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Communication

# On domination in connected cubic graphs

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**Abstract**

In 1996, Reed proved that the domination number $\gamma(G)$ of every $n$-vertex graph $G$ with minimum degree at least 3 is at most $3n/8$. Also, he conjectured that $\gamma(H) \leqslant \lceil n/3 \rceil$ for every connected 3-regular (cubic) $n$-vertex graph $H$. In this note, we disprove this conjecture. We construct a connected cubic graph $G$ on 60 vertices with $\gamma(G) = 21$ and present a sequence $\{G_k\}_{k=1}^{\infty}$ of connected cubic graphs with

$$\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geqslant \frac{8}{23} = \frac{1}{3} + \frac{1}{69}.$$



*Keywords:* Domination; Cubic graphs; Dominating set

## 1. Introduction

A set $D$ of vertices is *dominating* in a graph $G$ if every vertex of $G$ is at distance at most 1 from $D$. A set $A$ of vertices in a graph $G$ *dominates* itself and the vertices at distance one from it. The *domination number* of a graph $G$ is the minimum size of a dominating set in $G$.

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Let $\gamma(G)$ and $\delta(G)$ denote the domination number and the minimum degree of a graph $G$, respectively. It is natural that graphs $G$ with high minimum degree have small domination number. Ore [4] proved that $\gamma(G) \leqslant n/2$ for every $n$-vertex graph without isolated vertices (i.e., with $\delta(G) \geqslant 1$). Blank [1] proved that $\gamma(G) \leqslant 2n/5$ for every $n$-vertex graph with $\delta(G) \geqslant 2$. Reed [6] proved that $\gamma(G) \leqslant 3n/8$ for every $n$-vertex graph with $\delta(G) \geqslant 3$. All these bounds are sharp. However, Reed [6] conjectured that the domination number of each connected 3-regular (cubic) $n$-vertex graph is at most $\lceil n/3 \rceil$. Kawarabayashi et al. [3] proved that this conjecture is at least close to the truth for cubic graphs with large girth by showing that every cubic $n$-vertex graph $G$ with a 2-factor of girth at least $3k$ satisfies $\gamma(G) \leqslant (3k+2)/(9k+3)n$.

Reed's conjecture is obviously true for Hamiltonian graphs. Plummer [5] suggested that for such graphs on $n$ vertices with $n > 8$, the slightly stronger bound $\gamma(H) \leqslant \lfloor n/3 \rfloor$ holds. In [2], this was confirmed for $n \equiv 1 \pmod 3$ and disproved for $n \equiv 2 \pmod 3$. In this note, we disprove Reed's conjecture itself. We construct a connected cubic graph $G$ on 60 vertices with $\gamma(G) = 21$ and present a sequence $\{G_k\}_{k=1}^{\infty}$ of connected cubic graphs with

$$\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geqslant \frac{1}{3} + \frac{1}{69}.$$

## 2. Examples

The following easy observation will be helpful.

**Claim 1.** *If $D$ is a dominating set in a graph $G$ with maximum degree at most 3 and $C$ is a 4-cycle in $G$, then at least two vertices of $D$ are within distance one from $C$.*

Our basic building block is the graph $H_1$ in Fig. 1.

**Claim 2.** $\gamma(H_1) = \gamma(H_1 - v_6) = \gamma(H_1 - v_7) = 3$.

**Proof.** Let $D$ be a dominating set in $H_1$ or $H_1 - v_6$ or $H_1 - v_7$. If $D$ contains $v_6$ (or $v_7$, or $v_8$), then applying Claim 1 to the 4-cycle $(v_2, v_5, v_3, v_1)$ (or $(v_2, v_4, v_3, v_1)$, or $(v_2, v_4, v_3, v_5)$) implies that $|D| \geqslant 3$. If none of $v_6$, $v_7$, and $v_8$ is in $D$, then, to dominate $v_8$, we need $v_1 \in D$. There is no vertex that dominates both $v_4$ and $v_5$ and at the same time at least one of $v_6$ and $v_7$. Thus, $|D| \geqslant 3$. □

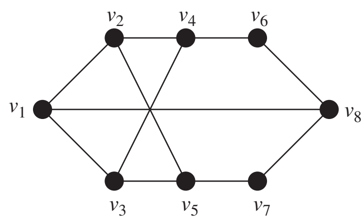


Fig. 1. Graph $H_1$.

This claim has the following immediate consequence.

**Corollary 1.** *Let G be a graph with maximum degree at most* 3 *containing* $H_1$. *For any dominating set D of G, either* $|D \cap V(H_1)| \geqslant 3$ *or both* $v_6$ *and* $v_7$ *are dominated by vertices outside* $H_1$.

