graphs that are not 3-list-colorable

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Abstract

In 1995, Voigt constructed a planar triangle-free graph that is not 3-list-colorable. It has 166 vertices. Gutner then constructed such a graph with 164 vertices. We present two more graphs with these properties. The first graph has 97 vertices and a failing list assignment using triples from a set of six colors, while the second has 109 vertices and a failing list assignment using triples from a set of five colors.

Keywords: Planar graphs; Triangle-free graphs; List coloring

1. Introduction

A *list assignment* L for a graph G is an assignment of a set $L(v)$ of “admissible colors” to each vertex v of G. An *L-coloring* of G is a proper vertex coloring f of G in which $f(v) \in L(v)$ for every vertex v. A graph is called *k-list-colorable* if G has an L-coloring for each list assignment L such that $|L(v)| = k$ for every $v \in V(G)$.

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The well-known theorem of Grötzsch [2] states that every planar triangle-free graph is 3-colorable. This theorem was later slightly sharpened by Grünbaum [3] and Aksionov [1], who showed that every planar graph with at most 3 triangles is still 3-colorable. Somewhat unexpectedly, the situation with list coloring is different. After an example [5] of a planar graph that is not 4-list-colorable, Voigt [6] also gave an example of a planar triangle-free graph that is not 3-list-colorable. This example has 166 vertices. Later, Gutner [4] found a graph with the same properties having 164 vertices. In this note, we construct yet smaller examples of planar triangle-free graphs that are not 3-list-colorable. One of the examples has 97 vertices and a failing list assignment using triples from a set of six colors and the other has 109 vertices and a failing list assignment using triples from a set of five colors. We do not know similar examples (with any number of vertices) with failing list assignments using triples from a set of four colors. The statement that such examples do not exist would be a generalization of Grötzsch Theorem.

2. The examples

We build the first example in four steps.

Let $F(a, b, \alpha, \beta, \gamma)$ denote the graph in Fig. 1, where each vertex is assigned the list as in the figure and $\alpha, \beta, \gamma, \delta, 1, 2$ are all distinct colors. Note that $F(a, b, \alpha, \beta, \gamma)$ has 16 vertices.

Lemma 1. *With list assignment L as in Fig.* 1, *the color-restricted graph* $F(a, b, \alpha, \beta, \gamma)$ *has no L-coloring.*

Proof. Let $F = F(a, b, \alpha, \beta, \gamma)$. Suppose that F has an L-coloring f. We have $f(a) = \alpha$, $f(b) = \beta$, and hence $f(u) = f(v) = \gamma$. Furthermore, $\{f(z_1), f(z_2)\} = \{\gamma, \delta\}$.

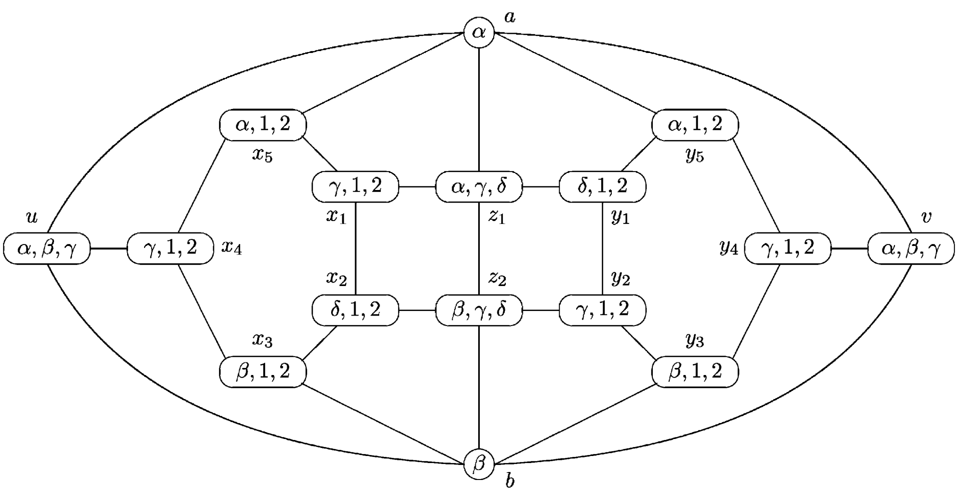

Fig. 1. Graph $F(a, b, \alpha, \beta, \gamma)$.

