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Short communication

Disjoint K_r-minors in large graphs with given average degree

Thomas Böhme[a,*], Alexandr Kostochka[b,c]

[a] *Institut für Mathematik, Technische Universität Ilmenau, Ilmenau, Germany*
[b] *Department of Mathematics, University of Illinois, Urbana, IL 61801-2975, USA*
[c] *Institute of Mathematics, 630090 Novosibirsk, Russia*

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Abstract

It is proved that there are functions $f(r)$ and $N(r, s)$ such that for every positive integer r, s, each graph G with average degree $d(G) = 2|E(G)|/|V(G)| \geq f(r)$, and with at least $N(r, s)$ vertices has a minor isomorphic to $K_{r,s}$ or to the union of s disjoint copies of K_r.

1. Introduction

In this note all graphs are finite and do not have loops or multiple edges. A graph H is a *minor* of another graph G if H can be obtained from a subgraph of G by contracting edges. In 1968, Mader [2] showed that for every $r \in \mathbb{N}$ there exists a positive integer $h(r)$ such that every graph with average degree $d(G) \geq h(r)$ contains a minor isomorphic to K_r. Mader proved that $h(r) = O(r \log r)$. The order of magnitude of $h(r)$ was determined independently by Kostochka [4] and Thomason [5] (see also [1, p. 178]). They proved that $h(r) = \Theta(r\sqrt{\log r})$. Recently, Thomason [6] found the asymptotics of $h(r)$.

The main result of the present note is the following theorem conjectured by Mohar [3].

Theorem 1.1. *There exist a function $f : \mathbb{N} \to \mathbb{N}$ and a function $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that every graph with average degree at least $f(r)$ and order at least $N(r, s)$ has a minor isomorphic to s disjoint copies of K_r or a minor isomorphic to $K_{r,s}$.*

The order of magnitude of $f(r)$ in the proof below is the same as that of $h(r)$. On the other hand, Mohar observed that $f(5)$ in Theorem 1.1 is larger than $h(5) = 6$, as

* Tel.: +49-3677-69-3630; fax: +49-3677-69-3272.
E-mail addresses: tboehme@theoinf.tu-ilmenau.de (T. Böhme), kostochk@math.uiuc.edu (A. Kostochka).

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follows. Every graph that can be embedded into the torus surface does not contain a minor isomorphic to two disjoint copies of K_5 or $K_{5,3}$. Since there are arbitrarily large 6-regular graphs on the torus surface this implies $f(5) > h(5) = 6$.

For a graph G, we will use $|G|$ and $\|G\|$ to denote the number of vertices and edges of G, respectively.

2. Proof of Theorem 1.1

We will prove the following variation of Theorem 1.1.

Theorem 2.1. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ and a function $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that every graph with n vertices and at least $f(r)(n/2) + (N(r,s) - n)$ edges contains a minor isomorphic to s disjoint copies of K_r or a minor isomorphic to $K_{r,s}$.*

For the proof of Theorem 2.1 we need the following fact.

Proposition 2.2. *Let $\mathcal{X} = \{X_1, \dots, X_l\}$ be a family of sets such that $l \geq A^r s^{r+1}$ and for every $i \in \{1, \dots, l\}, r \leq |X_i| \leq A$. Then there is a subfamily $\{X_{i_1}, \dots, X_{i_s}\}$ of $\mathcal{X}$ such that either*

(i) $|\bigcap_{j=1}^{s} X_{ij}| \geq r$, *or*

(ii) *for every $j \in \{1, \dots, s\}$ there is a subset $\tilde{X}_{i_j}$ of X_{i_j} such that $|\tilde{X}_{i_j}| \geq |X_{i_j}| - r + 1$ and for $1 \leq \mu < \nu \leq s$, $\tilde{X}_{i_\mu} \cap \tilde{X}_{i_\nu} = \emptyset$.*

