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Irreducible hypergraphs for Hall-type conditions, and arc-minimal digraph expanders

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Abstract

Suppose that a hypergraph $\mathcal{H} = (V, \mathcal{E})$ satisfies a Hall-type condition of the form $|\bigcup \mathcal{F}| \geqslant r|\mathcal{F}| + \delta$ whenever $\emptyset \neq \mathcal{F} \subseteq \mathcal{E}$, but that this condition fails if any vertex (element) is removed from any edge (set) in $\mathcal{E}$. How large an edge can $\mathcal{H}$ contain? It is proved here that there is no upper bound to the size of an edge if r is irrational, but that if $r = p/q$ as a rational in its lowest terms then $\mathcal{H}$ can have no edge with more than $\max\{p, p + \lceil\delta\rceil\}$ vertices (and if $\delta < 0$ then $\mathcal{H}$ must have an edge with at most $\lceil(p-1)/q\rceil$ vertices). If $\delta \leqslant 0$ then the upper bound p is sharp, but if $\delta > 0$ then the bound $p + \lceil\delta\rceil$ can be improved in some cases (we conjecture, in most cases). As a generalization of this problem, suppose that a digraph $D = (V, A)$ satisfies an expansion condition of the form $|N^+(X) \setminus X| \geqslant r|X| + \delta$ whenever $\emptyset \neq X \subseteq S$, where S is a fixed subset of V, but that this condition fails if any arc is removed from D. It is proved that if $r = p/q$ as a rational in its lowest terms, then every vertex of S has outdegree at most $\max\{p + q, p + q + \lceil\delta\rceil - 1\}$, and at most $\max\{p, p + \lceil\delta\rceil\}$ if S is independent, but that if r is irrational then the vertices of S can have arbitrarily large outdegree.

Keywords: Irreducible hypergraph; Hall-type condition; Bipartite expander; Arc-minimal expander; Arc-minimal digraph; Digraph expander

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1. Introduction

Throughout the paper, $\mathbb{N}$ denotes the set of positive integers, and we assume that $p \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{N}$, $d \in \mathbb{Z}$, $r, \delta \in \mathbb{R}$ and $r \geqslant 0$. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, i.e. a family $\mathcal{E}$ of subsets of a set V; the elements of V and $\mathcal{E}$ are called *vertices* and *edges* respectively. If $\mathcal{F} \subseteq \mathcal{E}$, we write $\bigcup \mathcal{F}$ as a shorthand for $\bigcup_{F \in \mathcal{F}} F$. Let $\mathcal{C}(r, \delta)$ be the class of all hypergraphs $\mathcal{H} = (V, \mathcal{E})$ for which

$$\left|\bigcup \mathcal{F}\right| \geqslant r|\mathcal{F}| + \delta \qquad \text{whenever } \emptyset \neq \mathcal{F} \subseteq \mathcal{E} \text{ and } |\mathcal{F}| < \infty, \tag{1.1}$$

and let $\mathcal{C}(p, q, d) = \mathcal{C}(p/q, d/q)$ be the class of all those for which

$$\left|\bigcup \mathcal{F}\right| \geqslant \frac{p|\mathcal{F}| + d}{q} \qquad \text{whenever } \emptyset \neq \mathcal{F} \subseteq \mathcal{E} \text{ and } |\mathcal{F}| < \infty. \tag{1.2}$$

We say that a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is *irreducible* in a class $\mathcal{C}$ if $\mathcal{H} \in \mathcal{C}$ but if $\mathcal{H}'$ is obtained by removing any vertex from any edge of $\mathcal{H}$ then $\mathcal{H}' \notin \mathcal{C}$. If $\mathcal{H} \in \mathcal{C}(r, \delta)$ and every edge in $\mathcal{E}$ is finite, then clearly (if $\mathcal{E}$ is finite) or by a standard compactness argument (if $\mathcal{E}$ is infinite) one can reduce $\mathcal{H}$ to an irreducible member of $\mathcal{C}(r, \delta)$ by removing vertices from some edges if necessary. (We allow our hypergraphs to have multiple edges, i.e. edges that are equal as sets, although Theorem 6.2 shows that this is unnecessary if $r \geqslant 1$.) This is not necessarily true if some edge $E \in \mathcal{E}$ is infinite. However, in that case one can remove any finite number of vertices from E without violating (1.1). Hence the irreducible hypergraphs in $\mathcal{C}(r, \delta)$ can have no infinite edges.

We started looking at irreducible hypergraphs in the hope of proving results about colourings [3]. Although we had some success with this approach, we found that usually it does not work, and the present paper arose from our attempt to understand why.

We shall see in Theorem 4.3 that if r is irrational then irreducible hypergraphs in $\mathcal{C}(r, \delta)$ can contain arbitrarily large edges. However, Theorem 2.2 shows that if $r = p/q$ as a fraction in its lowest terms, then an irreducible hypergraph in $\mathcal{C}(r, \delta)$ can contain no edge with more than $\max\{p, p + \lceil \delta \rceil\}$ vertices. In this case it suffices to consider the case when $\delta = d/q$ for some integer d, so that (1.1) reduces to (1.2); taking $|\mathcal{F}| = 1$, this clearly implies that every edge contains at least $\lceil (p + d)/q \rceil$ vertices. A hypergraph that is irreducible in $\mathcal{C}(p, q, d)$ will be called (p, q, d)-*irreducible*.

It is easy to see from Theorem 2.2 that if $d \geqslant 0$ then every $(p, 1, d)$-irreducible hypergraph is $(p + d)$-*uniform* (i.e. every edge has exactly $p + d$ vertices), and a $(0, q, d)$-irreducible hypergraph is $\lceil d/q \rceil$-uniform. But (p, q, d)-irreducible hypergraphs are not uniform in general. Let $\mathrm{maxmod}(p, q, d)$ denote the largest edge-size that is possible in a (p, q, d)-irreducible hypergraph. Theorems 2.2 and 4.4 show that $\mathrm{maxmod}(p, q, d) = p$ if $d \leqslant 0$, and

$$\max\{p, \lceil (p + d)/q \rceil\} \leqslant \mathrm{maxmod}(p, q, d) \leqslant p + \lceil d/q \rceil \qquad \text{if } d > 0. \tag{1.3}$$

We know of no examples where this lower bound is exceeded by more than one. Theorems 2.2 and 4.5 show that $\mathrm{maxmod}(p, q, 1) = p + 1$ for all p and q. In Theorem 2.3 we determine $\mathrm{maxmod}(1, q, d)$ for all q and d; for $d \geqslant 0$ it always equals either $\lceil (1+d)/q \rceil$

or $\lceil(1 + d)/q\rceil + 1$. Our construction giving the value $\lceil(p + d)/q\rceil + 1$ is described in Theorem 4.6, but it works only for certain ranges of values of d, for all of which $\lceil(p + d)/q\rceil < q$. Thus we make the following conjectures.

Conjecture 1.1. *For all values of p, q and $d \geqslant 0$, $\mathrm{maxmod}(p, q, d)$ is equal to either $\max\{p, \lceil(p + d)/q\rceil\}$ or one more than this.*

Conjecture 1.2. *For fixed values of p and q, $\mathrm{maxmod}(p, q, d) = \lceil(p+d)/q\rceil$ if d is large enough.*

These results and conjectures can be restated in the language of expanders. Let us say that a bipartite graph G with partite sets S, T (in that order) is an (r, δ)-*expander* if $|N_G(X)| \geqslant r|X| + \delta$ for every nonempty subset $X \subseteq S$. Then $\mathrm{maxmod}(p, q, d)$ is the largest possible degree of a vertex $s \in S$ in an edge-minimal $(p/q, d/q)$-expander. This is because a hypergraph $\mathcal{H} = (V, \mathcal{E})$ can be represented by a bipartite graph G with partite sets S, T, where $T = V$, the vertices in S are (in $1 : 1$ correspondence with) the edges in $\mathcal{E}$, and a vertex $s \in S$ is adjacent to a vertex $t \in T$ if and only if t belongs to (the edge in $\mathcal{E}$ corresponding to) s. Conversely, given a bipartite graph G with partite sets S, T, one can represent it by a hypergraph $\mathcal{H} = (V, \mathcal{E})$ satisfying the above description. In either case, the degree of a vertex in S is equal to the cardinality of the corresponding edge in $\mathcal{E}$, and (1.1) says precisely that $|N_G(X)| \geqslant r|X| + \delta$ for every nonempty $X \subseteq S$. We use this bipartite-graph representation in Section 3 to get an alternative proof of Theorem 2.2 when $d \leqslant 0$, and also to get further information about (p, q, d)-irreducible hypergraphs in this case; in particular, we prove that any such hypergraph must contain an edge with at most $\lceil p/q\rceil$ vertices if $d = 0$, and with at most $\lceil(p - 1)/q\rceil$ vertices if $d < 0$, and these bounds are sharp. (It seems likely that if $d > 0$ then there is always an edge with at most $\lceil(p + d)/q\rceil$ vertices, but we do not have a proof of this.)

In Section 5 we generalize this idea from bipartite graphs to digraphs. Suppose a digraph $D = (V, A)$ satisfies an expansion condition of the form $|N^+(X) \setminus X| \geqslant r|X| + \delta$ whenever $\emptyset \neq X \subseteq S$, where S is a fixed subset of V, but that this condition fails if any arc is removed from D. If D is bipartite with bipartition (S, T) and all arcs directed from S towards T, then we recover the bipartite model of hypergraphs described in the previous paragraph. It follows from the corresponding examples for hypergraphs (Theorem 4.3) that if r is irrational then there are bipartite digraphs D with this property in which S contains vertices with arbitrarily large outdegree. In Theorems 5.1 and 5.3, which are the digraph analogues of Theorems 2.2 and 4.4 for hypergraphs, we prove that if $r = p/q$ as a rational in its lowest terms, then the largest possible outdegree for a vertex in S is exactly $p + q$ if $\delta \leqslant 1$, and lies between $\max\{p + q, \lceil(p + d)/q\rceil\}$ and $p + q + \lceil\delta\rceil - 1$ if $\delta > 1$. The difference between these bounds and those in (1.3) reflects the extra complexity in the situation for nonbipartite digraphs compared with bipartite ones.

