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Tree representations of graphs

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Abstract

A graph is chordal if and only if it is the intersection graph of some family of subtrees of a tree. Applying "tolerance" allows larger families of graphs to be represented by subtrees. A graph G is in the family $[\Delta, d, t]$ if there is a tree with maximum degree Δ and subtrees corresponding to the vertices of G such that each subtree has maximum degree at most d and two vertices of G are adjacent if and only if the subtrees corresponding to them have at least t common vertices.

It is known that both $[3, 3, 1]$ and $[3, 3, 2]$ are equal to the family of chordal graphs. Furthermore, one can easily observe that every graph G belongs to $[3, 3, t]$ for some t. Denote by $t(G)$ the minimum t so that $G \in [3, 3, t]$. In this paper, we study $t(G)$ and parameters

$$t(n) = \min\{t : G \in [3, 3, t] \text{ for every } G \subseteq K_n\}$$

and

$$t_{\text{bip}}(n) = \min\{t : G \in [3, 3, t] \text{ for every } G \subseteq K_{n,n}\}.$$

In particular, our results imply that $\log n < t_{\text{bip}}(n) \leq 5n^{1/3} \log_2 n$ and $\log(n/2) < t(n) \leq 20n^{1/3} \log_2 n$.

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1. Intersection representations of graphs

One of important and interesting topics in graph theory is the representation of a graph using intersections of finite sets. Here, each vertex of a given graph is assigned a finite set, and two vertices are adjacent if and only if the corresponding sets intersect. More generally, a p-intersection representation of a graph G with vertices $v_1, v_2, \ldots, v_n$ is a collection of sets $\{S_1, S_2, \ldots, S_n\}$ such that $v_i v_j$ is an edge of G if and only if $|S_i \cap S_j| \geq p$. The *p-intersection number* $\theta_p(G)$ of G is the smallest cardinality of $\bigcup_{i=1}^{n} S_i$, taken over all p-intersection representations $\{S_1, S_2, \ldots, S_n\}$ of G.

Erdős et al. [7] proved that for all G on n vertices, the intersection number of G, $\theta_1(G)$, is at most $n^2/4$. For $p > 1$, p-intersection numbers have been studied [3,5,6,8], yet many questions remain open.

Since each graph has an intersection representation, we can impose additional restrictions on sets allowed in the intersection representation and investigate what families can be obtained. The best known example is the family of *interval graphs* for which we are allowed to choose only sets that are intervals on the real line or, alternatively, subpaths of a path. This is further generalized in the following definition.

Definition 1. For three positive integers Δ, d, and t, we say that a graph G has a (Δ, d, t)-representation (and write $G \in [\Delta, d, t]$) if the following is true. There exists a tree T with maximum degree $\Delta(T) \leq \Delta$ for which there are subtrees $T_1, \ldots, T_n$ such that

(a) $\Delta(T_i) \leq d$ for every $i = 1, \ldots, n$,
(b) $v_i v_j \in E(G)$ if and only if $|V(T_i) \cap V(T_j)| \geq t$ for all $1 \leq i < j \leq n$.

We will use ∞ in place of a maximum degree when no limit is given.

As mentioned above, $[2, 2, 1]$ is the family of interval graphs, and the interval graphs have been characterized by Lekkerkerker and Boland [14]. It is not hard to show that for $t \geq 1$, $G \in [2, 2, t]$ if and only if $G \in [2, 2, t+1]$. Thus, for all $t \in \mathbb{N}$, the graphs in $[2, 2, t]$ are the interval graphs, a proper subfamily of the chordal graphs.

A graph is called a *subtree graph* if it is in $[\infty, \infty, 1]$. In the early 1970's, it was shown that a graph is a subtree graph if and only if it is a chordal graph. This result is due separately to Buneman [2], Gavril [9], and Walter [17]. An improvement was found by McMorris and Scheinerman [15] who showed that $[3, 3, 1]$ is the family of chordal graphs. Later, Golumbic and Jamison [11] proved that $[3, 3, 1] = [3, 3, 2]$.

1.1. Tree representations

As was observed by Jamison and Mulder [13], the family $[n^2/4, n^2/4, 2]$ contains all graphs on n or fewer vertices. This follows from the already mentioned fact that for all G on n vertices, $\theta_1(G)$ is at most $n^2/4$; see [7]. Then one can construct a tree representation of G using a star with $\theta_1(G)$ leaves as the host tree. The substar assigned to a vertex corresponds to the center node of the star plus the leaves corresponding to its set in the intersection representation of G.

