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On $K_{s,t}$-minors in graphs with given average degree☆

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Abstract

Let $D(H)$ be the minimum d such that every graph G with average degree d has an H-minor. Myers and Thomason found good bounds on $D(H)$ for almost all graphs H and proved that for 'balanced' H random graphs provide extremal examples and determine the extremal function. Examples of 'unbalanced graphs' are complete bipartite graphs $K_{s,t}$ for a fixed s and large t. Myers proved upper bounds on $D(K_{s,t})$ and made a conjecture on the order of magnitude of $D(K_{s,t})$ for a fixed s and $t \to \infty$. He also found exact values for $D(K_{2,t})$ for an infinite series of t. In this paper, we confirm the conjecture of Myers and find asymptotically (in s) exact bounds on $D(K_{s,t})$ for a fixed s and large t.

Keywords: Graph minors; Average degree; Complete bipartite graphs

1. Introduction

Recall that a graph H is a *minor* of a graph G if one can obtain H from G by a sequence of edge contractions and vertex and edge deletions. In other words, H is a minor of G if there is $V_0 \subset V(G)$ and a mapping $f : (V(G) - V_0) \to V(H)$ such that for every $v \in V(H)$, the set $f^{-1}(v)$ induces a nonempty connected subgraph in G and for every $uv \in E(H)$, there is an edge in G connecting $f^{-1}(u)$ with $f^{-1}(v)$.

Mader [4] proved that for each positive integer t, there exists a $D(t)$ such that every graph with average degree at least $D(t)$ has a K_t-minor. Kostochka [1,2] and Thomason [11] determined the order of magnitude of $D(t)$, and recently Thomason [12] found the asymptotics of $D(t)$. Furthermore, Myers and Thomason [9,6], for a general graph H, studied the minimum number $D(H)$ such that every graph G with average degree at least $D(H)$ has an H-minor, i.e., a minor isomorphic to H. They showed that for almost all graphs H, random graphs are bricks for constructions of extremal graphs. On the other hand, they observed that for fixed s and very large t, the union of many K_{s+t-1} with $s-1$ common vertices does not have any $K_{s,t}$-minor and has a higher average degree than a construction obtained as a union of random subgraphs.

In view of this, Myers [8,7] considered $D(K_{s,t})$ for fixed s and large t. The above example of the union of many K_{s+t-1} with $s-1$ common vertices shows that $D(K_{s,t}) \geqslant t + 2s - 3$. Myers proved

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Theorem 1 (*Myers [8]*). *Let $t > 10^{29}$ be a positive integer. Then every graph $G = (V, E)$ with more than $((t + 1)/2)(|V| - 1)$ edges has a $K_{2,t}$-minor.*

This bound is tight for $|V| \equiv 1 \pmod t$. Myers noted that probably the average degree that provides the existence of a $K_{s,t}$-minor, provides also the existence of a $K^*_{s,t}$-minor, where $K^*_{s,t} = K_s + \overline{K_t}$ is the graph obtained from $K_{s,t}$ by adding all edges between vertices in the smaller partite set. In other words, $K^*_{s,t}$ is the graph obtained from K_{s+t} by deleting all edges of a subgraph on t vertices. Myers also conjectured that for every positive integer s, there exists $C = C(s)$ such that for each positive integer t, every graph with average degree at least $C\,t$ has a $K_{s,t}$-minor.

Preparing this paper, we have learned that Kühn and Osthus [3] proved the following refinement of Myers' conjecture.

Theorem 2 (*Kühn and Osthus [3]*). *For every $\varepsilon > 0$ and every positive integer s there exists a number $t_0 = t_0(s, \varepsilon)$ such that for all integers $t \geqslant t_0$ every graph of average degree at least $(1 + \varepsilon)t$ contains $K^*_{s,t}$ as a minor.*

In this paper, we prove a stronger statement but under stronger assumptions: We find asymptotically (in s) exact bounds on $D(K_{s,t})$ for t much larger than s. Our main result is

Theorem 3. *Let s and t be positive integers with $t > (180s \log_2 s)^{1+6s \log_2 s}$. Then every graph $G = (V, E)$ with $|E| \geqslant ((t + 3s)/2)(|V| - s + 1)$ has a $K^*_{s,t}$-minor. In particular, $D(K^*_{s,t}) \leqslant t + 3s$. On the other hand, for arbitrarily large n, there exist graphs with at least n vertices and average degree at least $t + 3s - 5\sqrt{s}$ that do not have a $K_{s,t}$-minor.*

This confirms the insight of Myers that $D(K^*_{s,t})$ and $D(K_{s,t})$ are essentially the same for fixed s and large t. It follows from our theorem that the above described construction giving $D(K_{s,t}) \geqslant t + 2s - 3$ is not optimal for $s > 100$.

In the next section we describe a construction giving the lower bound for $D(K_{s,t})$. In Section 3 we handle graphs with few vertices. Then in Section 4 we derive a couple of technical statements on contractions and in Section 5 we finish the proof of Theorem 3.

Throughout the paper, $N(x) = \{v \in V : xv \in E\}$ is the open neighborhood of the vertex x, and $N[x] = N(x) \cup \{x\}$ is the closed neighborhood of x. If $X \subseteq V$, then $N(X) = \bigcup_{x \in X} N(x) - X$ and $N[X] = \bigcup_{x \in X} N[x]$. We denote the minimum degree of G by $\delta(G)$.

2. Lower bound

We will need the following old result of Sauer [10]:

Lemma 1 (*Sauer [10]*). *Let $g \geqslant 5$ and $m \geqslant 4$. Then for every even $n \geqslant 2(m - 1)^{g-2}$, there exists an n-vertex m-regular graph of girth at least g.*

If $2 \leqslant s \leqslant 18$, then $3s - 5\sqrt{s} < 2s - 3$ and the construction above described by Myers and Thomason gives the lower bound. Let $s \geqslant 19$.