We prove the main results about the size of the largest edge in an irreducible hypergraph in Section 2, although the constructions needed for the lower bounds are left until Section 4. An alternative proof of the upper bound using bipartite graphs, and results about the size of the smallest edge, are given in Section 3. Arc-minimal digraph expanders are discussed in Section 5. In Section 6 we tidy up a couple of loose ends.

2. The upper bounds

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and let $p, q \in \mathbb{N}$ and $d \in \mathbb{Z}$. For each finite subset $X \subseteq V$, let $e(X) = e_{\mathcal{H}}(X)$ be the number of edges of $\mathcal{H}$ contained in X, and define

$$\mathrm{sur}(X) = \mathrm{sur}_{\mathcal{H}}(X) := q|X| - pe_{\mathcal{H}}(X) - d, \tag{2.1}$$

so that $\mathrm{sur}_{\mathcal{H}}(X) \geqslant 0$ if $\mathcal{H} \in \mathcal{C}(p, q, d)$ and $e_{\mathcal{H}}(X) \neq 0$. Let $\mathcal{E}^+(X, Y)$ denote the set of edges of $\mathcal{H}$ that are contained in $X \cup Y$ but not in X or Y. The following result is easy to see.

Lemma 2.1. *If $X, Y \subseteq V$ then*

$$\mathrm{sur}_{\mathcal{H}}(X) + \mathrm{sur}_{\mathcal{H}}(Y) - \mathrm{sur}_{\mathcal{H}}(X \cup Y) - \mathrm{sur}_{\mathcal{H}}(X \cap Y) = p|\mathcal{E}^+(X, Y)|. \tag{2.2}$$

Proof. By (2.1), the LHS of (2.2) is equal to

$$p[e_{\mathcal{H}}(X \cup Y) + e_{\mathcal{H}}(X \cap Y) - e_{\mathcal{H}}(X) - e_{\mathcal{H}}(Y)]. \tag{2.3}$$

An edge that is contained in $X \cap Y$ contributes $p(2 - 2) = 0$ to (2.3). An edge that is contained in X or Y but not $X \cap Y$ contributes $p(1 - 1) = 0$ to (2.3). An edge that is contained in $X \cup Y$ but not X or Y contributes $p(1 - 0) = p$ to (2.3). $\square$

The following theorem is our main upper bound. It is not necessary to assume here that p and q are coprime, although naturally the bound is strongest when they are.

Theorem 2.2. $\mathrm{maxmod}(p, q, d) \leqslant \max\{p, p + \lceil d/q \rceil\}$.

Proof. Suppose that $\mathcal{H} = (V, \mathcal{E})$ is (p, q, d)-irreducible and that $\mathcal{E}$ contains an edge $E_0 = \{v_1, \ldots, v_t\}$ where $t \geqslant p + 1$. By the irreducibility of $\mathcal{H}$, there are finite sets $X_1, \ldots, X_t \subseteq V$ such that, for each i,

$$X_i \cap E_0 = E_0 \setminus \{v_i\} \qquad \text{and} \qquad \mathrm{sur}(X_i) \leqslant p - 1 \tag{2.4}$$

(so that, if v_i were removed from E_0, then $e(X_i)$ would increase by 1, and $\mathrm{sur}(X_i)$ would become negative). For $i = 1, \ldots, t$, let $Y_i := \bigcap_{j=1}^{i} X_j$. Evidently

$$Y_i \cap E_0 = \{v_{i+1}, \ldots, v_t\}. \tag{2.5}$$

It is easy to prove by induction on i that

$$\mathrm{sur}(Y_i) \leqslant p - i. \tag{2.6}$$

For, this holds by (2.4) if $i = 1$. And if $i \geqslant 2$ then $Y_i = Y_{i-1} \cap X_i$, $\mathrm{sur}(Y_{i-1}) \leqslant p - i + 1$ by the induction hypothesis, $\mathrm{sur}(X_i) \leqslant p - 1$ by (2.4), and $E_0 \in \mathcal{E}^+(Y_{i-1}, X_i)$ so that $|\mathcal{E}^+(Y_{i-1}, X_i)| \geqslant 1$ and $\mathrm{sur}(Y_{i-1} \cup X_i) \geqslant 0$ by (1.2); thus $\mathrm{sur}(Y_i) \leqslant (p - i + 1) + (p - 1) - 0 - p = p - i$ by Lemma 2.1.

Suppose now that $t = |E_0| \geqslant p + \lceil d/q \rceil + 1$. By (2.6), $\mathrm{sur}(Y_{p+1}) < 0$. Since $\mathcal{H} \in \mathcal{C}(p, q, d)$, it follows from (1.2) that $e(Y_{p+1}) = 0$. But, by (2.5), $|Y_{p+1}| \geqslant t - p - 1 \geqslant \lceil d/q \rceil$, and so $\mathrm{sur}(Y_{p+1}) = q|Y_{p+1}| - d \geqslant q\lceil d/q \rceil - d \geqslant 0$. This contradiction proves Theorem 2.2. $\square$

Theorems 4.4 and 4.5 show that Theorem 2.2 is sharp whenever $d \leqslant 1$; specifically, $\mathrm{maxmod}(p, q, d) = p$ if $d \leqslant 0$ and $p + 1$ if $d = 1$. The next theorem completely determines the value of $\mathrm{maxmod}(p, q, d)$ when $p = 1$, and it shows that Theorem 2.2 is not sharp in general.

Theorem 2.3. *If* $d \leqslant 0$ *then* $\mathrm{maxmod}(1, q, d) = 1$. *For each integer* $k \geqslant 2$, $\mathrm{maxmod}(1, q, d) = k$ *if*

$$\frac{(k-1)^2 - 1}{k-1} q < d \leqslant \frac{k^2 - 1}{k} q.$$

Proof. For $d \leqslant 0$ the result follows from Theorem 2.2, since clearly $\mathrm{maxmod}(1, q, d) \geqslant 1$. (If all edges of $\mathcal{H}$ are empty, then (1.2) must fail if the number of edges is large enough.) For $d > 0$ the theorem states, more precisely, that

$$\mathrm{maxmod}(1, q, d) = k = 1 + \lceil d/q \rceil \qquad \text{if } \frac{(k-1)^2 - 1}{k-1} q < d \leqslant (k-1)q \tag{2.7}$$

and

$$\mathrm{maxmod}(1, q, d) = k = \lceil d/q \rceil \qquad \text{if } (k-1)q < d \leqslant \frac{k^2 - 1}{k} q. \tag{2.8}$$

The lower bound $\mathrm{maxmod}(1, q, d) \geqslant k$ in (2.8), or in (2.7) when $d = (k-1)q$, is shown by a hypergraph comprising a single edge of k vertices. The lower bound in the rest of (2.7) is shown by Theorem 4.6. The upper bound $\mathrm{maxmod}(1, q, d) \leqslant k = 1 + \lceil d/q \rceil$ in (2.7) follows directly from Theorem 2.2. We must prove that $\mathrm{maxmod}(1, q, d) \leqslant k = \lceil d/q \rceil$ in (2.8).

So suppose that d is as in (2.8) and $\mathcal{H} = (V, \mathcal{E})$ is a (p, q, d)-irreducible hypergraph containing an edge $E_0 = \{v_1, \ldots, v_t\}$ with $t \geqslant k + 1$. Let the sets X_i be defined as in the proof of Theorem 2.2. Then $\mathrm{sur}(X_i) \leqslant p - 1 = 0$ for each i, and $|X_i| \geqslant t - 1 \geqslant k$. If $e(X_i) = 0$ then $\mathrm{sur}(X_i) = q|X_i| - d \geqslant qk - d > 0$ (since $d < qk$). This contradiction shows that $e(X_i) \neq 0$ and so $\mathrm{sur}(X_i) \geqslant 0$ by (1.2). Thus $\mathrm{sur}(X_i) = 0$, for each i.

As in the proof of Theorem 2.2, $\mathrm{sur}(X_i \cap X_j) \leqslant -1$ whenever $i \neq j$, so that $e(X_i \cap X_j) = 0$. If $|X_i \cap X_j| \geqslant k$ then we get the contradiction $\mathrm{sur}(X_i \cap X_j) = q|X_i \cap X_j| - d \geqslant qk - d > 0$. But $E_0 \setminus \{v_i, v_j\} \subseteq X_i \cap X_j$, and so $|X_i \cap X_j| \geqslant |E_0| - 2 = t - 2 \geqslant k - 1$; thus equality holds throughout, and

$$X_i \cap X_j = E_0 \setminus \{v_i, v_j\} \qquad \text{whenever } i \neq j. \tag{2.9}$$

Therefore $\mathrm{sur}(X_i \cap X_j) = q(k-1) - d$, and $e(E_0 \setminus \{v_i, v_j\}) = e(X_i \cap X_j) = 0$ whenever $i \neq j$.