We can further improve this by taking a path P of length $\theta_p(G)$ and adding one leaf to each vertex of the path. The subtree assigned to a vertex is path P and the leaves corresponding to its set in the p-intersection representation of G. It is easy to see that this is a $(3, 3, \theta_p(G) + p)$-representation of G. One may therefore ask what is the minimum $t = t(G)$ such that $G \in [3, 3, t]$.

Since $t(G) = 1$ for every chordal graph G, we turn our attention to the complete bipartite graph $K_{n,n}$, which is not chordal for $n \geq 2$. We have already observed that $t(K_{n,n}) \leq \theta_p(K_{n,n}) + p$. Now we recall the following result of Füredi.

Proposition 2 (*Cf. Proposition 3.3 in [8]*). *If* $p = 2^{k+1} - 1$, *then* $\theta_p(K_{p+1,p+1}) = 4p$.

Since each interval $[n, 2n]$ contains a power of 2, we obtain $t(K_{n,n}) \leq 10n$. For arbitrary graphs G, the value of $\theta_p(G)$ is generally not known and, therefore, $t(G) \leq \theta_p(G) + p$ only yields $O(n^2)$ bounds.

In this paper we improve this further and prove the following theorem.

Theorem 3. *For all n,* $t_{\text{bip}}(n) = \min\{t : G \in [3, 3, t]$ *for every* $G \subseteq K_{n,n}\}$ *satisfies*

$$t_{\text{bip}}(n) \leq 7(2n^{1/3} + 4)(\log_2 n/3 + 2) - 6.$$

As a corollary we obtain an upper bound on $t(G)$ for an arbitrary graph G.

Corollary 4. *For all n,* $t(n) = \min\{t : G \in [3, 3, t]$ *for every* $G \subseteq K_n\}$ *satisfies*

$$t(n) \leq 28(2n^{1/3} + 1)(\log_2 n/3 + 2) - 24.$$

For the lower bound, we obtain the following result.

Theorem 5. $t_{\text{bip}}(n) \geq t(K_{n,n}) > \log_2 n$ *for all n. Hence,* $t(n) > \log_2(n/2)$.

We remark that it is not obvious that a graph having a (Δ, d, t)-representation has also a $(\Delta, d, t + 1)$-representation. This was, indeed, conjectured by Jamison and Mulder, and it has been proved only for some special cases ($t = 2, 3, 4$) in [13].

Conjecture 6. *For* $\Delta, d, t \in \mathbb{N}$, $t > 1$, *we have* $[\Delta, d, t] \subseteq [\Delta, d, t + 1]$.

In Section 5, we prove the following special case of the conjecture.

Proposition 7. *For* $\Delta, d, t \in \mathbb{N}$, $t > 1$,

$$[\Delta, d, t] \subseteq [\Delta, \min(d + 1, \Delta), t + 1].$$

We see from Proposition 7 that $K_{n,n} \in [\Delta, \Delta, t] \Longrightarrow K_{n,n} \in [\Delta, \Delta, t + 1]$, and in particular,

$$K_{n,n} \in [3, 3, t] \Longrightarrow K_{n,n} \in [3, 3, t + 1]. \tag{1}$$

Finally, Theorem 3 provides a partial answer to the following question raised by Mulder (see [10]).

Problem 8. For which $n \geq 3$, $K_{n,n} \notin [3, 3, n - 1]$ and $K_{n,n} \in [3, 3, n]$?

In view of (1), we can restate this question as finding all n for which $t(K_{n,n}) = n$. This is known to be true for $n = 3$ and $n = 4$ by the results and constructions of Jamison and Mulder [13]. On the other hand, Theorem 3 shows the existence of n_0 such that $t(K_{n,n}) < n$ for $n > n_0$. Careful analysis (outlined in the Appendix) reveals that $4 \leq n_0 \leq 589$.

2. The upper bound: A reduction

In this section, we show that the upper bounds in Theorem 3 and Corollary 4 follow from the same construction.

2.1. Reduction

For given positive integers n and t let K be any set of size n, T be any binary tree with root r, and let L be the set of its leaves. Suppose that for every $\boldsymbol{a} \in K$ there are two subtrees $T_A(\boldsymbol{a})$ and $T_B(\boldsymbol{a})$ of T rooted in r and satisfying the following properties:

(i) $|T_A(\boldsymbol{a}) \cap T_A(\boldsymbol{a}')| < t$ for all $\boldsymbol{a} \neq \boldsymbol{a}',\ \boldsymbol{a}, \boldsymbol{a}' \in K$,
(ii) $|T_B(\boldsymbol{a}) \cap T_B(\boldsymbol{a}')| < t$ for all $\boldsymbol{a} \neq \boldsymbol{a}'\ \boldsymbol{a}, \boldsymbol{a}' \in K$,
(iii) $T_A(\boldsymbol{a}) \cap T_A(\boldsymbol{a}') \cap L = T_B(\boldsymbol{a}) \cap T_B(\boldsymbol{a}') \cap L = \emptyset$ for all $\boldsymbol{a} \neq \boldsymbol{a}'\ \boldsymbol{a}, \boldsymbol{a}' \in K$,
(iv) $T_A(\boldsymbol{a}) \cap T_B(\boldsymbol{a}') \cap L \neq \emptyset$ for all $\boldsymbol{a}, \boldsymbol{a}' \in K$,
(v) $|T_A(\boldsymbol{a}) \cap T_B(\boldsymbol{a}')| < t$ for all $\boldsymbol{a}, \boldsymbol{a}' \in K$.

Then we construct a $(3, 3, t)$-representation of $G \subset K_{n,n}$ as follows. Suppose that $A \cup B$ is the bipartition of G. Since $|A| = |B| = |K| = n$, we can associate every vertex in A and B with one distinct element of K. For every edge $\boldsymbol{ab}$ of G, fix one leaf $v(\boldsymbol{ab})$ in $T_A(\boldsymbol{a}) \cap T_B(\boldsymbol{b}) \cap L \neq \emptyset$ (cf. (iv)). It follows from (iii) that $v(\boldsymbol{ab}) \neq v(\boldsymbol{a}'\boldsymbol{b}')$ for distinct edges $\boldsymbol{ab}$ and $\boldsymbol{a}'\boldsymbol{b}'$. Set

$$L(G) = \{v(\boldsymbol{ab}) : \boldsymbol{ab} \in E(G)\}.$$

We obtain the host tree T' by appending a distinct path P_v with t vertices to every leaf $v \in L(G)$. For $\boldsymbol{a} \in A$ ($\boldsymbol{b} \in B$, respectively), we construct a subtree $T'(\boldsymbol{a})$ of T' ($T'(\boldsymbol{b})$ of T', respectively) by taking $T_A(\boldsymbol{a})$ ($T_B(\boldsymbol{b})$, respectively) and all paths P_v for every leaf $v \in T_A(\boldsymbol{a}) \cap L(G)$ (all $v \in T_B(\boldsymbol{b}) \cap L(G)$, respectively). In other words,

$$T'(\boldsymbol{a}) = T_A(\boldsymbol{a}) \cup \{P_v : v \in T_A(\boldsymbol{a}) \cap L(G)\} \qquad \text{for } \boldsymbol{a} \in A \tag{2a}$$

and

$$T'(\boldsymbol{b}) = T_B(\boldsymbol{b}) \cup \{P_v : v \in T_B(\boldsymbol{b}) \cap L(G)\} \qquad \text{for } \boldsymbol{b} \in B. \tag{2b}$$

For $\boldsymbol{a} \neq \boldsymbol{a}' \in A$, we have

$$T'(\boldsymbol{a}) \cap T'(\boldsymbol{a}') = (T_A(\boldsymbol{a}) \cap T_A(\boldsymbol{a}')) \cup \left\{P_v : v \in T_A(\boldsymbol{a}) \cap T_A(\boldsymbol{a}') \cap L(G)\right\}.$$

By (iii), $\{P_v : v \in T_A(\boldsymbol{a}) \cap T_A(\boldsymbol{a}') \cap L(G)\}$ is an empty set, therefore, by (i),

$$|T'(\boldsymbol{a}) \cap T'(\boldsymbol{a}')| = |T_A(\boldsymbol{a}) \cap T_A(\boldsymbol{a}')| < t. \tag{3a}$$

Similarly, using (ii) and (iii), we get

$$|T'(\boldsymbol{b}) \cap T'(\boldsymbol{b}')| = |T_B(\boldsymbol{b}) \cap T_B(\boldsymbol{b}')| < t \tag{3b}$$

for every $\boldsymbol{b} \neq \boldsymbol{b}' \in B$.