The bigger block $H_2$ in Fig. 2 is constructed using two copies of $H_1$ and two additional vertices.

**Claim 3.** $\gamma(H_2) = \gamma(H_2 - v_{10}) = \gamma(H_2 - v_9 - v_{10}) = 6$. *In particular, every dominating set in any graph with maximum degree at most* 3 *containing* $H_2$ *has at least* 6 *vertices in* $V(H_2) - v_{10}$.

**Proof.** The fact that $\gamma(H_2 - v_9 - v_{10}) = 6$ immediately follows from Corollary 1. Suppose that $v_9$ belongs to a dominating set $D$. If neither of $v_6$ and $v'_6$ is in $D$, then by Corollary 1, $|D| \geqslant 7$. If, say, $v_6 \in D$, then by Claim 1 applied to the 4-cycles $(v_1, v_2, v_5, v_3)$, $(v'_1, v'_2, v'_5, v'_3)$, we have $|D| \geqslant 6$. This proves the claim. □

Our yet bigger block $H_3$ on 36 vertices is obtained from two copies $H_2$ and $H'_2$ of $H_2$ by identifying $v_{10}$ with $v'_{10}$ into a new vertex $v^*_{10}$ and adding a new vertex $v_0$ adjacent only to $v^*_{10}$. The following property immediately follows from Claim 3.

**Claim 4.** *Every dominating set in any graph with maximum degree at most* 3 *containing* $H_3$ *has at least* 12 *vertices in* $V(H_3) - v^*_{10} - v_0$.

Construct $H_4$ from copies of $H_1$, $H_2$, and $H_3$ by identifying the vertex $v_6$ of $H_1$ with the vertex $v_{10}$ of $H_2$ and the vertex $v_7$ of $H_1$ with the vertex $v_0$ of $H_3$ (see Fig. 3).

**Theorem 1.** *Graph* $H_4$ *has* 60 *vertices and* $\gamma(H_4) \geqslant 21$.

**Proof.** Clearly, $|V(H_4)| = |V(H_1)| + |V(H_2)| + |V(H_3)| - 2 = 60$. Let $D$ be a smallest dominating set in $H_4$. By Claim 3, $D$ has at least 6 vertices in $V(H_2) - v_{10}$. By Claim 4,

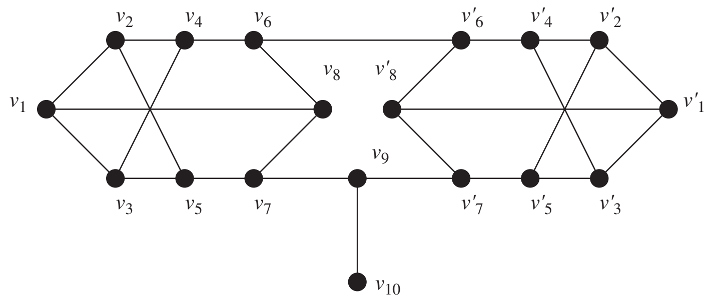


Fig. 2. The graph $H_2$.

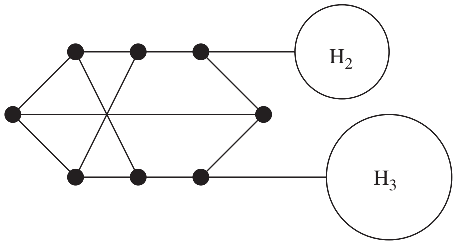


Fig. 3. Graph $H_4$.

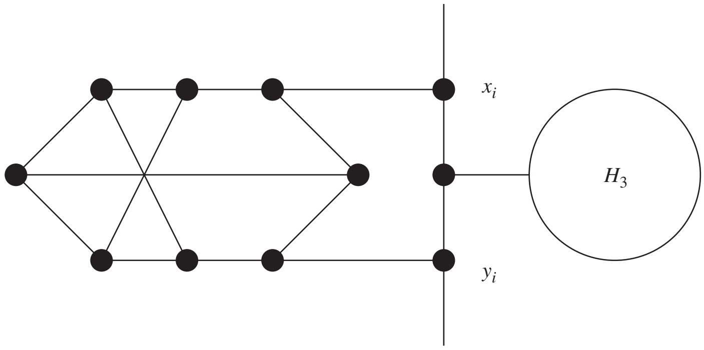


Fig. 4. Graph $F_i$.