For $i = 1, \ldots, t$, let $U_i := \bigcup_{j=1}^{i} X_j$, and let x_i denote the number of edges of $\mathcal{H}$ that are contained in $E_0 \setminus \{v_i\}$; note that these edges are all equal (as sets) to $E_0 \setminus \{v_i\}$, by the last remark of the previous paragraph. We shall prove by induction that

$$\mathrm{sur}(U_i) \leqslant (i-1)(d - qk) + q - 1 - \sum_{j=i+1}^{t} x_j \tag{2.10}$$

for $i = 2, \ldots, t$. This holds if $i = 2$ since, by Lemma 2.1, $\mathrm{sur}(X_1 \cup X_2) = 0 + 0 - [q(k-1)-d] - |\mathcal{E}^+(X_1, X_2)|$, and $|\mathcal{E}^+(X_1, X_2)| \geqslant 1 + \sum_{j=3}^{t} x_j$ (since $E_0 \in \mathcal{E}^+(X_1, X_2)$). So suppose $i \geqslant 3$. Then $U_{i-1} \cap X_i = E_0 \setminus \{v_i\}$ by (2.9), and so $\mathrm{sur}(U_{i-1} \cap X_i) = qk - px_i - d = qk - d - x_i$. By Lemma 2.1 and the induction hypothesis,

$$\begin{aligned}\mathrm{sur}(U_i) = \mathrm{sur}(U_{i-1} \cup X_i) &\leqslant (i-2)(d-qk) + q - 1 - \sum_{j=i}^{t} x_j + 0 - (qk - d - x_i) \\ &= (i-1)(d-qk) + q - 1 - \sum_{j=i+1}^{t} x_j.\end{aligned}$$

This proves (2.10).

Finally, applying (2.10) when $i = t = k+1$ we find that $\mathrm{sur}(U_{k+1}) \leqslant k(d-qk) + q - 1 \leqslant -1$ by the upper limit for d in the statement of (2.8). But this contradicts (1.2) since $E_0 \subseteq U_{k+1}$ and so $e(U_{k+1}) > 0$; and this contradiction completes the proof of Theorem 2.3. □

Theorem 2.3 shows that, for fixed q, $\mathrm{maxmod}(1, q, d)$ is a nondecreasing function of d. However, we can prove that if p is even and $d > 0$ and equality holds in Theorem 2.2, then $d - 1$ is divisible by q. (The proof of this is too long to include here.) This shows that $\mathrm{maxmod}(p, q, d)$ is not nondecreasing if p is even, since then $\mathrm{maxmod}(p, q, d) = p$ whenever $d \leqslant q$, except that $\mathrm{maxmod}(p, q, 1) = p + 1$ as remarked before Theorem 2.3.

3. An alternative approach

In this section we adopt an alternative approach using bipartite graphs. We give an alternative proof of Theorem 2.2 when $d \leqslant 0$, and we then use the same idea to obtain further information about (p, q, d)-irreducible hypergraphs, particularly about the size of a *smallest* edge. We can use this method to prove Theorem 2.2 also when $d > 0$, but we omit the proof since it is longer and we have not managed to use it to obtain the same further information in this case.

We write $G = (S, T; E)$ to denote that G is a bipartite graph with vertex-set $V(G) = S \cup T$ and edge-set $E(G) = E$, where the order in which the partite sets S, T are written is significant. If $X \subseteq V(G)$ then $N(X) = N_G(X)$ denotes the set of all vertices that are adjacent to vertices in X. Then G is an (r, δ)-expander, i.e. it represents a hypergraph in $\mathcal{C}(r, \delta)$ as explained in Section 1, if and only if

$$|N(X)| \geqslant r|X| + \delta \tag{3.1}$$

for every nonempty finite subset $X \subseteq S$. If $\delta \leqslant 0$ then of course (3.1) holds even if $X = \emptyset$. We say that G is (r, δ)-*irreducible* if (3.1) holds in G but fails whenever any edge is removed from G.

We can now prove the following result.

Theorem 3.1. *Let p, q, d be integers such that $p, q > 0$ and $d \leqslant 0$, and let $G = (S, T; E)$ be a $(p/q, d/q)$-irreducible bipartite graph. Then:*

(a) *every vertex in* S *has degree at most* p;
(b) *every vertex in* T *has degree at most* q;
(c) *some vertex in* S *has degree at most* $\lceil p/q \rceil$;
(d) *some vertex in* T *has degree at most* $\lceil q/p \rceil$;
(e) *if* $d < 0$ *then some vertex in* S *has degree at most* $\lceil (p-1)/q \rceil$.

Proof. Because G is $(p/q, d/q)$-expanding,

$$|N_G(X)| \geqslant (p|X| + d)/q \tag{3.2}$$

for every finite subset $X \subseteq S$. Let $G_1 = (S, T^+; E_1)$ be obtained from G by replacing every $t_i \in T$ by a set $T_i = \{t_{i,1}, \ldots, t_{i,q}\}$ containing q copies of t_i, all of which are adjacent in G_1 to precisely the neighbours of t_i in G. Then (3.2) gives

$$|N_{G_1}(X)| \geqslant p|X| + d \tag{3.3}$$

for every finite subset $X \subseteq S$. Let $G_2 = (S^+, T^+; E_2)$ be obtained from G_1 by replacing every $s_i \in S$ by a set $S_i = \{s_{i,1}, \ldots, s_{i,p}\}$ containing p copies of s_i, all of which are adjacent in G_2 to precisely the neighbours of s_i in G_1. Then (3.3) gives

$$|N_{G_2}(X)| \geqslant |X| + d \tag{3.4}$$

for every finite subset $X \subseteq S^+$. Finally, form G_3 by adding $-d$ new vertices to G_2 that are adjacent to all vertices in S^+. Then $|N_{G_3}(X)| \geqslant |X|$ for every finite subset $X \subseteq S^+$, and so by Hall's theorem [2] or its transfinite extension [1] G_3 has a matching covering S^+. (For reasons explained in Section 1 in the language of irreducible hypergraphs, every vertex of S has finite degree in G, and so the result of [1] applies.)

It follows that G_2 has a matching covering all but $-d$ vertices of S^+; call a matching with this property a d-*good* matching in G_2. For a d-good matching P, let $G_2' = G_2'(P) := (S^+, T^+; P)$; then (3.4) still holds for G_2'. Let $G_1' = G_1'(P)$ be obtained from G_2' by merging the p copies of every $s_i \in S$ back into s_i. Then (3.3) still holds for G_1', since if X^+ is the subset of S^+ comprising all p copies of every vertex in X, then

$$|N_{G_1'}(X)| = |N_{G_2'}(X^+)| \geqslant |X^+| + d = p|X| + d.$$

Now let $G' = G'(P)$ be the bipartite multigraph obtained from G_1' by merging the q copies of every $t_i \in T$ back into t_i, and finally let $G'' = G''(P)$ be the simple bipartite graph obtained by identifying parallel edges in G'. Evidently (3.2) holds in G'', and G'' is a subgraph of G. Since G is $(p/q, d/q)$-irreducible, therefore $G''(P) = G$; and this holds whichever d-good matching P is chosen in G_2.

It is clear from this construction that every vertex of S has degree at most p and every vertex of T has degree at most q in $G'' = G$. This proves (a) and (b). We now turn to the proof of (c) and (d).

Claim 3.1.1. *G is a forest.*

Proof. This is obvious if $p = 1$ or $q = 1$, so suppose $p \geqslant 2$ and $q \geqslant 2$. Suppose G contains a circuit $C : s_1, t_1, \ldots, s_k, t_k, s_1$. Let $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be any function that increases sufficiently rapidly that $f(n+1) - f(n) > k[f(i) - f(i-1)]$ whenever $1 \leqslant i \leqslant n \in \mathbb{N}$ (e.g., $f(n) := 2^{kn}$ for all n). For each edge e of G, let $\mu_{G'(P)}(e)$

denote the multiplicity of the set of edges of $G'(P)$ corresponding to e, and among all d-good matchings in G_2 let P be one that maximizes the sum $\sum_{e\in C} f(\mu_{G'(P)}(e))$. W.l.o.g. assume $\mu_{G'(P)}(s_1t_1) \geqslant \mu_{G'(P)}(e)$ for all other edges e of C. For each edge e of C, choose an edge of P that maps into e when $G = G''(P)$ is constructed as above from $G'_2(P)$; let the chosen edges be

$$s_{1,h_1}t_{1,i_1},\ t_{1,j_1}s_{2,l_2},\ s_{2,h_2}t_{2,i_2},\ t_{2,j_2}s_{3,l_3},\dots,s_{k,h_k}t_{k,i_k},\ t_{k,j_k}s_{1,l_1}.$$

Replacing the edges

$$t_{1,j_1}s_{2,l_2},\ t_{2,j_2}s_{3,l_3},\dots,t_{k,j_k}s_{1,l_1}$$

of P by the edges

$$s_{1,l_1}t_{1,j_1},\ s_{2,l_2}t_{2,j_2},\dots,s_{k,l_k}t_{k,j_k}$$

gives another d-good matching P' in G_2 such that $\sum_{e\in C} f(\mu_{G'(P')}(e)) > \sum_{e\in C} f(\mu_{G'(P)}(e))$. (This replacement is possible since every edge e of G corresponds to a copy of $K_{p,q}$ in G_2; thus since $s_{1,h_1}t_{1,i_1} \in E_2$ it follows that $s_{1,l_1}t_{1,j_1} \in E_2$, etc.) This contradiction shows that there can be no such circuit C, and so Claim 3.1.1 is proved. □

In proving (c) we may assume that all isolated vertices and endvertices of the forest G are in T, since otherwise (c) clearly holds. Choose a vertex $s_0 \in S$ such that at most one neighbour of s_0 in G is not an endvertex of G; such a vertex s_0 must exist, in any forest. Suppose $N_G(s_0) = \{t_1,\dots,t_k\}$, where $t_1,\dots,t_{k-1}$ are endvertices of G. Start with an arbitrary d-good matching P_1 in G_2. For $i = 1,\dots,k-1$ in turn, if $\nu_i := \max\{p,q\} - \mu_{G'(P_i)}(s_0t_i) > 0$, then form a d-good matching P_{i+1} from P_i by removing ν_i edges of P_i between S_0 and T_k and replacing them with ν_i edges between the same vertices of S_0 and vertices of T_i that are not matched by P_i. (If there are not as many as ν_i edges of P_i between S_0 and T_k then replace all there are, and observe that $G''(P_{i+1})$ is then a proper subgraph of G (missing the edge s_0t_k), which is a contradiction.) Then $\mu_{G'(P_k)}(s_0t_i) = \max\{p,q\}$ for $i = 1,\dots,k-1$. If $p < q$ then in $G'(P_k)$ all p edges from s_0 go to t_1, and so s_0 has degree $1 = \lceil p/q\rceil$ in G. Otherwise q of the p edges incident with s_0 in $G'(P_k)$ go to t_i for each $i = 1,\dots,k-1$ and so s_0 has degree $k = \lceil p/q\rceil$ in G.