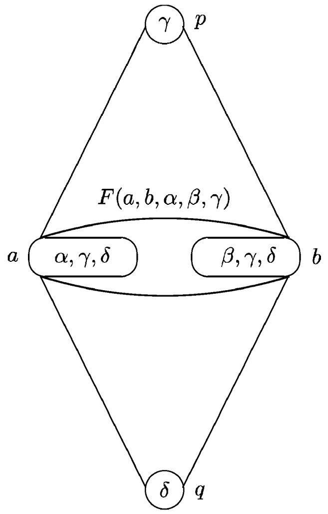

Fig. 2. Graph $\widetilde{F}(p, q, \alpha, \beta, \gamma, \delta)$.

Case 1: $f(z_1) = \gamma$ and $f(z_2) = \delta$. Each of the vertices $x_1, \ldots, x_5$ can be colored only with 1 or 2; but these vertices form a 5-cycle. This is a contradiction.

Case 2: $f(z_1) = \delta$ and $f(z_2) = \gamma$. Each of the vertices $y_1, \ldots, y_5$ can be colored only with 1 or 2; but these vertices form a 5-cycle. This proves the lemma. □

Let $\widetilde{F}(p, q, \alpha, \beta, \gamma, \delta)$ be the graph in Fig. 2, obtained from $F(a, b, \alpha, \beta, \gamma)$ by adding vertices p and q adjacent to a and b and assigning $L(a) = \{\alpha, \gamma, \delta\}$ and $L(b) = \{\beta, \gamma, \delta\}$. Note that $\widetilde{F}(p, q, \alpha, \beta, \gamma, \delta)$ has 18 vertices. The following is an easy consequence of Lemma 1.

Lemma 2. *With list assignment L as in Fig.* 2, *the color-restricted graph* $\widetilde{F}(p, q, \alpha, \beta, \gamma, \delta)$ *has no L-coloring.*

Proof. If we color one of p and q with γ and the other with δ, then we can color a only with α and b only with β. Applying Lemma 1 finishes the proof. □

Let $H(p, q, \beta, \gamma, \delta)$ be the graph in Fig. 3, obtained from three copies of $\widetilde{F}(p, q, \alpha, \beta, \gamma, \delta)$, named as $\widetilde{F}(p_1, q_1, \alpha, \beta, \gamma, \delta)$, $\widetilde{F}(p_2, q_2, \alpha, \delta, \beta, \gamma)$, and $\widetilde{F}(p_3, q_3, \alpha, \gamma, \beta, \delta)$ by merging p_1, p_2 and p_3 into the new vertex p with $L(p) = \{\beta, \gamma, \delta\}$, merging q_1, q_2 and q_3 into the new vertex q with $L(q) = \{\beta, \gamma, \delta\}$ and identifying the copies b_1 and b_2 of b in $\widetilde{F}(p_1, q_1, \alpha, \beta, \gamma, \delta)$ and $\widetilde{F}(p_2, q_2, \alpha, \delta, \beta, \gamma)$, respectively. Note that $H(p, q, \beta, \gamma, \delta)$has 49 vertices.

Lemma 3. *In each L-coloring of* $H(p, q, \beta, \gamma, \delta)$ *the colors of p and q are the same.*

Proof. If we color p and q with different colors $x, y \in \{\beta, \gamma, \delta\}$, then, in view of Lemma 2, we cannot color properly the subgraph $\widetilde{F}(p, q, \alpha, z, x, y)$ with $\{z, x, y\} = \{\beta, \gamma, \delta\}$. □

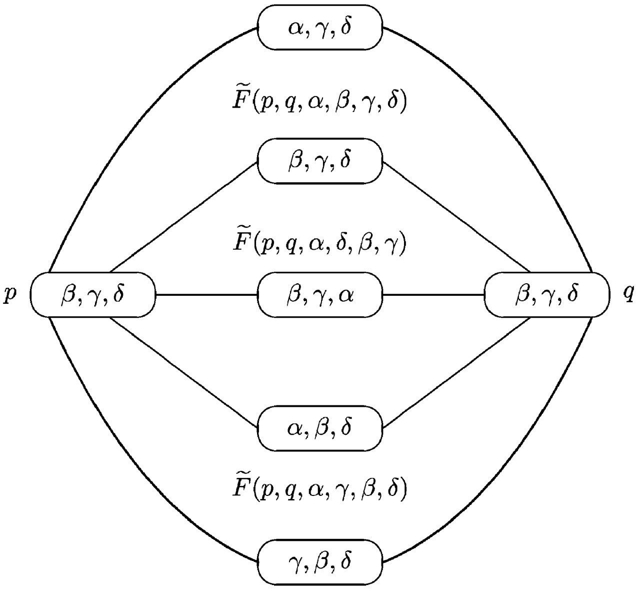

Fig. 3. Graph $H(p, q, \beta, \gamma, \delta)$.

