Chvátal's Condition Cannot Hold for Both a Graph and Its Complement

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Abstract

Chvátal's Condition is a sufficient condition for a spanning cycle in an n-vertex graph. The condition is that when the vertex degrees are $d_1, \ldots, d_n$ in nondecreasing order, $i < n/2$ implies that $d_i > i$ or $d_{n-i} \geq n - i$. We prove that this condition cannot hold in both a graph and its complement, and we raise the problem of finding its asymptotic probability in the random graph with edge probability 1/2.

This note is motivated by a discussion in the book of Palmer [7, page 81–85]. A theorem is strong if the conclusion is satisfied only when the hypothesis is satisfied, because then the hypotheses cannot be weakened. Palmer defines the *strength* of a theorem to be the probability that its hypotheses hold divided by the probability that its conclusion holds.

We use the standard random graph model for generating n-vertex simple graphs: the vertex set is $\{1, \ldots, n\}$, and edge ij occurs with probability p, independently of other edges. Let $\mathbb{G}_{n,p}$ denote the random variable for the resulting graph. In general, "$\mathbb{G}_{n,p}$ almost always satisfies Q" means that the probability of $\mathbb{G}_{n,p}$ satisfying Q tends to 1 as $n \to \infty$. We restrict our attention to constant p.

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A graph is *Hamiltonian* if it has a spanning cycle. When p is constant, $\mathbb{G}_{n,p}$ is almost always Hamiltonian. Dirac [3] proved that an n-vertex graph is Hamiltonian when every vertex degree is at least $n/2$. When $p > 1/2$, this condition holds almost always; when $p \leq 1/2$, it fails almost always. Hence the asymptotic strength of Dirac's Theorem is 0 when $p \leq 1/2$. Ore [6] proved the stronger theorem that an n-vertex graph is Hamiltonian when the degrees of any two nonadjacent vertices sum to at least n. The asymptotic strength of Ore's Theorem also is 0 when $p \leq 1/2$.

The strongest possible result using the degree list alone is that of Chvátal [2]. Chvátal proved that if the vertex degrees of an n-vertex graph are $d_1, \ldots, d_n$ in nondecreasing order, and for $i < n/2$ it holds that $d_i > i$ or $d_{n-i} \geq n - i$, then the graph is Hamiltonian. (If the condition just barely fails anywhere, then the graph can fail to be Hamiltonian.)

Palmer [7, page 85] states that Chvátal's Condition almost always fails when $p < 1/2$. It is implied by Dirac's Condition and hence almost always holds when $p > 1/2$.

Question: What is the asymptotic probability of Chvátal's Condition for the random graph with edge probability 1/2?

There are other sufficient conditions for Hamiltonian cycles that almost always hold when $p = 1/2$ and hence yield stronger theorems. Given a graph G with n vertices, let $C(G)$ be the graph obtained by adding to G all edges joining two nonadjacent vertices in G whose degrees sum to at least n. Ore's Condition is that $C(G) = K_n$. Let $C^*(G)$ be the result of repeatedly applying the operator C until no further change occurs; Chvátal's Condition implies that $C(G) = K_n$. Bondy and Chvátal [1] observed that G is Hamiltonian if and only if $C^*(G)$ is Hamiltonian.

The value $p = 1/2$ is critical here, as shown by Gimbel, Kurtz, Lesniak, Scheinerman, and Wierman [4]. If $p < 1/2$, then almost always $C(\mathbb{G}_{n,p}) = \mathbb{G}_{n,p}$. If $p > 1/2$, then almost always $C(\mathbb{G}_{n,p}) = K_n$. If $p = 1/2$, then almost always $C(C(C(\mathbb{G}_{n,p})))$ is complete but $C(C(\mathbb{G}_{n,p}))$ is not. Hence the theorem that $C^*(G) = K_n$ suffices for Hamiltonicity of n-vertex graphs has asymptotic strength 1 when $p \geq 1/2$.

By proving that Chvátal's Condition cannot hold for both a graph and its complement, we prove that the condition always has probability less than 1/2 (strict inequality, because it fails for both G and $\overline{G}$ when each has a vertex of degree at most 1). Since the degree list is almost always constant for a while near the middle, the asymptotic probability may rest on the probability that Chvátal's Condition fails at the middle values. When n is even, by complementation $d_{n/2-1} \geq n/2$ with probability nearly 1/2. When n is odd, however, Brendan McKay observes that the asymptotic probability of $d_{(n-1)/2} = d_{(n+1)/2} = d_{(n+3)/2} = (n-1)/2$ (for both G and $\overline{G}$) is a nontrivial constant that can be determined by the method in [5]; this bounds the asymptotic probability of Chvátal's Condition below 1/2 when n is odd.

Theorem 1 *If a graph G with at least three vertices satisfies Chvátal's Condition, then $\overline{G}$ does not.*

Proof. Let n be the order of G. Let $d_1, \ldots, d_n$ be the vertex degrees, indexed so that $d_1 \leq \cdots \leq d_n$. Let $d'_1, \ldots, d'_n$ be the vertex degrees of $\overline{G}$, also indexed in nondecreasing order, so that $d'_{n+1-i} = n - 1 - d_i$. Chvátal's Condition for G states that if $i < n/2$, then $d_i > i$ or $d_{n-i} \geq n - i$.

The claim is easy to show for odd n, so we show this first in order to simplify notation for the other case. Consider $i = (n-1)/2$, so $n - i = (n+1)/2$. If $d_{(n-1)/2} > (n-1)/2$, then $d_{(n+1)/2} \geq (n+1)/2$, so we may assume the latter condition. This in turn implies $d'_{(n+1)/2} \leq (n-3)/2$, which also implies $d'_{(n-1)/2} \leq (n-3)/2$. Hence $\overline{G}$ fails Chvátal's Condition at $i = (n-1)/2$.

Now consider n even. Let $j = \max\{i\colon d_i > i \text{ and } i < n/2\}$ Such an index exists, since $d_1 \leq 1$ implies $d_n \geq d_{n-1} \geq n - 1$ by Chvátal's Condition for G, but this contradicts $d_1 \leq 1$.

If $j = n/2 - 1$, then $d_{n/2-1} \geq n/2$, so complementation yields $d'_{n/2+2} \leq n/2 - 1$. Hence also $d'_{n/2+1} \leq n/2 - 1$ and $d'_{n/2-1} \leq n/2 - 1$, so Chvátal's Condition fails.

If $j < n/2 - 1$, then $d_{j+1} \leq j + 1$. By Chvátal's Condition, $d_{n-1-j} \geq n - 1 - j$. Now complementation yields $d'_{j+2} \leq j$, and hence also $d'_j \leq j$. If Chvátal's Condition holds for $\overline{G}$, then $d'_{n-j} \geq n - j$. Now complementation yields $d_{j+1} \leq j - 1$. This implies $d_j < j$, which contradicts the choice of j. ■

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