First, we describe the complement $\overline{G(s, t)}$ of a brick $G(s, t)$ for the construction. Let q be the number in $\{\lceil\sqrt{3s}\rceil, 1 + \lceil\sqrt{3s}\rceil\}$ such that $t - q$ is even. Observe that for $s \geqslant 18$,

$$2.5\sqrt{s} \geqslant 2 + \lceil\sqrt{3s}\rceil \geqslant q + 1, \tag{1}$$

and $q \geqslant \lceil\sqrt{3s}\rceil \geqslant 8$.

By Lemma 1, if $2s + t - q > (q - 3)^{2s-1}$, then there exists a $(q - 2)$-regular graph $F(s, t)$ of girth at least $2s + 1$ with $2s + t - q$ vertices. Since $t > (180s \log_2 s)^{1+6s \log_2 s}$ and $2s > q$, the condition $2s + t - q > (q - 3)^{2s-1}$ holds. Let $G(s, t) = \overline{F(s, t)}$.

Claim 2.1. $|E(G(s, t))| \geqslant 0.5(t + 3s - 2q)(|V(G(s, t))| - s + 1) + (s - 1)^2/4.$

Proof. Since $|V(G(s, t))| = 2s + t - q$ and $F(s, t)$ is $(q - 2)$-regular, the statement of the claim is equivalent to the inequality

$$(2s + t - q)(2s + t - 2q + 1) \geqslant (t + 3s - 2q)(s + t - q + 1) + (s - 1)^2/2.$$

Open the parentheses: all factors of t cancel out and we get the inequality $s^2 - s \geqslant q(s-1) + (s-1)^2/2$ which reduces to $s+1 \geqslant 2q$. The last inequality holds for $s \geqslant 18$. □

Claim 2.2. *$G(s,t)$ has no $K_{s,t}$-minor.*

Proof. Suppose to the contrary that there exist $V_0 \subset V(G(s,t))$ and a mapping $f : (V(G(s,t)) - V_0) \to V(K_{s,t})$ as in the definition of a minor. Let X be the set of vertices $x \in V(K_{s,t})$ with $|f^{-1}(x)| \geqslant 2$ and let $V' = V_0 \cup f^{-1}(X)$. Since $|V(G(s,t))| = 2s + t - q$, we have $|V'| \leqslant 2(s-q)$.

Let S denote the partite set of s vertices in $K_{s,t}$ and $V'' = f^{-1}(S - X) = f^{-1}(S) - V'$. Then $|V''| \geqslant q$. Since every $v \in V''$ is adjacent in $G(s,t)$ to every vertex outside of $V'' \cup V'$, the subgraph F' of $F(s,t)$ on $V'' \cup V'$ contains all edges incident with V''. Since the girth of $F(s,t)$ is at least $2s+1$, F' has at most $|V''| - 1$ edges inside V''. Therefore, F' has at least $(q-2)|V''| - (|V''| - 1)$ edges of $F(s,t)$ incident with V''. If the subgraph F_0 of F' induced by these edges has a cycle, at least half of the vertices of this cycle should be in V'' and therefore, the length of this cycle should be at most $2|V''| \leqslant 2s$, a contradiction to the definition of $F(s,t)$. If F_0 has no cycles, then, by the above, $|V'' \cup V'| \geqslant 2 + (q-3)|V''|$. Recall that $|V'' \cup V'| \leqslant |V''| + 2(s-q)$, and therefore we have $2(s-q) \geqslant 2 + (q-4)|V''| \geqslant 2 + (q-4)q$, i.e., $2s \geqslant 2 + q(q-2)$. But this does not hold if $s \geqslant 18$ and $q \geqslant \sqrt{3s}$. □

Claim 2.3. *$F(s,t)$ has an independent set of size $s-1$.*

Proof. We can construct such a set greedily, since $F(s,t)$ is $(q-2)$-regular and the number of vertices of $F(s,t)$ is greater than $(s-1)(q-1)$. □

Let I be a clique of size $s-1$ in $G(s,t)$ that exists by Claim 2.3. Define $G(s,t,1) = G(s,t)$ and for $r = 2, \ldots,$ let $G(s,t,r)$ be the union of $G(s,t,r-1)$ and $G(s,t)$ with the common vertex subset I. In other words, we glue every vertex of I in $G(s,t,r-1)$ with its copy in $G(s,t)$.

Claim 2.4. *For every $r \geqslant 1$,*

(a) $|V(G(s,t,r))| = s - 1 + r(s + t - q + 1)$;
(b) $|E(G(s,t,r))| \geqslant 0.5(t + 3s - 2q)(|V(G(s,t,r))| - s + 1) + \binom{s-1}{2} - r(s^2/4)$;
(c) *$G(s,t,r)$ has no $K_{s,t}$-minor.*

Proof. Statement (a) is immediate and we will prove (b) and (c) by induction on r. For $r = 1$, (b) is clear from Claim 2.1 and (c) is equivalent to Claim 2.2. Suppose that the claim holds for $r \leqslant r_0 - 1$.

Suppose first that $G(s,t,r_0)$ contains a $K_{s,t}$-minor G'. Since the common part of $G(s,t,r_0-1)$ and $G(s,t)$ is a clique of size $s-1$ and neither of these graphs has a $K_{s,t}$-minor, each of $G(s,t,r_0-1) - I$ and $G(s,t) - I$ must contain a branching vertex of $K_{s,t}$. But then there are no s internally disjoint paths between these vertices, a contradiction.