Suppose that $\boldsymbol{a} \in A$ and $\boldsymbol{b} \in B$ are such that $\boldsymbol{ab} \notin E(G)$. It follows from the definition of $L(G)$ and (iii) that no $v \in L(G)$ belongs to $T_A(\boldsymbol{a}) \cap T_B(\boldsymbol{b})$, and thus,

$$|T'(\boldsymbol{a}) \cap T'(\boldsymbol{b})| = |T_A(\boldsymbol{a}) \cap T_B(\boldsymbol{b})| \overset{\text{(v)}}{<} t. \tag{3c}$$

By (iv), there is a leaf $v = v(\boldsymbol{ab}) \in L(G)$ for every $\boldsymbol{a} \in A$ and $\boldsymbol{b} \in B$, $\boldsymbol{ab} \in E(G)$, such that $v \in T_A(\boldsymbol{a}) \cap T_B(\boldsymbol{b})$. Consequently,

$$T'(\boldsymbol{a}) \cap T'(\boldsymbol{b}) \supseteq (T_A(\boldsymbol{a}) \cap T_B(\boldsymbol{b})) \cup P_v$$

and, therefore,

$$|T'(\boldsymbol{a}) \cap T'(\boldsymbol{b})| \geq |P_v| = t. \tag{4}$$

What remains is to describe the construction of subtrees $T_A(\boldsymbol{a})$ and $T_B(\boldsymbol{a})$ and prove that they satisfy conditions (i)–(v). This is done in Section 3.

2.2. *Proof of Corollary 4*

We prove that any construction satisfying (i)–(v) above can be turned into a $(3, 3, 4t)$-representation of an arbitrary n-vertex graph.

Let $G = (V, E)$ be an arbitrary graph on n vertices. Without loss of generality we may assume $V = K$. We construct an auxiliary bipartite graph Γ with bipartition $A \cup B$, $A = B = V$, and edge set

$$E(\Gamma) = \{\boldsymbol{ab} : \boldsymbol{a} \in A, \boldsymbol{b} \in B, \boldsymbol{ab} \in E\}.$$

Consider a $(3, 3, t)$-representation of Γ given by (2a) and (2b) in which we append paths P_v with $4t$ (not t) vertices. We set $T''(\boldsymbol{v}) = T'(\boldsymbol{a}) \cup T'(\boldsymbol{b})$, where $\boldsymbol{a} = \boldsymbol{v} \in A$ and $\boldsymbol{b} = \boldsymbol{v} \in B$. Note that this is a tree because $T'(\boldsymbol{a})$ and $T'(\boldsymbol{b})$ share the root of T'. We prove this is a $(3, 3, 4t)$-representation of G.

If $\boldsymbol{vv}' \in E$, then, similarly to (4),

$$|T''(\boldsymbol{v}) \cap T''(\boldsymbol{v}')| \geq |T'(\boldsymbol{a}) \cap T'(\boldsymbol{b}')| \geq |P_v| = 4t,$$

where $\boldsymbol{a} = \boldsymbol{v} \in A$, $\boldsymbol{b}' = \boldsymbol{v}' \in B$, and v is a leaf belonging to $T_A(\boldsymbol{a}) \cap T_B(\boldsymbol{b}')$.

If $\boldsymbol{vv}' \notin E$, then

$$\begin{aligned} |T''(\boldsymbol{v}) \cap T''(\boldsymbol{v}')| \leq\ & |T'(\boldsymbol{a}) \cap T'(\boldsymbol{a}')| + |T'(\boldsymbol{a}) \cap T'(\boldsymbol{b}')| \\ & + |T'(\boldsymbol{b}) \cap T'(\boldsymbol{a}')| + |T'(\boldsymbol{b}) \cap T'(\boldsymbol{b}')|, \end{aligned}$$

where $\boldsymbol{a} = \boldsymbol{v} \in A$, $\boldsymbol{a}' = \boldsymbol{v}' \in A$, $\boldsymbol{b} = \boldsymbol{v} \in B$, and $\boldsymbol{b}' = \boldsymbol{v}' \in B$. We have $|T'(\boldsymbol{a}) \cap T'(\boldsymbol{a}')|$, $|T'(\boldsymbol{b}) \cap T'(\boldsymbol{b}')| < t$ because $\boldsymbol{a}$ and $\boldsymbol{a}'$ are in the same partition class and $\boldsymbol{b}$ and $\boldsymbol{b}'$ are in the same partition class of Γ (see (3a) and (3b)). The other two terms on the right-hand side are also bounded by t by (3c). Hence,

$$|T''(\boldsymbol{v}) \cap T''(\boldsymbol{v}')| < 4t.$$

3. Upper bound: Trees defined by PLG $(2, q)$

In this section we provide a construction satisfying conditions (i)–(v) in Section 2.1.

3.1. *Preliminaries*

Let p be a prime, q be any power of p, and $\mathbb{F}_q$ be the field with elements $0, 1, \ldots, q-1$. We say that $(a_1, a_2) \in \mathbb{F}_q^2$ and $(b_1, b_2) \in \mathbb{F}_q^2$ are *equivalent* if $a_1 = \lambda b_1$ and $a_2 = \lambda b_2$ for some non-zero $\lambda \in \mathbb{F}_q$. Then $\mathbb{F}_q^2 \setminus \{(0,0)\}$ splits into $q+1$ classes that are represented by $(0,1), (1,0), (1,1), \dots, (1, q-1)$ and the set X of these representatives is called the 1-*dimensional projective space over* $\mathbb{F}_q$.