Proof. Consider a maximal subfamily $\mathcal{X}' = \{X_{i_1}, \dots, X_{i_t}\}$ of $\mathcal{X}$ such that for every $j \in \{1, \dots, t\}$ there is a subset $\tilde{X}_{i_j}$ of X_{i_j} with the property that $|\tilde{X}_{i_j}| \geq |X_{i_j}| - r + 1$, and $\tilde{X}_{i_\mu} \cap \tilde{X}_{i_\nu} = \emptyset$ for $1 \leq \mu < \nu \leq t$. If $t \geq s$, then $\{X_{i_1}, \dots, X_{i_s}\}$ is the desired subfamily. Otherwise, every member of $\mathcal{X}$ has at least r elements in $M = \bigcup_{j=1}^{t} X_{i_j}$. Note that $|M| \leq tA \leq (s-1)A$. Hence some r-subset is contained in at least $|\mathcal{X}|/\binom{|M|}{r} \geq A^r s^{r+1}/A^r s^r = s$ members of $\mathcal{X}$. This proves the proposition. □

Proof of Theorem 2.1. For $r, s, n \geq 2$, let $f(r) = 2h(r) + 2r$,

$$N(r,s) = (2(s+r)h(r+s))^{r+1},$$

and

$$F(n,r,s) = f(r)\frac{n}{2} + (N(r,s) - n).$$

Suppose that the statement of the theorem is not true, and let n be the smallest positive integer such that there exists a graph G with the properties

(a) G does not contain a minor isomorphic to $K_{r,s}$ or s disjoint copies of K_r,

(b) $\|G\| \geq F(n,r,s)$.

We derive further properties of such G in a series of claims.

Claim 1. $2\|G\| < h(r+s)n$.

Proof. Otherwise, by the definition of h, G has a minor isomorphic to K_{r+s}, a contradiction to (a).

Claim 2. $n > N(r,s)/h(r+s)$.

Proof. Suppose that $n \le N(r,s)/h(r+s)$. Since $h(r+s) \ge h(4) \ge 2$ and $f(r) \ge 4$, we deduce from (b) that

$$\|G\| \ge F(n,r,s) = f(r)\frac{n}{2} + N(r,s) - n \ge N(r,s) \ge nh(r+s),$$

a contradiction to Claim 1.

Claim 3. *Every edge of G belongs to at least $0.5f(r) - 1$ triangles.*

Proof. Let $e \in E(G)$ and G/e be the graph obtained from G by contracting e. If e belongs to $t(e) \le 0.5f(r) - 2$ triangles, then

$$\|G/e\| = \|G\| - 1 - t(e) \ge F(n,r,s) - 1 - 0.5f(r) + 2 = F(n-1,r,s).$$

By the minimality of n, G/e satisfies the theorem, and hence G does, a contradiction. This proves the claim.

For every $v \in V(G)$, let G_v be the subgraph induced by $\{v\} \cup N(v)$. Claim 3 yields that

$$\delta(G_v) \ge 0.5f(r) \qquad \text{for every } v \in V. \tag{1}$$

The next claim is directly implied by Claim 1.

Claim 4. *The cardinality of the set X of vertices in G of degree less than $2h(r+s)$ is at least $n/2$.*

Let $\mathcal{X} = \{\{x\} \cup N(x) : x \in X\}$. By Claims 2 and 4, $|\mathcal{X}| \ge n/2 > N(r,s)/2h(r+s)$. Note that $N(r,s)/2h(r+s) = (2h(r+s))^r(r+s)^{r+1}$. Hence by Proposition 2.2, there is a subset X' of X of cardinality at least $s+r$ such that either

(i) $\left|\bigcap_{x\in X'}(\{x\} \cup N(x))\right| \ge r$ or

(ii) for every $x \in X'$ there is a subset $N'(x)$ of $\{x\} \cup N(x)$ such that $|N'(x)| \ge |\{x\} \cup N(x)| - r + 1$ and $N'(x) \cap N'(y) = \emptyset$ whenever $x \ne y$.

Case (i). Let R be a subset of $\bigcap_{x\in X'}(\{x\} \cup N(x))$ of cardinality r. Then the subgraph of G induced by $R \cup X'$ contains $K_{r,s}$ with partite sets R and $X' - R$. This contradicts (a).

Case (ii). Consider the subgraphs H_x of G induced by $N'(x)$ for $x \in X'$. By (ii) and (1),

$$\delta(H_x) \ge \delta(G_x) - r + 1 \ge 0.5f(r) - r + 1 > h(r) \qquad \text{for every } x \in X'.$$

It follows that every H_x contains a minor isomorphic to K_r. Consequently, G contains a minor isomorphic to the union of s disjoint copies of K_r. This contradicts (a), and so the theorem is proved. □

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