This proves (c). The proof of (d) is exactly the same but with the roles of S and T interchanged.

To prove (e), let P be a d-good matching in G_2, where now $d < 0$. Choose a vertex $s_1 \in S$ such that the corresponding set $S_1 = \{s_{1,1},\dots,s_{1,p}\}$ of vertices in S^+ contains one of the $-d$ vertices that is not matched by P; then s_1 has degree at most $p-1$ in the multigraph $G'(P)$. If $p = 1$ then s_1 has degree 0 in $G'(P)$ and hence in $G''(P) = G$, which is all we have to prove; so we may assume $p > 1$. Then, as in the proof of (c), we may assume that all isolated vertices and endvertices of the forest G are in T. Let s_0 be a vertex in the same component of G as s_1 such that at most one neighbour of s_0 in G is not an endvertex of G. If $s_0 \neq s_1$, let the path from s_1 to s_0 in G have vertices $s_1, t_1, \dots, s_k, t_k, s_0$. For each edge e of this path, choose an edge of P that maps into e when $G = G''(P)$ is constructed as above from $G'_2(P)$; let the chosen edges be

$$s_{1,h_1}t_{1,i_1},\ t_{1,j_1}s_{2,l_2},\ s_{2,h_2}t_{2,i_2},\ t_{2,j_2}s_{3,l_3},\dots,s_{k,h_k}t_{k,i_k},\ t_{k,j_k}s_{0,l_0}.$$

Let s_{1,l_1} be a vertex of S_1 that is not matched by P. Replacing the edges

$$t_{1,j_1}s_{2,l_2},\ t_{2,j_2}s_{3,l_3},\ldots,t_{k,j_k}s_{0,l_0}$$

of P by the edges

$$s_{1,l_1}t_{1,j_1},\ s_{2,l_2}t_{2,j_2},\ldots,s_{k,l_k}t_{k,j_k}$$

gives another d-good matching P' in G_2 in which s_{1,l_1} is matched but s_{0,l_0} is not. Thus, in $G_2'(P')$, there are at most $p-1$ edges incident with vertices in S_0. If we now apply the argument used above to prove (c), then we find that s_0 has degree at most $k = \lceil (p-1)/q \rceil$ in G. This completes the proof of (e), and so of Theorem 3.1. $\square$

Note that there is no part '(f)' in Theorem 3.1, saying that if $d < 0$ then there is a vertex in T with degree at most $\lceil (q-1)/p \rceil$. For example, the path *abcde* with $S = \{a, c, e\}$ and $T = \{b, d\}$ is a $(3/4, -1/4)$-irreducible bipartite graph, and each of b, d has degree 2, but $2 > \lceil (4-1)/3 \rceil = 1$.

The following corollary states that parts (a), (c) and (e) of Theorem 3.1 are sharp. Let us write $\mathrm{maxminmod}(p, q, d)$ for the maximum value of the minimum degree of all vertices in S, where the maximum is taken over all $(p/q, d/q)$-irreducible bipartite graphs $(S, T; E)$. Equivalently, $\mathrm{maxminmod}(p, q, d)$ is the maximum size of the smallest edge in $\mathcal{E}$, where the maximum is taken over all (p, q, d)-irreducible hypergraphs $\mathcal{H} = (V, \mathcal{E})$.

Corollary 3.2. *Let p, q, d be integers such that p and q are positive and coprime and $d \leqslant 0$. Then* $\mathrm{maxmod}(p, q, d) = p$, $\mathrm{maxminmod}(p, q, 0) = \lceil p/q \rceil$, *and* $\mathrm{maxminmod}(p, q, d) = \lceil (p-1)/q \rceil$ *if $d < 0$.*

Proof. The first statement follows from Theorem 3.1(a) and Theorem 4.4, and the third follows from Theorem 3.1(e) and Theorem 4.1. The second statement follows from Theorem 3.1(c), since it is clear from putting $X = \{v\}$ in (3.2) that $\mathrm{maxminmod}(p, q, d) \geqslant (p+d)/q = p/q$ if $d = 0$. $\square$

Finally, we consider the case of irrational r.

Corollary 3.3. *If r is irrational and $\delta \leqslant 0$, then for every finite (r, δ)-irreducible bipartite graph $G = (S, T; E)$ there is a vertex in S with degree at most $\lceil r \rceil$. This is sharp.*

Proof. The set of numbers

$$\{\lceil r \rceil - r\} \cup \{ri + \delta - j : i = 0, \ldots, |S|,\ j = 0, \ldots, |T|\}$$

is a discrete set that may or may not contain 0. Choose an integer q sufficiently large that every positive number in the set is greater than $(|S|+1)/q$. Choose integers p and d such that

$$(p-1)/q < r < p/q \qquad \text{and} \qquad (d+|S|)/q \leqslant \delta < (d+|S|+1)/q.$$

Then $\lceil r \rceil = \lceil p/q \rceil$, since $\lceil r \rceil - r > 1/q$. Also, for $i = 0, \ldots, |S|$ and $j = 0, \ldots, |T|$,

$$(pi+d)/q + (|S|-i)/q < ri + \delta < (pi+d)/q + (|S|+1)/q,$$

so that $(pi+d)/q < ri+\delta$, and $j \geqslant (pi+d)/q$ if and only if $j \geqslant ri+\delta$ (since if $j \geqslant (pi+d)/q$ then $ri+\delta-j < (|S|+1)/q$ and so $ri+\delta-j \leqslant 0$). So G is

$(p/q, d/q)$-irreducible. Thus it follows from Theorem 3.1(c) that there is a vertex in S with degree at most $\lceil p/q \rceil = \lceil r \rceil$. The sharpness of this result is proved in Theorem 4.2. □

If $\delta < 0$ then Corollary 3.3 holds for infinite (r, δ)-irreducible bipartite graphs as well. In this case we can prove that for some δ', $\delta \leqslant \delta' < 0$, G has an (r, δ')-irreducible subgraph G_0 that is the union of finitely many finite components of G. The result then follows on applying Corollary 3.3 to G_0. This does not seem to work if $\delta = 0$, since then we might have to take $\delta' > 0$, and we have not proved anything about the minimum degree of vertices in S when $\delta > 0$.

4. The lower bounds

The hypergraphs that we construct in this section may apparently have multiple edges. The question of whether they can be taken to be simple is discussed in the final section, in and before Theorem 6.2.

We start with the lower bounds on the maximum size of a smallest edge when $\delta < 0$. For positive integers t, m and n, let $\mathcal{H} = (V, \mathcal{E}) = \mathcal{H}(t, m, n)$ be a hypergraph in which V is the union of t disjoint sets $Z_1, \ldots, Z_t$, each of cardinality n, and $\mathcal{E}$ comprises m copies of every set Z_i; thus $|V| = tn$ and $|\mathcal{E}| = tm$.

Theorem 4.1. *If p and q are positive coprime integers and $d < 0$, then there is a (p, q, d)-irreducible hypergraph in which every edge has at least $\lceil (p-1)/q \rceil$ vertices.*

Proof. Since p and q are coprime, there exist positive integers m and n such that $qn = pm - 1$. Let $\mathcal{H} = \mathcal{H}(-d, m, n)$. To prove that $\mathcal{H} \in \mathcal{C}(p, q, d)$, it suffices to consider the set $\mathcal{F}_i$ of edges contained in $Z_1 \cup \cdots \cup Z_i$ $(1 \leqslant i \leqslant -d)$. But $|\mathcal{F}_i| = im$ and

$$\left|\bigcup \mathcal{F}_i\right| = in = \frac{i(pm-1)}{q} = \frac{p|\mathcal{F}_i| - i}{q} \geqslant \frac{p|\mathcal{F}_i| + d}{q}.$$

It follows that $\mathcal{H} \in \mathcal{C}(p, q, d)$. So one can form an irreducible hypergraph $\mathcal{H}' \in \mathcal{C}(p, q, d)$ by removing vertices from edges of $\mathcal{H}$. Suppose that $\mathcal{H}'$ contains an edge e with fewer than $(p-1)/q$ vertices. W.l.o.g. $e \in Z_1$, so let $\mathcal{F}$ be the set of all edges of $\mathcal{H}'$ contained in $e \cup Z_2 \cup \cdots \cup Z_{-d}$. Then $|\mathcal{F}| \geqslant 1 + m(-d-1)$ and

$$\left|\bigcup \mathcal{F}\right| < \frac{p-1}{q} + n(-d-1) = \frac{p - 1 + (pm-1)(-d-1)}{q} \leqslant \frac{p|\mathcal{F}| + d}{q},$$

which is impossible since $\mathcal{H}' \in \mathcal{C}(p, q, d)$. Thus every edge of $\mathcal{H}'$ has at least $(p-1)/q$ vertices, as required. □

The next theorem is the analogous result for irrational r. Theorem 3.1(e) and Corollary 3.3 show that Theorems 4.1 and 4.2, respectively, are best possible.