It is a well-known fact that any non-singular 2×2 matrix $\mathbb{A} \in \mathbb{F}_q^{2\times 2}$ (the group of all such matrices is denoted by $\mathrm{GL}(2,q)$) acts on X as a permutation by mapping $\boldsymbol{x}$ to $\boldsymbol{x}\mathbb{A}$ (see, e.g., [4, 12]). Clearly matrices $\mathbb{A}$ and $\lambda\mathbb{A}$ define the same permutation for $\lambda \neq 0 \in \mathbb{F}_q$, hence all these permutations are defined by matrices $\begin{bmatrix} 1 & c \\ b & d \end{bmatrix}$ with $bc \neq d$ and $\begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix}$ with $c \neq 0$. (We just remark that these matrices correspond to the *projective group* $\mathrm{PLG}(2,q)$ and that the number of these matrices is $(q+1)q(q-1)$ (see [12]).)

Let K be the set of all vectors $(b,c,d) \in \mathbb{F}_q^3$ such that $b,c,d \neq 0$ and $bc \neq d$. By subtracting from $(q+1)q(q-1)$ the number of matrices of types $\begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix}$ with $c \neq 0$, $\begin{bmatrix} 1 & c \\ b & 0 \end{bmatrix}$ with $bc \neq 0$, $\begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix}$ with $d \neq 0$, and $\begin{bmatrix} 1 & 0 \\ b & d \end{bmatrix}$ with $b, d \neq 0$, we obtain the following.

Fact 9. *K has $(q-2)(q-1)^2$ elements.*

Now for each $(b,c,d) \in K$ we define a mapping $\pi_{(b,c,d)} : X \to X$ by

$$\begin{aligned} &\pi_{(b,c,d)}((0,1)) = (1, b^{-1}d),\\ &\pi_{(b,c,d)}((1,-b^{-1})) = (0,1)\\ &\pi_{(b,c,d)}((1,\lambda)) = (1, (c+d\lambda)(1+b\lambda)^{-1}) \quad \text{for } \lambda \neq -b^{-1}. \end{aligned} \tag{5}$$

Since $\pi_{(b,c,d)}$ corresponds to the action of $\begin{bmatrix} 1 & c \\ b & d \end{bmatrix}$ on X, we have immediately that $\pi_{(b,c,d)}$ is a permutation of X for every $(b,c,d) \in K$. We will need the following fact observed by Deza and Frankl [4].

Fact 10. *For all (b,c,d) and $(b',c',d') \in K$, $\pi_{(b,c,d)}(\boldsymbol{x}) = \pi_{(b',c',d')}(\boldsymbol{x})$ has at most two solutions $\boldsymbol{x} \in X$.*

Before we define a family of subtrees satisfying the conditions from Section 2.1, we need one more operation on X. For $c, d \in \mathbb{F}_q$ and $\boldsymbol{x} \in X$ we set

$$c\boldsymbol{x} + d = \begin{cases} (1, cx+d) & \text{if } \boldsymbol{x} = (1,x)\\ (1,d) & \text{if } \boldsymbol{x} = (0,1). \end{cases} \tag{6}$$

Clearly, $c\boldsymbol{x}+d$ acts on $X^* = X \setminus \{(0,1)\}$ in the same way as $cx+d$ does on $\mathbb{F}_q$.

For each $(b,c,d) \in K$ we define a mapping $\kappa_{(b,c,d)} : X \times X \to X$ by

$$\kappa_{(b,c,d)}(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} d\boldsymbol{y}+b & \text{if } \boldsymbol{x} = (1,0) \text{ or } \pi_{(b,c,d)}(\boldsymbol{x}) = (1,0),\\ d\boldsymbol{y}+c & \text{otherwise.} \end{cases} \tag{7}$$

The following fact follows from (5) and (7).