By construction, $|V(G(s,t,r_0))| - |V(G(s,t,r_0-1))| = s + t - q + 1$ and by Claim 2.1,

$$|E(G(s,t,r_0))| - |E(G(s,t,r_0-1))| = |E(G(s,t))| - \binom{s-1}{2}$$
$$\geqslant 0.5(t + 3s - 2q)(s + t - q + 1) - \frac{s^2}{4}.$$

This together with the induction assumption proves (b). □

Now, by part (b) of Claim 2.4, if $|V(G(s,t,r))| \geqslant st + 4s^2$ (to be crude), then $|E(G(s,t,r))| > 0.5(t + s - 2q - 2)|V(G(s,t,r))|$. Since this happens whenever $r \geqslant s+1$, we conclude from (1) that for large r, $G(s,t,r)$ has average degree greater than

$$t + 3s - 2q - 2 \geqslant t + 3s - 5\sqrt{s}.$$

This proves the lower bound.

3. Graphs with few vertices

In this section, we prove the upper bound of Theorem 3 for graphs with at most $10t/9$ vertices.

Lemma 2. *Let m, s, and n be positive integers such that*

$$n > 10s(30m)^m. \tag{2}$$

Let $G=(V,E)$ be a graph with $|V|=n$ and $|E|\leqslant 0.5mn$ such that

$$\deg(v)\leqslant 0.6n \quad \forall v\in V. \tag{3}$$

Then there exist an $L\subset V$ with $|L|\leqslant m-1$ and s disjoint pairs (x_i,y_i) of vertices in $G-L$ such that $\mathrm{dist}_{G-L}(x_i,y_i)>2$ *for all $i=1,\dots,s$.*

Proof. For every two distinct vertices x, y in G, let $A(x,y)$ denote the set of common neighbors of x and y and $a(x,y)=|A(x,y)|$. For $a(G)=\sum_{x,y\in V}a(x,y)$, we have

$$a(G)\leqslant \sum_{v\in V}\binom{\deg(v)}{2}\leqslant \binom{0.6n}{2}\frac{mn}{0.6n}<0.3n(n-1)m. \tag{4}$$

Let $V_0=\{v\in V:\deg_G(v)\geqslant 0.1n/m\}$ and $V_1=V-V_0$. For every two distinct vertices x, y in G and $i=0,1$, let $A_i(x,y)=A(x,y)\cap V_i$ and $a_i(x,y)=|A_i(x,y)|$. Also, for $i=0,1$, let $a_i(G)=\sum_{x,y\in V}a_i(x,y)$. Similarly to (4),

$$a_1(G)\leqslant \sum_{v\in V_1}\binom{\deg(v)}{2}\leqslant \binom{0.1n/m}{2}\frac{mn}{0.1n/m}<0.05n(n-1). \tag{5}$$

Let $W=\{(x,y)\in\binom{V}{2}: xy\notin E,\ a_1(x,y)=0,\ \text{and } a_0(x,y)\leqslant m-1\}$. Then $|W|\geqslant\binom{n}{2}-|E|-a_1(G)-a(G)/m$. Hence, by (5) and (4),

$$|W|\geqslant\binom{n}{2}-\frac{mn}{2}-\frac{n(n-1)}{20}-0.3n(n-1)=\frac{n}{2}(0.3(n-1)-m)>\frac{n(n-1)}{9}. \tag{6}$$

Consider the auxiliary graph H with the vertex set V and edge set W. By (6), H has a matching M with $|M|\geqslant n/9$. Since the number of distinct subsets of V_0 of size at most $m-1$ is $\sum_{k=0}^{m-1}\binom{10m^2}{k}<\binom{10m^2}{m}<(10em)^m$, there exists an $L\subset V_0$ with $|L|\leqslant m-1$ such that for the set $M_L=\{xy\in M: A_0(x,y)=L\}$ we have (remembering (2))

$$|M_L|\geqslant\frac{n/9}{(10em)^m}>s.$$

But then L and the pairs in M_L are what we need. □

A graph G is (s,t)-*irreducible* if

(i) $v(G)\geqslant s$;
(ii) $e(G)\geqslant 0.5(t+3s)(v(G)-s+1)$;
(iii) G has no minor G' possessing (i) and (ii).

For an edge e of a graph G, $t_G(e)$ denotes the number of triangles in G containing e.

Lemma 3. *If G is an (s,t)-irreducible graph and $t>s^2$, then*

(a) $v(G)\geqslant t+2s+1$;
(b) $t_G(e)\geqslant 0.5(t+3s-1)$ *for every $e\in E(G)$*;

(c) *if* $W \subset V(G)$ *and* $v(G) - |W| \geqslant s$, *then* W *is incident with at least* $0.5(t+3s)|W|$ *edges*; *in particular*, $\delta(G) \geqslant 0.5(t+3s)$;
(d) G *is s-connected*;
(e) $e(G) < 0.5(t+3s)v(G)$.

Proof. The number n of vertices of G should satisfy the inequality $n(n-1)/2 \geqslant 0.5(t+3s)(n-s+1)$. The roots of the polynomial $f(n) = n^2 - n - (t+3s)(n-s+1)$ are

$$n_{1,2} = \frac{1}{2}\left(t+3s+1 \pm \sqrt{(t+3s+1)^2 - 4(t+3s)(s-1)}\right).$$

Observe that $(t+3s+1)^2 - 4(t+3s)(s-1) > (t+s+1)^2$ for $t \geqslant s^2$. Therefore, either $n < s$ or $n > t+2s+1$. This together with (i) proves (a).

Let G_e be obtained from G by contracting e. Then $e(G_e) = e(G) - t_G(e) - 1$. By (iii), $e(G_e) \leqslant 0.5(t+3s)(v(G_e) - s+1) - 0.5 = 0.5(t+3s)(v(G)-s) - 0.5$. This together with (ii) yields

$$t_G(e) = e(G) - e(G_e) - 1 \geqslant 0.5(t+3s) + 0.5 - 1 = 0.5(t+3s-1),$$

i.e., (b) holds.

Observe that (c) follows from the fact that $G - W$ does not satisfy (ii).