Theorem 4.2. *If r is a positive irrational number and $\delta \leqslant 0$, then there is an irreducible hypergraph in $\mathcal{C}(r, \delta)$ in which every edge has at least $\lceil r \rceil$ vertices.*

Proof. This is obvious if $\delta = 0$ (take $|\mathcal{F}| = 1$ in (1.1)), so suppose $\delta < 0$. Let $\alpha := r - \lfloor r \rfloor > 0$. Let q be a positive integer sufficiently large that $1/q < \alpha/(-\delta)$, so that

$(q-1)/q > (-\alpha-\delta)/(-\delta)$. Since r is irrational, numbers of the form $mr-n$ $(m, n \in \mathbb{N})$ are dense in $\mathbb{R}$. So let m, n be positive integers such that

$$\frac{-\alpha-\delta}{q-1} < mr-n < \frac{-\delta}{q}.$$

The proof now follows the argument of Theorem 4.1. Let $\mathcal{H} = \mathcal{H}(q, m, n)$. If $\mathcal{F}_i$ is the set of edges contained in $Z_1 \cup \cdots \cup Z_i$ $(1 \leqslant i \leqslant q)$ then $|\mathcal{F}_i| = im$ and

$$\left|\bigcup \mathcal{F}_i\right| = in > imr + i\delta/q \geqslant imr + \delta = r|\mathcal{F}_i| + \delta.$$

Thus $\mathcal{H} \in \mathcal{C}(r, \delta)$. Forming an irreducible hypergraph $\mathcal{H}' \in \mathcal{C}(r, \delta)$ from $\mathcal{H}$ as in the proof of Theorem 4.1, we see that if $\mathcal{H}'$ contains an edge with $\lfloor r \rfloor$ or fewer vertices then $\mathcal{H}'$ contains a set $\mathcal{F}$ of at least $1 + m(q-1)$ edges such that

$$\begin{aligned}\left|\bigcup \mathcal{F}\right| \leqslant \lfloor r \rfloor + n(q-1) < \lfloor r \rfloor + mr(q-1) + \alpha + \delta &= r[1 + m(q-1)] + \delta \\ &\leqslant r|\mathcal{F}| + \delta,\end{aligned}$$

which is impossible since $\mathcal{H}' \in \mathcal{C}(r, \delta)$. Thus every edge of $\mathcal{H}'$ has at least $\lceil r \rceil$ vertices, as required. $\square$

We now turn to the lower bounds on the maximum size of a largest edge. For nonnegative integers t, m, n, m', n', where $t > 0$ and $n > 0$, we construct a hypergraph $\mathcal{H} = (V, \mathcal{E}) = \mathcal{H}(t, m, n, m', n')$ as follows. Let V be the disjoint union of sets $Y, Z_1, \ldots, Z_t$, where $|Y| = n'$ and $|Z_i| = n$ for each i. Let $\mathcal{E}$ comprise the following edges: m' copies of Y, m copies of $Y \cup Z_i$ for each i, and an edge E_0 containing one vertex from each set Z_i. Then $\mathcal{H}(t, m, n, m', n')$ has $tn + n'$ vertices and $tm + m' + 1$ edges, and $|E_0| = t$.

We first use this construction to deal with the case when r is irrational.

Theorem 4.3. *If r is a positive irrational number and δ is an arbitrary real number, then irreducible hypergraphs in $\mathcal{C}(r, \delta)$ can contain arbitrarily large edges.*

Proof. Let $t \in \mathbb{N}$, $t \geqslant \max\{2, r+\delta\}$. We shall prove that there is an irreducible hypergraph in $\mathcal{C}(r, \delta)$ containing an edge with t vertices. Since r is irrational, numbers of the form $n - rm$ $(m, n \in \mathbb{N})$ are dense in $\mathbb{R}$. So let m, n, m', n' be positive integers and define $\epsilon := r + \delta - (n' - rm')$, where m, n, m', n' are chosen so that

$$0 < \epsilon < \min\{r, t\} \qquad \text{and} \qquad \frac{\epsilon}{t} < n - rm < \frac{\epsilon}{t-1}. \tag{4.1}$$

Let $\mathcal{H} := \mathcal{H}(t, m, n, m', n')$. Note that $|E_0| = t \geqslant r + \delta$.

We shall prove first that $\mathcal{H} \in \mathcal{C}(r, \delta)$. $\mathcal{H}$ has m' edges that are copies of Y, and $|Y| = n' = r + \delta + rm' - \epsilon > rm' + \delta = re_{\mathcal{H}}(Y) + \delta$. To complete the proof that $\mathcal{H} \in \mathcal{C}(r, \delta)$ it suffices to consider the set $\mathcal{F}_i$ of all edges contained in $Y \cup Z_1 \cup \cdots \cup Z_i$ $(1 \leqslant i \leqslant t-1)$ and the set $\mathcal{F}_i' := \mathcal{F}_i \cup \{E_0\}$ $(1 \leqslant i \leqslant t)$. Note that, by (4.1),

$$in + n' > i\left(\frac{\epsilon}{t} + rm\right) + (r + \delta + rm' - \epsilon) = r(im + m' + 1) + \delta - \frac{(t-i)\epsilon}{t}.$$

Now, $|\mathcal{F}_i| = im + m'$ if $i < t$, and

$$\left|\bigcup \mathcal{F}_i\right| = in + n' > r|\mathcal{F}_i| + r + \delta - \frac{(t-i)\epsilon}{t} > r|\mathcal{F}_i| + \delta$$

since $\epsilon < r$; and $|\mathcal{F}_i'| = im + m' + 1$ and

$$\left|\bigcup \mathcal{F}_i'\right| = in + n' + t - i > r|\mathcal{F}_i'| + \delta + \frac{(t-i)(t-\epsilon)}{t} \geqslant r|\mathcal{F}_i'| + \delta$$

since $\epsilon < t$. It follows that $\mathcal{H} \in \mathcal{C}(r, \delta)$.

Now let $\mathcal{H}'$ be obtained from $\mathcal{H}$ be deleting one vertex from the edge E_0, say the vertex in $E_0 \cap Z_t$, and let $\mathcal{F}$ consist of all edges contained in $V(\mathcal{H}') \setminus Z_t$. Then $|\mathcal{F}| = (t-1)m + m' + 1$ and, by (4.1),

$$\left|\bigcup \mathcal{F}\right| = (t-1)n + n' < (\epsilon + (t-1)rm) + (r + \delta + rm' - \epsilon) = r|\mathcal{F}| + \delta.$$

It follows that $\mathcal{H}' \notin \mathcal{C}(r, \delta)$.

Now, $\mathcal{H}$ is not an irreducible member of $\mathcal{C}(r, \delta)$, but one can form an irreducible member $\mathcal{H}''$ of $\mathcal{C}(r, \delta)$ by removing vertices from edges of $\mathcal{H}$. Since $\mathcal{H}' \notin \mathcal{C}(r, \delta)$, $\mathcal{H}''$ must contain the edge E_0 with t vertices. This completes the proof of Theorem 4.3. $\square$

We now use the same construction to prove a universal lower bound for rational r.

Theorem 4.4. *If p and q are positive coprime integers and d is an arbitrary integer, then* $\text{maxmod}(p, q, d) \geqslant p$.

Proof. The structure of the proof is very similar to that of the previous theorem. Since p and q are coprime, there exist nonnegative integers m, n, m', n' such that $qn = pm + 1$ and $qn' = pm' + d$. Let $\mathcal{H} := \mathcal{H}(p, m, n, m', n')$. We may assume that $|E_0| = p > (p+d)/q$, since clearly $\text{maxmod}(p, q, d) \geqslant (p+d)/q$ and so the result of the theorem is obvious if $p \leqslant (p+d)/q$.

We shall prove first that $\mathcal{H} \in \mathcal{C}(p, q, d)$. $\mathcal{H}$ has m' edges that are copies of Y, and $|Y| = n' = (pm' + d)/q = (pe_{\mathcal{H}}(Y) + d)/q$. To complete the proof that $\mathcal{H} \in \mathcal{C}(p, q, d)$, as in Theorem 4.3 it suffices to consider the set $\mathcal{F}_i$ of edges contained in $Y \cup Z_1 \cup \cdots \cup Z_i$ $(1 \leqslant i \leqslant p-1)$ and the set $\mathcal{F}_i' := \mathcal{F}_i \cup \{E_0\}$ $(1 \leqslant i \leqslant p)$. Now, $|\mathcal{F}_i| = im + m'$ if $i < p$, and

$$\begin{aligned}\left|\bigcup \mathcal{F}_i\right| &= in + n' = \frac{i(pm+1) + (pm'+d)}{q} > \frac{p(im+m') + d}{q}\\ &= \frac{p|\mathcal{F}_i| + d}{q};\end{aligned} \tag{4.2}$$

and $|\mathcal{F}_i'| = im + m' + 1$ and

$$\begin{aligned}\left|\bigcup \mathcal{F}_i'\right| &= in + n' + p - i = \frac{i(pm+1) + (pm'+d) + q(p-i)}{q}\\ &= \frac{p(im+m'+1) + d + (q-1)(p-i)}{q} \geqslant \frac{p|\mathcal{F}_i'| + d}{q}.\end{aligned}$$

It follows that $\mathcal{H} \in \mathcal{C}(p, q, d)$.