Fact 11. *For all (b,c,d) and $(b',c',d) \in K$, if $\pi_{(b,c,d)}(\boldsymbol{x}) = \pi_{(b',c',d)}(\boldsymbol{x})$ and $\kappa_{(b,c,d)}(\boldsymbol{x},\boldsymbol{y}) = \kappa_{(b',c',d)}(\boldsymbol{x},\boldsymbol{y})$ for some $\boldsymbol{x},\boldsymbol{y} \in X$, then $b = b'$ and $c = c'$.*

3.2. Construction

For a vertex v and integer ℓ, we denote by $T_\ell(v)$ the full binary tree of height ℓ rooted at v and by $\tilde{T}_\ell(v)$ the tree constructed as follows: the root v is adjacent to two vertices v_1 and v_2 and we append the tree $T_\ell(v_1)$ at v_1. In other words, $\tilde{T}_\ell(v) = \{vv_1\} \cup \{vv_2\} \cup T_\ell(v_1)$. We call the edge vv_2 the "special branch". We also set $\tilde{T}_\ell^-(v) = \{vv_1\} \cup T_{\ell-1}(v_1)$. (We obtain $\tilde{T}_\ell^-(v)$ from $\tilde{T}_\ell(v)$ by removing all its leaves.)

Note that $T_\ell(v)$ has $2^{\ell+1} - 1$ vertices and 2^ℓ leaves, $\tilde{T}_\ell(v)$ has $2^{\ell+1} + 1$ vertices and $2^\ell + 1$ leaves, and $\tilde{T}_\ell^-(v)$ has 2^ℓ vertices.

We now describe the host tree T with root r. Let h be a positive integer, $q = 2^h$, and $t = 7(2^h + 1)(h + 1) - 6$. In $\tilde{T}_h(r) = \{rr_1\} \cup \{rr_2\} \cup T_h(r_1)$ we label the leaf r_2 by $(0, 1)$ and the leaves of $T_h(r_1)$ by vectors $(\boldsymbol{x}) \in X^* = X \setminus \{(0, 1)\}$.

For $i = 1, \ldots, 5$, and for each $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_i) \in X^i$, we label the leaf in the special branch of $\tilde{T}_h((\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_i))$ by $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_i, (0, 1))$ and the other q leaves by $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_i, (1, 0)), \ldots, (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_i, (1, q - 1))$. Now we let T be the tree formed by the union of $\tilde{T}_h(r)$ and all trees

$$\tilde{T}_h((\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_i)) \quad \text{for } i = 1, \ldots, 5 \text{ and } (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_i) \in X^i.$$

Note that T has maximum degree 3. For two vertices u and v in T, we denote by $P(u, v)$ the vertex set of a unique path from u to v and we also set $P^-(u, v) = P(u, v) \setminus \{v\}$.

Recall now that K is the set of all vectors $(b, c, d) \in \mathbb{F}_q^3$ such that $b, c, d \neq 0$, and $bc \neq d$. For each $(b, c, d) \in K$ we shall define subtrees $T_A(b, c, d)$ and $T_B(b, c, d)$ of T and prove that they satisfy conditions (i)–(v) from Section 2.1.

We define $T_A(b, c, d)$ as the union over all $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in X$ of

- $\tilde{T}_h(r)$,
- $P\big((\boldsymbol{x}_1), (\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1))\big) \cup \tilde{T}_h\big((\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1))\big)$,
- $P\big((\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1), \boldsymbol{x}_2), (\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2))\big)$,
- $\tilde{T}_h\big((\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2))\big)$, and
- $P\big((\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{x}_3), (\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{x}_3, (1, d))\big)$.

We define $T_B(b, c, d)$ as the union over all $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3 \in X$ of

- $P(r, (1, d)) \cup \tilde{T}_h((1, d))$,
- $P\big(((1, d), \boldsymbol{y}_1), ((1, d), \boldsymbol{y}_1, \pi_{(b,c,d)}(\boldsymbol{y}_1))\big)$,
- $\tilde{T}_h\big(((1, d), \boldsymbol{y}_1, \pi_{(b,c,d)}(\boldsymbol{y}_1))\big)$,
- $P\big(((1, d), \boldsymbol{y}_1, \pi_{(b,c,d)}(\boldsymbol{y}_1), \boldsymbol{y}_2), ((1, d), \boldsymbol{y}_1, \pi_{(b,c,d)}(\boldsymbol{y}_1), \boldsymbol{y}_2, \kappa_{(b,c,d)}(\boldsymbol{y}_1, \boldsymbol{y}_2))\big)$,
- $\tilde{T}_h\big(((1, d), \boldsymbol{y}_1, \pi_{(b,c,d)}(\boldsymbol{y}_1), \boldsymbol{y}_2, \kappa_{(b,c,d)}(\boldsymbol{y}_1, \boldsymbol{y}_2))\big)$.

We prove that the trees above satisfy conditions (i)–(v) in the next section.