Assume that there is a partition (V_1, V_0, V_2) of $V(G)$ such that $|V_0| \leqslant s-1$ and G has no edges connecting V_1 with V_2. By (c), $|V_1|, |V_2| \geqslant 0.5(t+3s) - (s-1)$. Let G_i be the subgraph of G induced by $V_0 \cup V_i$, $n_i = v(G_i)$, and $e_i = e(G_i)$, $i = 1, 2$. Since G_1 and G_2 are minors of G, (iii) yields $e_i < 0.5(t+3s)(n_i - s+1)$ for $i = 1, 2$. But then

$$e(G) \leqslant e_1 + e_2 < \tfrac{1}{2}(t+3s)((n_1 - s+1) + (n_2 - s+1)).$$

Since $n_1 + n_2 - s + 1 = v(G) + |V_0| - s + 1 \leqslant v(G)$, this contradicts (ii).

If (e) does not hold for G, then for any $e \in E(G)$, $G - e$ satisfies (ii), a contradiction to (iii). □

Lemma 4. *Suppose that* $t > (180 s \log_2 s)^{1+6s\log_2 s}$. *If* H *satisfies* (i) *and* (ii) *and* $v(H) \leqslant t + 6s\log_2 s + 2s$, *then* H *has a* $K^*_{s,t}$*-minor.*

Proof. Let H_0 be an (s,t)-irreducible minor of H. H_0 also has at most $t + 6s\log_2 s + 2s$ vertices. Suppose that $v(H_0) = n = t + 2s + m$. By Lemma 3(a) and conditions of our lemma, $1 \leqslant m \leqslant 6s\log_2 s$. Let G be the complement of H_0. By (ii), we have

$$\begin{aligned} e(G) &\leqslant \binom{n}{2} - \frac{1}{2}(t+3s)(n-s+1) = \frac{1}{2}(n^2 - n - (n+s-m)(n-s+1)) \\ &= \frac{1}{2}((m-2)n + (s-1)(s-m)) < \frac{mn}{2}. \end{aligned}$$

By (c) of Lemma 3, the degree of every vertex in G is at most $n - 1 - 0.5(t+3s) = 0.5(t+s) + m - 1 < 0.6n$. Applying Lemma 2 to G we find an $L \subset V(G)$ with $|L| \leqslant m-1$ and s disjoint pairs of vertices (x_i, y_i), $i = 1, \ldots, s$ such that $\mathrm{dist}_{G-L}(x_i, y_i) > 2$ for all $i = 1, \ldots, s$. Then contracting the edges $x_i y_i$ in the graph $H_0' = H_0 - L$ we get a $K^*_{s,n-|L|-s}$-minor. □

Lemma 5. *Let* m, s, k, *and* n *be positive integers such that* $k \geqslant 10$, $s \geqslant 3$, $m \leqslant 0.1n$

$$n > 10sk^2 \quad \text{and} \quad (5/9)^{k-2} m < 1. \tag{7}$$

Let $G = (V, E)$ *be a graph with* $|V| = n$ *and* $|E| \leqslant 0.5mn$ *such that*

$$\deg(v) \leqslant \tfrac{5}{9} n \quad \forall v \in V. \tag{8}$$

Then there exist s pairwise disjoint k-tuples $X_i = \{x_{i,1}, \ldots, x_{i,k}\}$ *of vertices in G such that for every* $i = 1, \ldots, s$,

(q1) *no vertex is a common neighbor of all the vertices in* X_i;
(q2) $G(X_i)$ *does not contain any complete bipartite graph* $K_{j,k-j}$, $1 \leqslant j \leqslant k/2$.

Proof. First, we count all k-tuples not satisfying (q1), i.e., all $X=\{x_1,\ldots,x_k\}$ having a common neighbor. This number q_1 is at most

$$\sum_{v\in V}\binom{\deg(v)}{k}\leqslant\binom{\frac{5}{9}n}{k}\frac{mn}{5n/9}\leqslant(5/9)^{k-1}\binom{n}{k}m.$$

Thus by (7), $q_1<\frac{5}{9}\binom{n}{k}$.

Let $V_0=\{v\in V:\deg_G(v)\geqslant n/3\}$ and $V_1=V-V_0$. The number q_2' of k-tuples X that contain a complete bipartite graph $K_{j,k-j}$, $1\leqslant j\leqslant k/2$ such that the partite set of size j contains a vertex in V_1 does not exceed

$$\sum_{v\in V_1}\binom{\deg(v)}{\left\lceil\frac{k}{2}\right\rceil}\binom{n}{\left\lfloor\frac{k}{2}\right\rfloor-1}\leqslant\binom{n}{\left\lfloor\frac{k}{2}\right\rfloor-1}\binom{n/3}{\left\lceil\frac{k}{2}\right\rceil}\frac{mn}{n/3}.$$

Since $k\geqslant10$, $m\leqslant0.1n$, and $n>10sk^2\geqslant300k$, the last expression is at most

$$\binom{n}{k-1}3^{-k/2}3m\leqslant\binom{n}{k}3^{-0.5k+1}\frac{k}{n-k+1}m\leqslant\frac{1}{80}\binom{n}{k}.$$

Similarly, the number q_2'' of k-tuples X that contain a complete bipartite graph $K_{j,k-j}$, $1\leqslant j\leqslant k/2$ such that the partite set of size j contains only vertices in V_0 does not exceed

$$\sum_{j=1}^{\lfloor\frac{k}{2}\rfloor}\binom{|V_0|}{j}\binom{\frac{5}{9}n}{k-j}\leqslant\binom{|V_0|+\frac{5}{9}n}{k}\leqslant\left(\frac{3m+5n/9}{n}\right)^k\binom{n}{k}$$
$$\leqslant\left(\frac{77}{90}\right)^k\binom{n}{k}<0.211\binom{n}{k}.$$

Hence the total number q of k-tuples X not satisfying (q1) or (q2) is at most

$$q_1+q_2'+q_2''<\binom{n}{k}\left(\frac{5}{9}+\frac{1}{80}+0.211\right)<0.78\binom{n}{k}.$$