$D$ has at least 12 vertices in $V(H_3) - v_{10}^* - v_0$. By Corollary 1, $D$ has at least 3 vertices in $V(H_1 + v_{10}^*)$. This proves the theorem. □

**Theorem 2.** *There is a sequence* $\{G_k\}_{k=1}^{\infty}$ *of cubic connected graphs such that for every* $k$, $|V(G_k)| = 46k$ *and* $\gamma(G_k) \geqslant 16k$, *and thus*

$$\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geqslant \frac{8}{23} = \frac{1}{3} + \frac{1}{69}.$$

**Proof.** Our big block $F_i$ for constructing $G_k$ consists of the disjoint union of $H_1$ and $H_3$ plus two special vertices $x_i$ and $y_i$, where $x_i$ is adjacent to both $v_6$ in $H_1$ and $v_0$ in $H_3$, and $y_i$ is adjacent to both $v_7$ in $H_1$ and $v_0$ in $H_3$. This block has 46 vertices and exactly two of them, $x_i$ and $y_i$, are of degree two (see Fig. 4). The main property of $F_i$ that we will prove and use is:

(P1) *For every cubic graph* $G$ *containing* $F_i$ *and any dominating set* $D$ *in* $G$, *the set* $D$ *has at least* 16 *vertices in* $V(F_i)$.

Indeed, by Claim 4, $D$ has at least 12 vertices in $V(H_3) - v_{10}^* - v_0$. If $D$ has only two vertices in $V(H_1)$, then by Claim 2, $D$ must contain both $x_i$ and $y_i$, thus altogether at least $12+2+2=16$ vertices. If $D$ has three vertices in $V(H_1)$, then, to dominate $v_0$, $D$ contains also at least one of $x_i$, $y_i$, $v_0$, $v_{10}^*$. This proves (P1).

Now, the graph $G_k$ consists of vertex disjoint graphs $F_1, \ldots, F_k$, with added edges $x_1y_2$, $x_2y_3, \ldots, x_{k-1}y_k, x_ky_1$. In particular, for $k=1$, we add the edge $x_1y_1$. Clearly, $|V(G_k)|=46k$ and, by (P1), $\gamma(G_k) \geqslant 16k$. □

## 3. Comments

1. We have no guess what the supremum of $(\gamma(G))/(|V(G)|)$ over connected cubic graphs is.
2. All our examples have cut-edges, so Reed's conjecture may still be true for 2-connected cubic graphs.
3. For $k \geqslant 2$, the graph $G_k$ in Theorem 2 has girth 4. It is interesting whether there exist counterexamples to Reed's conjecture with larger girth.
4. Each graph $G_k$ in Theorem 2 has no 2-factor, since the vertex $v_{10}^*$ in $H_3$ is incident to three cut-edges. However, there are also counterexamples to Reed's conjecture that have 2-factors. A building block $F_i'$ for such an example consists of a path $(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4})$ with the chord $x_{i,2}x_{i,4}$ and two copies of $H_2$, where the vertex representing $v_{10}$ in one copy is identified with $x_{i,1}$ and the vertex representing $v_{10}$ in the other copy is identified with $x_{i,3}$. This graph has $4 + 2 \cdot 17 = 38$ vertices, of which exactly two ($x_{i,1}$ and $x_{i,4}$) have degree 2. Let $G$ be any graph with maximum degree at most 3 containing $F_i'$ as a subgraph, and let $D$ be any dominating set in $G$. By Claim 3, $D$ has at least 6 vertices in each copy of $H_2 - v_{10}$. Also $D$ must have at least one vertex in $\{x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}\}$ in order to dominate $x_{i,2}$. Thus, altogether $|D \cap V(F_i')| \geqslant 13$. Let $G_k'$ consist of disjoint graphs $F_1', \ldots, F_k'$, where $x_{i,4}$ is connected by an edge to $x_{i+1,1}$ for $i = 1, \ldots, k-1$, and $x_{k,4}$ is connected by an edge to $x_{1,1}$. By the above,

$$\frac{\gamma(G_k')}{|V(G_k')|} \geqslant \frac{13}{38} = \frac{1}{3} + \frac{1}{114}.$$

Since $H_2 - v_{10}$ has a Hamiltonian cycle $(v_9, v_7, v_5, v_3, v_4, v_2, v_1, v_8, v_6, u_6, u_8, u_1, u_2, u_4, u_3, u_5, u_7)$, for $k \geqslant 5$, every $G_k'$ has a 2-factor with the shortest cycle of length 17. By the Kawarabayashi–Plummer–Saito result [3],

$$\frac{\gamma(G_k')}{|V(G_k')|} \leqslant \frac{17}{48} = \frac{1}{3} + \frac{1}{48}.$$

It is interesting whether the bounds in [3] can be strengthened.

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