Now let $\mathcal{H}'$ be obtained from $\mathcal{H}$ be deleting one vertex from the edge E_0, say the vertex in $E_0 \cap Z_p$, and let $\mathcal{F}$ consist of all edges contained in $V(\mathcal{H}') \setminus Z_p$. Then $|\mathcal{F}| = (p-1)m + m' + 1$ and

$$\left|\bigcup \mathcal{F}\right| = (p-1)n + n' = \frac{(p-1)(pm+1) + (pm'+d)}{q}$$
$$= \frac{p((p-1)m + m' + 1) + d - 1}{q} < \frac{p|\mathcal{F}| + d}{q}.$$

It follows that $\mathcal{H}' \notin \mathcal{C}(p, q, d)$.

As in the proof of Theorem 4.3, one can form an irreducible member $\mathcal{H}''$ of $\mathcal{C}(p, q, d)$ by removing vertices from edges of $\mathcal{H}$, and since $\mathcal{H}' \notin \mathcal{C}(p, q, d)$, $\mathcal{H}''$ must contain the edge E_0 with p vertices. This completes the proof of Theorem 4.4. $\square$

We can improve slightly on the above lower bound in the case when $d \equiv 1 \pmod q$. If $d = 1$, then Theorems 2.2 and 4.5 together show that $\operatorname{maxmod}(p, q, 1) = p + 1$.

Theorem 4.5. *If p and q are positive coprime integers and $d \geqslant 1$ and $d \equiv 1 \pmod q$ then* $\operatorname{maxmod}(p, q, d) \geqslant p + 1$.

Proof. The proof is a simpler version of the previous proof. Since p and q are coprime, there exist positive integers m, n such that $qn = pm + 1$. Let $qn' = d - 1$ and $\mathcal{H} := \mathcal{H}(p + 1, m, n, 0, n')$. We may assume that $|E_0| = p + 1 > (p + d)/q$, since the result is obvious if $p + 1 \leqslant (p + d)/q$.

We shall first prove that $\mathcal{H} \in \mathcal{C}(p, q, d)$. Let $\mathcal{F}_i$ be the set of edges contained in $Y \cup Z_1 \cup \cdots \cup Z_i$ and let $\mathcal{F}_i' := \mathcal{F}_i \cup \{E_0\}$ $(1 \leqslant i \leqslant p + 1)$. Now, $|\mathcal{F}_i| = im$ if $i \leqslant p$, and

$$\left|\bigcup \mathcal{F}_i\right| = in + n' = \frac{i(pm+1) + (d-1)}{q} \geqslant \frac{p(im) + d}{q} = \frac{p|\mathcal{F}_i| + d}{q};$$

and $|\mathcal{F}_i'| = im + 1$ and

$$\left|\bigcup \mathcal{F}_i'\right| = in + n' + p + 1 - i = \frac{i(pm+1) + (d-1) + q(p+1-i)}{q}$$
$$= \frac{p(im+1) + d + (q-1)(p+1-i)}{q}$$
$$\geqslant \frac{p|\mathcal{F}_i'| + d}{q}.$$

It follows that $\mathcal{H} \in \mathcal{C}(p, q, d)$.

Now let $\mathcal{H}'$ by obtained from $\mathcal{H}$ be deleting one vertex from the edge E_0, say the vertex in $E_0 \cap Z_{p+1}$, and let $\mathcal{F}$ consist of all edges contained in $V(\mathcal{H}') \setminus Z_{p+1}$. Then $|\mathcal{F}| = pm + 1$ and

$$\left|\bigcup \mathcal{F}\right| = pn + n' = \frac{p(pm+1) + (d-1)}{q} < \frac{p|\mathcal{F}| + d}{q}.$$

It follows that $\mathcal{H}' \notin \mathcal{C}(p, q, d)$.

As before, one can form an irreducible member $\mathcal{H}''$ of $\mathcal{C}(p, q, d)$ by removing vertices from edges of $\mathcal{H}$, and since $\mathcal{H}' \notin \mathcal{C}(p, q, d)$, $\mathcal{H}''$ must contain the edge E_0 with $p + 1$ vertices. This completes the proof of Theorem 4.5. $\square$

The construction in the next theorem is somewhat different. It works for arbitrarily large d, but it is only interesting if $t > \lceil (p + d)/q \rceil$, which explains the upper bound given for d in the statement of the theorem. The theorem is nonvacuous (i.e. the range of values of d is nonempty) if and only if $t \leqslant q$, when $\lceil (p + d)/q \rceil < q$.

Theorem 4.6. *If p, q, d, t are positive integers such that p, q are coprime and $t > (p/q) + 1$ and*

$$qt - p - \frac{(q-1)t}{t-1} = q(t-1) - p + 1 - \frac{q-1}{t-1} \leqslant d \leqslant q(t-1) - p, \tag{4.3}$$

then $\operatorname{maxmod}(p, q, d) \geqslant t = \lceil (p + d)/q \rceil + 1$.

Proof. Since p and q are coprime, there exist positive integers m, n such that $qn = p(m + 1) + d - qt + q - 1$. Let V be the disjoint union of sets $E_0, Z_1, \ldots, Z_t$, where $E_0 = \{v_1, \ldots, v_t\}$ and $|Z_i| = n$ for each i. Let $\mathcal{H} = (V, \mathcal{E})$ where $\mathcal{E}$ comprises one copy of E_0 and m copies of $Z_i \cup E_0 \setminus \{v_i\}$ for each i.

We first prove that $\mathcal{H} \in \mathcal{C}(p, q, d)$. Let $\mathcal{F}_i$ be the set of edges contained in $E_0 \cup Z_1 \cup \cdots \cup Z_i$ $(1 \leqslant i \leqslant t)$ and let $\mathcal{F}_1'$ be the set of edges contained in $Z_1 \cup E_0 \setminus \{v_1\}$. Then $|\mathcal{F}_i| = im + 1$ and

$$\begin{aligned} q\left|\bigcup \mathcal{F}_i\right| = q(in + t) &= i[p(m+1) + d - qt + q - 1] + qt \\ &= p(im+1) + d + (i-1)(p + d - qt) + i(q-1) \\ &\geqslant p(im+1) + d - \frac{(i-1)(q-1)t}{t-1} + i(q-1) \qquad \text{by (4.3)} \\ &= p|\mathcal{F}_i| + d + \frac{(t-i)(q-1)}{t-1} \\ &\geqslant p|\mathcal{F}_i| + d \end{aligned}$$

since $i \leqslant t$. Also $|\mathcal{F}_1'| = m$ and

$$\left|\bigcup \mathcal{F}_1'\right| = n + t - 1 = \frac{p(m+1) + d - 1}{q} \geqslant \frac{pm + d}{q} = \frac{p|\mathcal{F}_1'| + d}{q}.$$

It follows that $\mathcal{H} \in \mathcal{C}(p, q, d)$.

Now let $\mathcal{H}'$ be obtained from $\mathcal{H}$ be deleting one vertex, say v_1, from the edge E_0, and let $\mathcal{F}$ comprise all edges of $\mathcal{H}'$ contained in $Z_1 \cup E_0 \setminus \{v_1\}$. Then $|\mathcal{F}| = m + 1$ and

$$\left|\bigcup \mathcal{F}\right| = n + t - 1 = \frac{p(m+1) + d - 1}{q} < \frac{p(m+1) + d}{q} = \frac{p|\mathcal{F}| + d}{q}.$$

It follows that $\mathcal{H}' \notin \mathcal{C}(p, q, d)$.

As before, one can form an irreducible member $\mathcal{H}''$ of $\mathcal{C}(p, q, d)$ by removing vertices from edges of $\mathcal{H}$, and since $\mathcal{H}' \notin \mathcal{C}(p, q, d)$, $\mathcal{H}''$ must contain the edge E_0 with t vertices. This completes the proof of Theorem 4.6. $\square$

5. Arc-minimal digraph expanders

Let $D = (V, A)$ be a digraph with vertex-set $V(D) = V$ and arc-set $A(D) = A$. We say that D is *arc-minimal* in a class $\mathcal{C}$ if $D \in \mathcal{C}$ but, for each arc $a \in A$, $D - a \notin \mathcal{C}$. If $v \in V$ and $X \subseteq V$, then $d^-(v)$ and $d^+(v)$ denote the indegree and outdegree of v, and $N^+(X)$ denotes the set of vertices w such that there exists an arc $\overrightarrow{uw} \in A$ with $u \in X$. We say that X is *independent* if no arc has both its head and its tail in X.

By an (r, δ)-*expanding digraph* we mean a triple $D = (V, A, S)$, where (V, A) is a digraph (which, by an abuse of terminology, we also call D), $S \subseteq V$, and

$$|N^+(X) \setminus X| \geqslant r|X| + \delta \qquad \text{whenever } \emptyset \neq X \subseteq S \text{ and } |X| < \infty. \tag{5.1}$$

(This might perhaps be described as a *regional expander*, since the condition (5.1) holds only for sets X in a certain region, namely for subsets of S, rather than for all subsets of V that are not too large, as is often the case in other contexts. However, S could be the whole of V, if V is infinite or if $r|V| + \delta \leqslant 0$.) As remarked in Section 1, if D is bipartite with bipartition (S, T) and all arcs directed from S towards T, then we recover the bipartite model of hypergraphs used in Section 3.

It is easy to see that every vertex in an arc-minimal (r, δ)-expanding digraph has finite outdegree, by an analogous argument to the one used in Section 1 to show that the irreducible hypergraphs in $\mathcal{C}(r, \delta)$ can have no infinite edges. Also, as already remarked in Section 1, it follows from the corresponding examples for hypergraphs (Theorem 4.3) that if r is irrational then there are bipartite arc-minimal (r, δ)-expanding digraphs $D = (V, A, S)$ in which S contains vertices with arbitrarily large outdegree. We shall see that this is not true if r is rational.