Therefore, there are at least $0.22\binom{n}{k}$ *good* k-tuples, i.e., k-tuples satisfying (q1) and (q2). Now, we choose disjoint good k-tuples $X_1,\ldots,X_s$ one by one in a greedy manner. Let X_1 be any good k-tuple. Suppose that we have chosen $1\leqslant i\leqslant s-1$ good k-tuples $X_1,\ldots,X_i$. The set $X=\bigcup_{j=1}^{i}X_j$ meets at most $\binom{n}{k}-\binom{n-k(s-1)}{k}$ good k-tuples. But by (7),

$$\binom{n}{k}-\binom{n-k(s-1)}{k}<\binom{n}{k}\left(1-\left(\frac{n-sk}{n-k}\right)^k\right)$$
$$<\binom{n}{k}\left(1-\left(1-\frac{sk^2}{n-k}\right)\right)<\frac{1}{10}\binom{n}{k}.$$

Thus, we can choose a good k-tuple X_{i+1} disjoint from X. $\square$

Lemma 6. *Suppose that* $s\geqslant3$, $t>(180s\log_2 s)^{1+6s\log_2 s}$. *If H satisfies* (i) *and* (ii) *and* $v(H)\leqslant10t/9$, *then H has a* $K^*_{s,t}$*-minor.*

Proof. Let H_0 be an (s,t)-irreducible minor of H. H_0 also has at most $10t/9$ vertices.

Let $v(H_0)=n=t+m$. By Lemma 4 and conditions of our lemma, $6s\log_2 s+2s\leqslant m\leqslant t/9$. Let G be the complement of H_0. We want to prove that G satisfies the conditions of Lemma 5 for $k=\max\{10,2+\lceil\log_{9/5}m\rceil\}$. Inequalities $k\geqslant10$, $s\geqslant3$, and $m\leqslant0.1n$ follow from the definitions under the conditions of our lemma. So does the second part of (7). The inequality $|E(G)| \leqslant 0.5mn$ follows from (ii) as in the proof of Lemma 4. By (c) of Lemma 3, the degree of every vertex in G is at most

$$n - 1 - 0.5(t + 3s) = 0.5(t - 3s) + m - 1 < 0.5n + (m - 3s)/2 < 5n/9.$$

Thus, we need only to verify the first part of (7), namely, $n > 10sk^2$. If $k = 10$, then this is implied by $n > t \geqslant (180s \log_2 s)^{1+6s \log_2 s} > 1000s$.

Suppose now that $k = 2 + \lceil \log_{9/5} m \rceil$. Since $m \leqslant t/9$,

$$k = 2 + \lceil \log_{9/5} m \rceil < 3 + \log_{9/5}(t/9) < \log_{9/5} t < 1.2 \log_2 t,$$

in order to verify $n > 10sk^2$, it is sufficient to check that

$$t > 10s(1.2 \log_2 t)^2. \tag{9}$$

Observe that the derivative of the RHS of (9) with respect to t is equal to $20s(1.2 \log_2 t)1.2/t \ln 2$ which is less than 1 for $t > (180s \log_2 s)^{1+6s \log_2 s}$. Therefore, it is enough to check (9) for $t = (180s \log_2 s)^{1+6s \log_2 s}$. Since $180s \log_2 s > 10s \times 1.2^2$, this would follow from

$$(180s \log_2 s)^{3s \log_2 s} > \log_2(180s \log_2 s)^{1+6s \log_2 s},$$

which is easy to verify. Thus we can apply Lemma 5 to G.

Let $X_1, \ldots, X_s$ be the k-tuples provided by Lemma 5. The conditions (q1) and (q2) mean that every X_i is a connected dominating set in H_0. Thus, H_0 has a $K^*_{s,n-sk}$-minor.

We need now only to check that $n - sk \geqslant t$, i.e., $sk \leqslant m$. Observe first that $m \geqslant 6s \log_2 s + 2s \geqslant s(6 \log_2 3 + 2) > 11s$. This verifies $sk \leqslant m$ for $k \leqslant 10$. Let $k = 2 + \lceil \log_{9/5} m \rceil$. As above, $k < 1.2 \log_2 m$ and it is enough to verify the inequality $1.2s < m/\log_2 m$ for $m = 6s \log_2 s$. In this case, the last inequality reduces to $1 < 5 \log_s / \log_2(6s \log_2 s)$ which in turn reduces to $s^5 > 6s \log_2 s$. This is true for $s \geqslant 3$. □

4. Auxiliary statements

Lemma 7. *Let G be a connected graph. If $\delta(G) \geqslant k$, $|V(G)| = n$, then there exists a partition $V(G) = W_1 \cup W_2 \cup \ldots$ of $V(G)$ such that for every i,*

(a) *the subgraph of G induced by $\bigcup_{j=1}^{i} W_j$ is connected*;
(b) $|W_i| \leqslant 3$;
(c)

$$V(G) - \bigcup_{j=1}^{i} N[W_j]| \leqslant n \left(\frac{n-k-1}{n} \right)^i. \tag{10}$$

Furthermore, one can have $|W_1| = 1$.