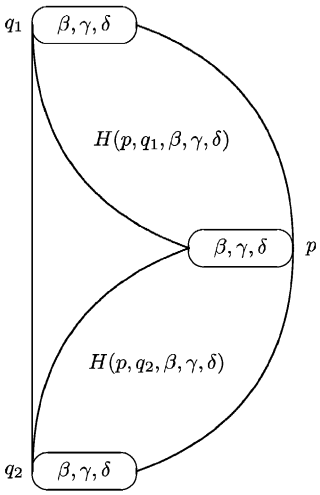

Fig. 4. Graph $\widetilde{H}$.

Our final construction, the graph $\widetilde{H}$ in Fig. 4, consists of two copies, $H_1(p, q_1, \beta, \gamma, \delta)$ and $H_2(p, q_2, \beta, \gamma, \delta)$, of $H(p, q, \beta, \gamma, \delta)$, plus one edge joining q_1 and q_2. Note that $\widetilde{H}$ has 97 vertices.

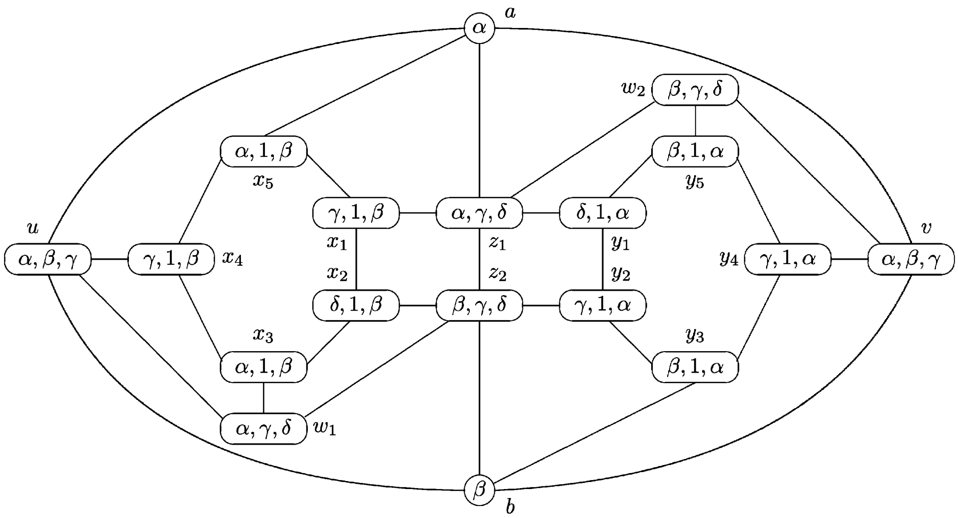

Fig. 5. Graph $F'(a, b, \alpha, \beta, \gamma)$.

If $\widetilde{H}$ has a proper coloring f from the lists, then, by Lemma 3, $f(q_1) = f(p)$ and $f(q_2) = f(p)$. But q_1 and q_2 are adjacent. This contradiction proves that the triangle-free planar graph $\widetilde{H}$ is not 3-choosable.

Note that the graph $F(a, b, \alpha, \beta, \gamma)$ and hence $\widetilde{H}$ has 6 different colors α, β, γ, δ, 1, 2 in its color lists. Let $F'(a, b, \alpha, \beta, \gamma)$ denote the graph in Fig. 5. Note that $F(a, b, \alpha, \beta, \gamma)$ has 16 vertices and 5 different colors in its color lists.

Lemma 4. *With list assignment L as in Fig.* 5, *the color-restricted graph* $F'(a, b, \alpha, \beta, \gamma)$ *has no L-coloring.*

Proof. The proof is similar to the proof of Lemma 1. □

Denote by $\widetilde{H}'$ the triangle-free planar graph obtained by replacing each subgraph $F(a, b, \alpha, \beta, \gamma)$ of $\widetilde{H}$ by a copy of the graph $F'(a, b, \alpha, \beta, \gamma)$. The graph $\widetilde{H}'$ has 109 vertices and is not list 3-colorable.

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