3.3. Proofs

From the above definition we conclude that the leaves of $T_A(b, c, d)$ are of the form

$$\big(\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{x}_3, (1, d)\big)$$

and the leaves of $T_B(b, c, d)$ are of the form

$$\big((1, d), \boldsymbol{y}_1, \pi_{(b,c,d)}(\boldsymbol{y}_1), \boldsymbol{y}_2, \kappa_{(b,c,d)}(\boldsymbol{y}_1, \boldsymbol{y}_2), \boldsymbol{y}_3\big).$$

If $T_A(b, c, d)$ and $T_A(b', c', d')$ (or $T_B(b, c, d)$ and $T_B(b', c', d')$) have the same leaf, then, by comparing coordinates, we obtain that $d = d'$, $\pi_{(b,c,d)}(\boldsymbol{x}_1) = \pi_{(b',c',d)}(\boldsymbol{x}_1)$ has a solution $\boldsymbol{x}_1 \in X$, and $\kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \kappa_{(b',c',d)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ for some $\boldsymbol{x}_2 \in X$. By Fact 11 we have $b = b'$ and $c = c'$. Thus $(b, c, d) = (b', c', d')$ and (iii) holds.

To prove (iv), we need to find $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3 \in X$ so that

$$\begin{aligned}&\left(\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{x}_3, (1, d)\right)\\ &\quad = \left((1, d'), \boldsymbol{y}_1, \pi_{(b',c',d')}(\boldsymbol{y}_1), \boldsymbol{y}_2, \kappa_{(b',c',d')}(\boldsymbol{y}_1, \boldsymbol{y}_2), \boldsymbol{y}_3\right).\end{aligned}$$

We see that this is satisfied for $\boldsymbol{x}_1 = (1, d')$, $\boldsymbol{y}_1 = \pi_{(b,c,d)}(\boldsymbol{x}_1)$, $\boldsymbol{x}_2 = \pi_{(b',c',d')}(\boldsymbol{y}_1)$, $\boldsymbol{y}_2 = \kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$, $\boldsymbol{x}_3 = \kappa_{(b',c',d')}(\boldsymbol{y}_1, \boldsymbol{y}_2)$, and $\boldsymbol{y}_3 = (1, d)$.

The definition of $T_A(b, c, d)$ and $T_B(b', c', d')$ implies that their intersection consists of the union of paths (this is because $T_B(b', c', d')$ has paths at places where $T_A(b, c, d)$ has trees and vice versa). From this we deduce that

$$T_A(b, c, d) \cap T_B(b', c', d') = P\left(r, (\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{x}_3, (1, d))\right),$$

where $\boldsymbol{x}_1 = (1, d')$, $\boldsymbol{y}_1 = \pi_{(b,c,d)}(\boldsymbol{x}_1)$, $\boldsymbol{x}_2 = \pi_{(b',c',d')}(\boldsymbol{y}_1)$, $\boldsymbol{y}_2 = \kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$, and $\boldsymbol{x}_3 = \kappa_{(b',c',d')}(\boldsymbol{y}_1, \boldsymbol{y}_2)$. Hence $|T_A(b, c, d) \cap T_B(b', c', d')| = 6h + 7 < t$ and (v) holds.

Consider the intersection of $T_A(b, c, d)$ and $T_A(b', c', d')$. To maximize this intersection (and reach beyond level $2h$) we must have $\pi_{(b,c,d)}(\boldsymbol{x}_1) = \pi_{(b',c',d')}(\boldsymbol{x}_1)$ for some $\boldsymbol{x}_1 \in X$. Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be (at most two) solutions (cf. Fact 10) of this equation.

We distinguish two cases. If $d = d'$, then $\kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \kappa_{(b',c',d)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ cannot have any solution for $\boldsymbol{x}_1 \in \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ because by Fact 11 we would have $(b, c, d) = (b', c', d)$. This implies that $T_A(b, c, d) \cap T_A(b', c', d')$ is contained in

$$\begin{aligned}&\tilde{T}_h^-(r) \cup \bigcup_{\boldsymbol{x}_1 \in X} P^-\left((\boldsymbol{x}_1), (\boldsymbol{x}_1, \pi_{(b,c,d)}(\boldsymbol{x}_1))\right)\\ &\quad \cup \tilde{T}_h^-\left((\boldsymbol{u}_1, \pi_{(b,c,d)}(\boldsymbol{u}_1))\right) \cup \tilde{T}_h^-\left((\boldsymbol{u}_2, \pi_{(b,c,d)}(\boldsymbol{u}_2))\right)\\ &\quad \cup \bigcup_{\boldsymbol{x}_2 \in X} P^-\left((\boldsymbol{u}_1, \pi_{(b,c,d)}(\boldsymbol{u}_1), \boldsymbol{x}_2), (\boldsymbol{u}_1, \pi_{(b,c,d)}(\boldsymbol{u}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{u}_1, \boldsymbol{x}_2))\right)\\ &\quad \cup \bigcup_{\boldsymbol{x}_2 \in X} P^-\left((\boldsymbol{u}_2, \pi_{(b,c,d)}(\boldsymbol{u}_2), \boldsymbol{x}_2), (\boldsymbol{u}_2, \pi_{(b,c,d)}(\boldsymbol{u}_2), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{u}_2, \boldsymbol{x}_2))\right). \qquad (8)\end{aligned}$$