Proof. For $i = 1$, $n((n-k-1)/n)^i = n - k - 1$, so we can take $W_1 = \{w_1\}$, where w_1 can be any vertex. Suppose that the lemma holds for $i = m - 1$ and let $X_m = V(G) - \bigcup_{j=1}^{m-1} N[W_j]$. Then

$$\sum_{v \in X_m} |N[v]| \geqslant (k+1)|X_m|$$

and hence there exists some w_m that belongs to at least $(k+1)|X_m|/n$ sets $N[v]$ for $v \in X_m$. We can choose w_m as close to $\bigcup_{j=1}^{i-1} W_j$ as possible. Since every vertex on distance 3 from $\bigcup_{j=1}^{i-1} W_j$ dominates at least $k+1$ vertices in $V(G) - \bigcup_{j=1}^{i-1} W_j$, the distance from $\bigcup_{j=1}^{i-1} W_j$ to w_m is at most 3. Therefore, we can form W_m from w_m and the vertices of a shortest path P_m from $\bigcup_{j=1}^{i-1} W_j$ to w_m. □

Lemma 8. *Let $\alpha \geqslant 2$. If G is a connected graph, $\delta(G) \geqslant k$, and $n \leqslant \alpha(k+1)$, then there exists a dominating set $A \subseteq V(G)$ such that $G[A]$ is connected and*

$$|A| \leqslant 3 \log_{\alpha/(\alpha-1)} n. \tag{11}$$

Proof. Let $V(G) = W_1 \cup W_2 \cup \ldots$ be a partition guaranteed by Lemma 7. Let $m = \lfloor \log_{\alpha/(\alpha-1)} n \rfloor$. Then $A' = \bigcup_{j=1}^{m} W_j$ does not dominate at most

$$n\left(1 - \frac{1}{\alpha}\right)^m = \left(\frac{\alpha}{\alpha - 1}\right)^x$$

vertices, where x is the fractional part of $\log_{\alpha/(\alpha-1)} n$. Since $\alpha \geqslant 2$, we have $(\alpha/(\alpha-1))^x < 2$. Thus, A' dominates all but at most one vertices in G. Suppose that the nondominated vertex (if exists) is w_0. Since G is connected, there is a common neighbor y_0 of w_0 and A'. Then $A = A' + y_0$ is a connected dominating set in G and $|A| = |A'| + 1 \leqslant 1 + 3(m-1) + 1 < 3 \log_{\alpha/(\alpha-1)} n$. □

Lemma 9. *Let s, k, and n be positive integers and $\alpha \geqslant 2$. Suppose that $n \leqslant \alpha(k+1)$. Let G be a $(3s \log_{\alpha/(\alpha-1)} n)$-connected graph with n vertices and $\delta(G) \geqslant k + 3(s-1)\log_{\alpha/(\alpha-1)} n$. Then $V(G)$ contains s disjoint subsets $A_1, \ldots, A_s$ such that for every $i = 1, \ldots, s$,*

(i) *$G[A_i]$ is connected*;
(ii) $|A_i| \leqslant 3 \log_{\alpha/(\alpha-1)} n$;
(iii) *A_i dominates $G - A_1 - \cdots - A_{i-1}$.*

Proof. Apply Lemma 8 s times. □

A subset X of vertices of a graph H is *k-separable* if $X \cup N(X) \neq V(H)$ and $|N(X) - X| \leqslant k$.

Lemma 10. *Let H be a graph and k be a positive integer. If C is an inclusionwise minimal k-separable set in H and $S = N(C) - C$, then the subgraph of H induced by $C \cup S$ is $(1 + \lceil \frac{k}{2} \rceil)$-connected.*

Proof. Assume that there is $D \subseteq S \cup C$ with $|D| \leqslant \lceil \frac{k}{2} \rceil$ that separates $H[S \cup C]$ into H_1 and H_2. Let H_1 be those of the two parts with fewer (or equal) vertices in S. Then the set $S_1 = D \cup (S \cap V(H_1))$ has at most k vertices and is a separating set in H. Moreover, a component of $H - S_1$ is a proper part of C, a contradiction. □

Lemma 11. *Let G be a $100s \log_2 t$-connected graph. Suppose that G contains a vertex subset U with $t + 100s \log_2 t \leqslant |U| \leqslant 3t$ such that $\delta(G[U]) \geqslant 0.4t + 100s \log_2 t$. Then G has a $K^*_{s,t}$-minor.*

Proof. Run the following procedure. Let S_1 be a smallest separating set in $G[U]$. If $|S_1| \geqslant 20s \log_2 t$, then stop. Otherwise, let $U'_1, U'_2, \ldots$ be the components of $G[U] - S_1$. If some of these components has a separating set S_2 with $|S_2| < 20s \log_2 t$, then let $U^2_1, U^2_2, \ldots$ be the components of $G[U] - S_1 - S_2$ and so on. Consider the situation after four such steps (if we did not stop earlier).

Claim 4.1. *If we did not stop after Step 3, then at most two components of $G[U] - S_1 - S_2 - S_3 - S_4$ are not $20s \log_2 t$-connected.*

Proof. Let $H = G[U] - S_1 - S_2 - S_3 - S_4$. By construction, H has at least 5 components and

$$\delta(H) \geqslant \delta(G[U]) - 4 \times 20s \log_2 t \geqslant 0.4t + 20s \log_2 t. \tag{12}$$

It follows that each component of H has more than $0.4t + 20s \log_2 t$ vertices. Moreover, if a component H' of H has fewer than $0.8t$ vertices, then each two vertices in H' have at least $40s \log_2 t$ common neighbors, and thus H'

is $40s\log_2 t$-connected. Therefore, if some three components of H are not $20s\log_2 t$-connected, then $|U|\geqslant|V(H)|\geqslant 3\cdot 0.8t+2\cdot 0.4t=3.2t$, a contradiction.

Claim 4.2. *For some* $1\leqslant m\leqslant 3$, *there are m vertex disjoint subgraphs* $H_1,\dots,H_m$ *of* $G[U]$ *such that*

(1) H_i *is* $20s\log_2 t$*-connected for* $i=1,\dots,m$;
(2) $\delta(H_i)\geqslant 0.4t+20s\log_2 t$ *for* $i=1,\dots,m$;
(3) $|V(H_1)|+\cdots+|V(H_m)|\geqslant t+m20s\log_2 t$.

Proof. Note that we stopped immediately after Step 4 or earlier. This implies (2). If we stopped before Step 4, then each component of $G[U]-S_1-\cdots$ is $20s\log_2 t$-connected. By Claim 4.1, if we stopped after Step 4, then at least three of the components are $20s\log_2 t$-connected. If we have at least three such components, then together they contain more than $3(0.4t+20s\log_2 t)>t+60s\log_2 t$ vertices. If we have at most two components, then we stopped before Step 2 and the total number of vertices in them is at least $|U|-20s\log_2 t\geqslant t+80s\log_2 t$. This proves the claim.