If $p \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{N}$ and $d \in \mathbb{Z}$, let $\text{maxdeg}(p, q, d)$ denote the largest outdegree that is possible for a vertex in the set S of an arc-minimal $(p/q, d/q)$-expanding digraph $D = (V, A, S)$, i.e. one that is arc-minimal subject to the condition

$$|N^+(X) \setminus X| \geqslant \frac{p|X| + d}{q} \qquad \text{whenever } \emptyset \neq X \subseteq S \text{ and } |X| < \infty. \tag{5.2}$$

Clearly (taking $|X| = 1$) (5.2) forces every vertex of S to have outdegree at least $\lceil (p + d)/q \rceil$.

The next two theorems are the digraph analogues of Theorems 2.2 and 4.4 for hypergraphs. They show that if p and q are coprime then $\text{maxdeg}(p, q, d) = p + q$ if $d \leqslant q$, and $\max\{p + q, \lceil (p + d)/q \rceil\} \leqslant \text{maxdeg}(p, q, d) \leqslant p + q + \lceil d/q \rceil - 1$ if $d > q$. The proof given below for Theorem 5.1(a) is an alternative proof of Theorem 2.2.

Theorem 5.1. *Let $D = (V, A, S)$ be an arc-minimal $(p/q, d/q)$-expanding digraph, and let $v \in S$.*

(a) *If $d^-(v) = 0$ then $d^+(v) \leqslant \max\{p, p + \lceil d/q \rceil\}$.*
(b) $\text{maxdeg}(p, q, d) \leqslant \max\{p + q, p + q + \lceil d/q \rceil - 1\}$.

Proof. Let $v_1, \ldots, v_t$ be the outneighbours of v. By the arc-minimality of D, there are finite sets $X_1, \ldots, X_t \subseteq S$ containing v such that, for each i,

$$|N^+(X_i) \setminus X_i| - 1 < (p|X_i| + d)/q, \tag{5.3}$$

$v_i \in N^+(X_i) \setminus X_i$, and v is the only vertex in X_i from which there is an arc going to v_i (so that, if the arc vv_i were removed from D, then (5.2) would fail; the sets X_i need not be distinct). If $1 \leqslant j \leqslant s \leqslant t$, let $W_j^{(s)}$ consist of all vertices of V that are in at least j of the sets $X_1, \ldots, X_s$. We claim that

$$\sum_{i=1}^{s} |N^+(X_i) \setminus X_i| \geqslant \sum_{j=1}^{s} |N^+(W_j^{(s)}) \setminus W_j^{(s)}|. \tag{5.4}$$

For, consider a typical vertex $w \in V$. Let there be h sets X_i such that $w \in X_i$, and k sets X_i such that $w \in N^+(X_i) \setminus X_i$ $(1 \leqslant i \leqslant s)$. Then w contributes k to the LHS of (5.4), and it contributes at most k to the RHS, since it can contribute to the RHS only when $j \in \{h+1, \ldots, h+k\}$. (If $j \leqslant h$ then $w \in W_j^{(s)}$, and if $\overrightarrow{uw} \in A$ then u can be in at most k sets X_i in addition to the h sets that contain w, so that $u \notin W_j^{(s)}$ if $j > h+k$.) This proves (5.4).

Now, for each i, (5.3) implies that

$$|N^+(X_i) \setminus X_i| \leqslant (p|X_i| + d + q - 1)/q;$$

and for each j, $|N^+(W_j^{(s)}) \setminus W_j^{(s)}| \geqslant (p|W_j^{(s)}| + d)/q$. Since $\sum_{i=1}^{s} |X_i| = \sum_{j=1}^{s} |W_j^{(s)}|$, it follows from (5.4) that

$$\begin{aligned} |N^+(W_s^{(s)}) \setminus W_s^{(s)}| &\leqslant \sum_{i=1}^{s} |N^+(X_i) \setminus X_i| - \sum_{j=1}^{s-1} |N^+(W_j^{(s)}) \setminus W_j^{(s)}| \\ &\leqslant \sum_{i=1}^{s} (p|X_i| + d + q - 1)/q - \sum_{j=1}^{s-1} (p|W_j^{(s)}| + d)/q \\ &= [p|W_s^{(s)}| + d + s(q-1)]/q. \end{aligned} \tag{5.5}$$

Note that $v \in W_s^{(s)}$ and $v_i \in N^+(\{v\}) \setminus X_i$ for each i $(1 \leqslant i \leqslant s)$, and so

$$\{v_1, \ldots, v_s\} \subseteq N^+(W_s^{(s)}) \setminus W_s^{(s)}. \tag{5.6}$$

To prove (a), suppose on the contrary that $t \geqslant \max\{p, p + \lceil d/q \rceil\} + 1$. In this case we take $s = p+1$. If $|W_s^{(s)}| = 1$ then $W_s^{(s)} = \{v\}$, $N^+(W_s^{(s)}) \setminus W_s^{(s)} = \{v_1, \ldots, v_t\}$, and (5.5) gives the contradiction

$$p + \lceil d/q \rceil + 1 \leqslant t \leqslant [p + d + (p+1)(q-1)]/q = p + (d-1)/q + 1.$$

Thus $|W_s^{(s)}| \geqslant 2$. Let $X := W_s^{(s)} \setminus \{v\} \neq \emptyset$. For each i $(1 \leqslant i \leqslant s)$, $W_s^{(s)} \subseteq X_i$, and v is the only vertex of X_i from which there is an arc going to v_i. It follows from this and (5.5) and (5.6) that

$$\begin{aligned} |N^+(X) \setminus X| \leqslant |N^+(W_s^{(s)}) \setminus W_s^{(s)}| - s &\leqslant (p|W_s^{(s)}| + d - s)/q \\ &= (p|X| + d - 1)/q \end{aligned}$$

since $s = p+1$. This contradicts (5.2), and this contradiction proves (a).

To prove (b), suppose on the contrary that v can be chosen so that $t \geqslant \max\{p+q, p + q + \lceil d/q \rceil - 1\} + 1$. In this case we take $s = p+q+1$. If $|W_s^{(s)}| = 1$ then

$W_s^{(s)} = \{v\}$, $N^+(W_s^{(s)}) \setminus W_s^{(s)} = \{v_1, \dots, v_t\}$, and (5.5) gives the contradiction

$$p + q + \lceil d/q \rceil \leqslant t \leqslant [p + d + (p + q + 1)(q - 1)]/q = p + q + (d - 1)/q.$$

Thus $|W_s^{(s)}| \geqslant 2$. Let $X := W_s^{(s)} \setminus \{v\} \neq \emptyset$. For each i $(1 \leqslant i \leqslant s)$, $W_s^{(s)} \subseteq X_i$, and v is the only vertex of X_i from which there is an arc going to v_i. However, it is possible now that $v \in N^+(X) \setminus X$. Thus, by (5.5) and (5.6),

$$\begin{aligned} |N^+(X) \setminus X| &\leqslant |N^+(W_s^{(s)}) \setminus W_s^{(s)}| - s + 1 \leqslant (p|W_s^{(s)}| + d - s + q)/q \\ &= (p|X| + d - 1)/q \end{aligned}$$

since $s = p + q + 1$. This again contradicts (5.2), and this completes the proof of Theorem 5.1. □

Corollary 5.2. *Let $D = (V, A, S)$ be an arc-minimal $(p/q, d/q)$-expanding digraph, where $d \geqslant 1$.*

(a) *If $q = 1$ then every vertex of S has outdegree exactly $p + d$.*
(b) *If $p = 0$ then every vertex of S has outdegree exactly $\lceil d/q \rceil$.*

Proof. (a) If $q = 1$ then Theorem 5.1 (b) says that every vertex of S has outdegree at most $p + d$. But clearly (taking $|X| = 1$ in (5.2)) every vertex has outdegree at least $p + d$, and the result follows.

(b) If $p = 0$ then (5.2) says that $|N^+(X) \setminus X| \geqslant \lceil d/q \rceil$ for every nonempty finite subset X of S. There is no loss of generality in assuming that $q = 1$, and so the result follows from (a). □

The following theorem is very similar to Theorem 4.4, but for digraphs rather than hypergraphs.

Theorem 5.3. *If p and q are positive coprime integers and d is an arbitrary integer, then* $\mathrm{maxdeg}(p, q, d) \geqslant p + q$.

Proof. Since p and q are coprime, there exist positive integers m, n, m', n' such that $qn = (p + q)m + 1$, $qn' = pm' + d$ and $n' > (p + q)m$. Let $\mathcal{H} = \mathcal{H}(p + q, m, n, 0, 0)$, which (by the proof of Theorem 4.4) belongs to $\mathcal{C}(p + q, q, 0)$, and let $\mathcal{H}'' = (V, \mathcal{E})$ be an irreducible member of $\mathcal{C}(p + q, q, 0)$ formed by removing vertices from edges of $\mathcal{H}$. By the proof of Theorem 4.4, $\mathcal{H}''$ contains the edge E_0 with $p + q$ vertices.