To finish the proof of the lemma, we consider 3 cases according to the smallest value of m for which Claim 4.2 holds.

Case 1: $m=1$. Since $|V(H_1)|\leqslant|U|\leqslant 3t$, we have $|V(H_1)|/0.4t\leqslant 7.5$ and

$$3\log_{\frac{7.5}{6.5}}3t=\frac{3}{\log_2\frac{75}{65}}\log_2 3t<15\log_2 3t\leqslant 20\log_2 t$$

whenever $t\geqslant 27$. It follows that we can apply Lemma 9 to H_1. By this lemma, there are s disjoint subsets $A_1,\dots,A_s$ of $V(H_1)$ such that for every $i=1,\dots,s$,

(i) $G[A_i]$ is connected;
(ii) $|A_i|\leqslant 3\log_{\frac{75}{65}}3t\leqslant 20\log_2 t$;
(iii) A_i dominates $H_1-A_1-\cdots-A_{i-1}$.

Since $|V(H_1)-A_1-\cdots-A_s|\geqslant t+20s\log_2 t-s\cdot 20\log_2 t=t$, H_1 has a $K^*_{s,t}$-minor.

Case 2: $m=2$. Since Case 1 does not hold, we know that Statement (3) of Claim 4.2 fails for both H_i, so $|V(H_i)|\leqslant t+20s\log_2 t\leqslant 1.2t$ for $i=1,2$. We can apply Lemma 9 to each of H_1 and H_2 with $\alpha=1.2t/0.4t=3$. Hence, there exist disjoint subsets $A^1_1,\dots,A^1_s$ of $V(H_1)$ and disjoint subsets $A^2_1,\dots,A^2_s$ of $V(H_2)$ such that for every $i=1,\dots,s$ and every $j=1,2$,

(i) $G[A^j_i]$ is connected;
(ii) $|A^j_i|\leqslant 3\log_{3/2}1.2t\leqslant 7\log_2 t$;
(iii) A^j_i dominates $H_j-A^j_1-\cdots-A^j_{i-1}$.

For $j=1,2$, let $A_j=\bigcup_{i=1}^s A^j_i$ and $V_j=V(H_j)-A_j$. Since the connectivity of $G-A_1-A_2$ is at least $100s\log_2 t-14s\log_2 t$, there are s vertex disjoint V_1,V_2-paths $P_1,\dots,P_s$ in $G-A_1-A_2$. We may assume that every P_i has exactly one vertex in V_1 and one vertex in V_2. For $i=1,\dots,s$, define $A^0_i=A^1_i\cup A^2_i\cup V(P_i)$. Then by (i), $G[A^0_i]$ is connected for every i. By (iii), each A^0_i dominates $U_0=(V_1\cup V_2)-\bigcup_{j=1}^s V(P_j)$ and A^0_k for $k>i$. Note that

$$\begin{aligned}|U_0|&\geqslant|V_1\cup V_2|-2s\geqslant|V(H_1)|+|V(H_2)|-|A_1|-|A_2|-2s\\&\geqslant t+40s\log_2 t-14\log_2 t-2s>t.\end{aligned}$$

Hence $G[V(H_1)\cup V(H_2)\cup\bigcup_{j=1}^s V(P_j)]$ has a $K^*_{s,t}$-minor.

Case 3: $m=3$. Since Cases 1 and 2 do not hold, we can assume that $|V(H_i)|\leqslant 0.8t$ for $i=1,2,3$. To see this, suppose without loss of generality that $|V(H_1)|\geqslant 0.8t$. Then $|V(H_2)|\geqslant\delta(H_2)>0.4t$, so

$$|V(H_1)|+|V(H_2)|\geqslant 1.2t>t+40s\log_2 t,$$

and Case 2 would apply, a contradiction.

Now we can apply Lemma 9 to each of H_1, H_2, and H_3 with $\alpha = 2$. Hence, there exist disjoint subsets $A_1^j, \ldots, A_s^j$ of $V(H_j)$, $j = 1, 2, 3$ such that for every $i = 1, \ldots, s$ and every $j = 1, 2, 3$,

(i) $G[A_i^j]$ is connected;
(ii) $|A_i^j| \leqslant 3\log_2 0.8t < 3\log_2 t$;
(iii) A_i^j dominates $H_j - A_1^j - \cdots - A_{i-1}^j$.

For $j = 1, 2, 3$, let $U_j = V(H_j) - \bigcup_{i=1}^s A_s^j$. Then

$$|U_1 \cup U_2 \cup U_3| \geqslant 3(0.4t + 20s\log_2 t) - 3s(3\log_2 t) = 1.2t + 51s\log_2 t.$$

For $j=1, 2, 3$, choose $X_j \subset U_j$ with $|X_1|=2s$ and $|X_2|=|X_3|=s$. The connectivity of the graph $H_0 = G - \bigcup_{j=1}^3 \bigcup_{i=1}^s A_s^j$ is at least $100s\log_2 t - 9s\log_2 t = 91s\log_2 t$. Hence there are $2s$ vertex disjoint $(X_1, X_2 \cup X_3)$-paths $P_1, \ldots, P_{2s}$ in H_0. Let us renumber the P_i-s so that every P_i for an odd i is an (X_1, X_2)-path (and every P_i for an even i is an (X_1, X_3)-path). Then we can find $2s$ subpaths $Q_1, \ldots, Q_{2s}$ of $P_1, \ldots, P_{2s}$ such that for every $k = 1, \ldots, s$,

(a) $Q_{2k-1} \cup Q_{2k} \subseteq P_{2k-1} \cup P_{2k}$;
(b) $|V(Q_{2k-1} \cup Q_{2k}) \cap (U_1 \cup U_2 \cup U_3)| \leqslant 4$;
(c) $V(Q_{2k-1} \cup Q_{2k}) \cap U_j \neq \emptyset$ for every $j = 1, 2, 3$.

For $i = 1, \ldots, s$ let $F_i = Q_{2i-1} \cup Q_{2i} \cup A_i^1 \cup A_i^2 \cup A_i^3$. Then

(i) $G[F_i]$ is connected for every i;
(ii) F_i-s are pairwise disjoint;
(iii) F_i dominates $U_1 \cup U_2 \cup U_3 - \bigcup_{k=1}^{2s} Q_k$ and F_j for $j > i$.

Since $|U_1 \cup U_2 \cup U_3 - \bigcup_{k=1}^{2s} Q_k| \geqslant 1.2t + 91s\log_2 t - 4s$, G has a $K_{s,t}^*$-minor. $\square$

5. Final argument

Below, $G = (V, E)$ is a minimum counterexample to Theorem 3. In particular, G is (s, t)-irreducible.

Case 1: G is $200s\log_2 t$-connected. If G has a vertex v with $t + 100s\log_2 t \leqslant \deg(v) \leqslant 3t - 1$, then G satisfies Lemma 11 with $U = N[v]$ and we are done. Thus, we can assume that every vertex in G has either 'small' ($< t + 100s\log_2 t$) or 'large' ($\geqslant 3t$) degree. Let V_0 be the set of vertices of 'small' degree. If $|V_0| > t + 100s\log_2 t$, then there is some $V_0' \subseteq V_0$ such that

$$t + 100s\log_2 t \leqslant \left| \bigcup_{v \in V_0'} N[v] \right| \leqslant 3t - 1.$$

In this case, we can apply Lemma 11 with $U = \bigcup_{v \in V_0'} N[v]$.

Now, let $|V_0| \leqslant t + 100s\log_2 t$. By Lemma 3(e), the average degree of G is less than $t + 3s$. Since every vertex outside of V_0 has degree at least $3t$, we get

$$0.5t|V_0| + 3t(n - |V_0|) < (t + 3s)n$$

and hence $n < 2.5|V_0|/(2 - 3s/t) < 3t$. If $n > t + 100s\log_2 t$, then we apply Lemma 11 with $U = V(G)$. If $n \leqslant t + 100s\log_2 t$, then we are done by Lemma 6.

Case 2: G is not $200s\log_2 t$-connected. Let S be a separating set with less than $k = \lceil 200s\log_2 t \rceil$ vertices and $V(G) - S = V_1 \cup V_2$ where vertices in V_1 are not adjacent to vertices in V_2. Then each of V_1 and V_2 is a k-separable set. For $j = 1, 2$, let W_j be an inclusion minimal k-separable set contained in V_j and $S_j = N(W_j) - W_j$. By Lemma 10, the graph $G_j = G[W_j \cup S_j]$ is $100s\log_2 t$-connected.

Case 2.1: $|W_j \cup S_j| \geqslant t + 100s\log_2 t$ for some $j \in \{1, 2\}$. Then we essentially repeat the argument of Case 1 with the restriction that the vertices v are taken only in W_j. Since by the minimality of G, the number of edges incident to W_j is less than $0.5(t + 3s)|W_j| + 200s\log_2 t|W_j|$, the argument goes through.

Case 2.2: $|W_j \cup S_j| < t + 100s\log_2 t$ for both $j \in \{1, 2\}$. By Lemma 3(c), we need $|W_j| \geqslant t - 400s\log_2 t$. Let $H_j = G(W_j)$.

Claim 5.1. (a) $\delta(H_j) \geqslant 0.5t - 200s\log_2 t$; (b) H_j *is* $400s\log_2 t$*-connected.*

Proof. The first statement follows from Lemma 3(c). If S_0 is a separating set in H_j with $|S_0| < 400s\log_2 t$, then the smaller part, say, H_0, of $H_j - S_0$ has at most $0.5t + 50s\log_2 t$ vertices and $|S_0 \cup S_j| \leqslant 600s\log_2 t$. This contradicts Lemma 3(c). □

By the above claim and Lemma 9 (for $k = 0.4t$ and $\alpha = 3$), $V(H_j)$ contains s disjoint subsets $A_1^j, \dots, A_s^j$ such that for every $i = 1, \dots, s$,

(i) $G[A_i^j]$ is connected;
(ii) $|A_i^j| \leqslant 3\log_{3/2}|W_j| < 6\log_2|W_j|$;
(iii) A_i^j dominates $W_j - A_1^j - \cdots - A_{i-1}^j$.

Since G is s-connected, $|S_j| \geqslant s$, $j = 1, 2$, and there are s pairwise vertex disjoint S_1, S_2-paths $P_1, \dots, P_s$. We may assume that the only common vertex of P_i with S_j is p_{ij}. By Lemma 3(b), each p_{ij} has at least $0.5t - 200s\log_2 t$ neighbors in W_j. Thus, we can choose $2s$ distinct vertices q_{ij} such that $q_{ij} \in W_j - \bigcup_{k=1}^{s} A_k^j$ and $p_{ij}q_{ij} \in E(G)$.

Define $F_i = A_i^1 \cup A_i^2 \cup V(P_i) + q_{ij}$, $i = 1, \dots, s$. Then for every $i = 1, \dots, s$,

(i) $G[F_i]$ is connected;
(ii) F_i-s are pairwise disjoint;
(iii) F_i *dominates* $\bigcup_{j=1}^{2} W_j - F_1 - \cdots - F_{i-1}$.

Since

$$\left|\bigcup_{j=1}^{2} W_j - F_1 - \cdots - F_{i-1}\right| \geqslant 2(t - 400s\log_2 t) - 12s\log_2 2t - 2s > t,$$

G has a $K_{s,t}^*$-minor. □

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