Let $\widehat{D}$ be the bipartite digraph with partite sets $\widehat{S}$, $\widehat{T}$ in which: $\widehat{T} = V$, the vertices in $\widehat{S}$ are (in 1:1 correspondence with) the edges in $\mathcal{E}$, and a vertex $s \in \widehat{S}$ is joined by an arc to a vertex $t \in \widehat{T}$ if and only if t belongs to (the edge in $\mathcal{E}$ corresponding to) s. Then $|\widehat{S}| = |\mathcal{E}| = (p + q)m + 1$ and $|\widehat{T}| = |V| = (p + q)n$. Moreover $\widehat{S}$ contains a vertex y_0 with outdegree $p + q$ (corresponding to the edge E_0 of $\mathcal{H}''$), and $\widehat{D}$ is arc-minimal with respect to the property that $|N^+_{\widehat{D}}(\widehat{X})| \geqslant (p + q)|\widehat{X}|/q$ for each nonempty subset $\widehat{X}$ of $\widehat{S}$.

Form a digraph expander $D = (V, A, S)$ by adding to $\widehat{D}$ a set $\widetilde{S}$ of m' vertices, a set $\widetilde{T}$ of $n' - |\widehat{S}|$ vertices, and arcs from all vertices in $\widetilde{S} \cup \widehat{S} \setminus \{y_0\}$ to all vertices in $\widehat{S} \cup \widetilde{T}$. Let $S := \widetilde{S} \cup \widehat{S}$. The result of the theorem is obvious if $p + q \leqslant (p + d)/q$, and so we may suppose that, in D, $d^+(y_0) = p + q \geqslant (p + d)/q$.

We shall prove first that D is $(p/q, d/q)$-expanding. Suppose that $\emptyset \neq X \subseteq S$. If $X \subseteq \widetilde{S}$ then

$$|N^+(X) \setminus X| = |\widehat{S} \cup \widetilde{T}| = n' = (pm' + d)/q = (p|\widetilde{S}| + d)/q \geqslant (p|X| + d)/q;$$

if $X \cap \widehat{S} = \{y_0\}$ and $X \neq \{y_0\}$ then

$$|N^+(X) \setminus X| = n' + p + q - 1 \geqslant [p(|\widetilde{S}| + 1) + d]/q \geqslant (p|X| + d)/q$$

since $p + q - 1 \geqslant p \geqslant p/q$; and if $X \cap \widehat{S} = \widehat{X} \notin \{\emptyset, \{y_0\}\}$ then

$$\begin{aligned}|N^+(X) \setminus X| = |N^+(\widehat{X}) \cap \widehat{T}| + n' - |\widehat{X}| &\geqslant (p+q)|\widehat{X}|/q + (p|\widetilde{S}| + d)/q - |\widehat{X}| \\ &= [p(|\widehat{X}| + |\widetilde{S}|) + d]/q \\ &\geqslant (p|X| + d)/q.\end{aligned}$$

Thus D is $(p/q, d/q)$-expanding.

Now let D' be obtained from D be deleting one arc out of y_0. By the arc-minimality of $\widehat{D}$, there is a subset $\widehat{X}$ of $\widehat{S}$ such that $|N^+_{D'}(\widehat{X}) \cap \widehat{T}| < (p+q)|\widehat{X}|/q$. Let $X := \widetilde{S} \cup \widehat{X}$. Then

$$\begin{aligned}|N^+(X) \setminus X| = |N^+(\widehat{X}) \cap \widehat{T}| + n' - |\widehat{X}| &< (p+q)|\widehat{X}|/q + (p|\widetilde{S}| + d)/q - |\widehat{X}| \\ &= (p|X| + d)/q.\end{aligned}$$

It follows that D' is not $(p/q, d/q)$-expanding.

Now, D is not an arc-minimal $(p/q, d/q)$-expander, but one can form an arc-minimal $(p/q, d/q)$-expander D'' by removing arcs from D. Since D' is not $(p/q, d/q)$-expanding, D'' must contain all $p + q$ arcs leaving y_0. This completes the proof of Theorem 5.3. $\square$

6. Two loose ends

If $D = (V, A)$ is a digraph and $X \subseteq S \subseteq V$, let $\partial(X)$ denote the set of arcs $\overrightarrow{uw}$ such that $u \in X$ and $w \notin X$. An analogue of (5.1) would be

$$|\partial(X)| \geqslant r|X| + \delta \qquad \text{whenever } \emptyset \neq X \subseteq S \text{ and } |X| < \infty. \tag{6.1}$$

In general, a digraph that is arc-minimal subject to (6.1) can have vertices with arbitrarily large outdegree. For example, if $r = 1, \delta = 0, S = \{u_1, \ldots, u_{n-1}, v\}, V = S \cup \{w_1, \ldots, w_n\}$ and $A = \{\overrightarrow{u_1v}, \ldots, \overrightarrow{u_{n-1}v}, \overrightarrow{vw_1}, \ldots, \overrightarrow{vw_n}\}$, then it is easy to see that D is arc-minimal subject to (6.1); but D has maximum outdegree n, which can be arbitrarily large. However, if (exceptionally) $r = 0$ then the maximum outdegree is bounded, as we see in the following analogue of Corollary 5.2 (b).

Theorem 6.1. *Let $D = (V, A)$ be a digraph (with parallel edges allowed) and $S \subseteq V$, and let $d \in \mathbb{N} \cup \{0\}$. Suppose that D is arc-minimal subject to the condition that, for each finite subset $X \subseteq S$, $|\partial(X)| \geqslant d$. Then every vertex of S has outdegree d.*

Proof. The proof is a simpler version of the proof of Theorem 5.1. If $v \in S$ then v has outdegree $d^+(v) = |\partial(\{v\})| \geqslant d$. We must prove that $d^+(v) \leqslant d$. It is clear that if $d = 0$ then $d^+(v) = 0$; so suppose $d > 0$.

Suppose if possible that $a_1, \ldots, a_{d+1}$ are distinct arcs with v as their tail. By the arc-minimality of D, there are sets $X_1, \ldots, X_{d+1} \subseteq S$ containing v such that, for each i,

$a_i \in \partial(X_i)$ and $|\partial(X_i)| = d$. (The sets X_i may not be distinct.) For $j = 1, \ldots, d+1$, let W_j be the set of vertices that are in at least j of the sets X_i. We claim that

$$\sum_{i=1}^{d+1} |\partial(X_i)| \geqslant \sum_{j=1}^{d+1} |\partial(W_j)|. \tag{6.2}$$

For, consider a typical arc $a = uw$. Let there be h sets X_i such that $w \in X_i$ and k sets X_i such that $u \in X_i$ and $w \notin X_i$. Then, exactly as in the proof of Theorem 5.1, a contributes k to the LHS of (6.2) and at most k to the RHS. This proves (6.2).

Since each summand on the LHS of (6.2) is equal to d, and each summand on the RHS is at least d, it follows that each summand on the RHS is exactly d. In particular, $|\partial(W_{d+1})| = d$. But $a_1, \ldots, a_{d+1} \in \partial(W_{d+1})$, and this contradiction completes the proof of Theorem 6.1. □

We now turn to the question of the simplicity of irreducible hypergraphs. The analogue of Theorem 2.2 is not true for simple hypergraphs (that is, ones in which the edges are distinct as sets). For example, if $\mathcal{H} = (V, \mathcal{E})$ where $|V| = 5$ and $\mathcal{E}$ comprises the ten 2-subsets of V, then $\mathcal{H}$ is simple and $|\bigcup \mathcal{F}| \geqslant |\mathcal{F}|/2$ for every subset $\mathcal{F} \subseteq \mathcal{E}$. By Theorem 2.2, an irreducible hypergraph with this property contains no edge with more than one vertex. But any *simple* hypergraph obtained by removing vertices from edges of $\mathcal{H}$ must contain an edge with two vertices. Thus Theorem 2.2 would no longer hold if $\mathrm{maxmod}(p, q, d)$ were redefined to refer to simple hypergraphs only. We now show that this problem cannot arise when $r \geqslant 1$.

Theorem 6.2. *If $\mathcal{H} = (V, \mathcal{E})$ is an irreducible hypergraph in $\mathcal{C}(r, \delta)$, where $r \geqslant 1$, then $\mathcal{H}$ is simple.*

Proof. Suppose not. Let E_1, E_2 be two edges that are equal as sets, let $x \in E_1$, let $E_1' := E_1 \setminus \{x\}$, and let $\mathcal{H}' = (V, \mathcal{E}')$ be the hypergraph obtained from $\mathcal{H}$ by substituting E_1' for E_1. By the irreducibility of $\mathcal{H}$, there is a nonempty subset $\mathcal{F}' \subseteq \mathcal{E}'$ such that $|\bigcup \mathcal{F}'| < r|\mathcal{F}'| + \delta$. If $\mathcal{F}$ is the corresponding set of edges in $\mathcal{H}$, so that $|\mathcal{F}| = |\mathcal{F}'|$, then

$$\left|\bigcup \mathcal{F}\right| \geqslant r|\mathcal{F}| + \delta = r|\mathcal{F}'| + \delta > \left|\bigcup \mathcal{F}'\right|.$$

It follows that $x \in \bigcup \mathcal{F}$ and $x \notin \bigcup \mathcal{F}'$, so that $|\bigcup \mathcal{F}| = |\bigcup \mathcal{F}'| + 1$, $E_1 \in \mathcal{F}$ and $E_2 \notin \mathcal{F}$. If now $\mathcal{F}'' := \mathcal{F} \cup \{E_2\}$, then

$$\begin{aligned}\left|\bigcup \mathcal{F}''\right| = \left|\bigcup \mathcal{F}\right| = \left|\bigcup \mathcal{F}'\right| + 1 < r|\mathcal{F}| + \delta + 1 = r|\mathcal{F}''| - r + \delta + 1 \\ \leqslant r|\mathcal{F}''| + \delta,\end{aligned}$$

contradicting the hypothesis that $\mathcal{H} \in \mathcal{C}(r, \delta)$. This contradiction completes the proof. □

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