If $d \neq d'$, then $\kappa_{(b,c,d)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \kappa_{(b',c',d)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ has at most two solutions for a fixed $\boldsymbol{x}_1$ (cf. (7) and (6)). Denote by $\boldsymbol{u}_{11}, \boldsymbol{u}_{12}$ the solutions for $\boldsymbol{x}_1 = \boldsymbol{u}_1$ and by $\boldsymbol{u}_{21}, \boldsymbol{u}_{22}$ the solutions for $\boldsymbol{x}_1 = \boldsymbol{u}_2$. This implies that $T_A(b, c, d) \cap T_A(b', c', d')$ is contained in (8) enlarged with

$$\begin{aligned}&\bigcup_{\boldsymbol{x}_2 \in \{\boldsymbol{u}_{11}, \boldsymbol{u}_{12}\}} \tilde{T}_h^-\left((\boldsymbol{u}_1, \pi_{(b,c,d)}(\boldsymbol{u}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{u}_1, \boldsymbol{x}_2))\right)\\ &\quad \cup \bigcup_{\boldsymbol{x}_2 \in \{\boldsymbol{u}_{21}, \boldsymbol{u}_{22}\}} \tilde{T}_h^-\left((\boldsymbol{u}_2, \pi_{(b,c,d)}(\boldsymbol{u}_2), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{u}_2, \boldsymbol{x}_2))\right)\\ &\quad \cup \bigcup_{\boldsymbol{x}_2 \in \{\boldsymbol{u}_{11}, \boldsymbol{u}_{12}\}} \bigcup_{\boldsymbol{x}_3 \in X} P^-\left((\boldsymbol{u}_1, \pi_{(b,c,d)}(\boldsymbol{u}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{u}_1, \boldsymbol{x}_2), \boldsymbol{x}_3),\right.\\ &\qquad \left.(\boldsymbol{u}_1, \pi_{(b,c,d)}(\boldsymbol{u}_1), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{u}_1, \boldsymbol{x}_2), \boldsymbol{x}_3, (1, d))\right)\\ &\quad \cup \bigcup_{\boldsymbol{x}_2 \in \{\boldsymbol{u}_{21}, \boldsymbol{u}_{22}\}} \bigcup_{\boldsymbol{x}_3 \in X} P^-\left((\boldsymbol{u}_2, \pi_{(b,c,d)}(\boldsymbol{u}_2), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{u}_2, \boldsymbol{x}_2), \boldsymbol{x}_3),\right.\\ &\qquad \left.(\boldsymbol{u}_2, \pi_{(b,c,d)}(\boldsymbol{u}_2), \boldsymbol{x}_2, \kappa_{(b,c,d)}(\boldsymbol{u}_2, \boldsymbol{x}_2), \boldsymbol{x}_3, (1, d))\right).\end{aligned}$$

Clearly, the second case yields a larger intersection whose size is bounded by

$$2^h + (2^h + 1) \cdot h + 2 \cdot 2^h + 2 \cdot (2^h + 1) \cdot h + 4 \cdot 2^h + 4 \cdot (2^h + 1) \cdot h < t.$$

Hence (i) holds.

Now we look at how $T_B(b,c,d)$ and $T_B(b',c',d')$ intersect. Clearly, $d = d'$ holds and $\pi_{(b,c,d)}(\boldsymbol{y}_1) = \pi_{(b',c',d)}(\boldsymbol{y}_1)$ has a solution in order to maximize this intersection. Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be (possibly two) solutions of this equation. By Fact 11, $\kappa_{(b,c,d)}(\boldsymbol{y}_1,\boldsymbol{y}_2) = \kappa_{(b',c',d)}(\boldsymbol{y}_1,\boldsymbol{y}_2)$ cannot have any solution $\boldsymbol{y}_2$ for $\boldsymbol{y}_1 \in \{\boldsymbol{v}_1,\boldsymbol{v}_2\}$ because otherwise we would have $(b,c,d) = (b',c',d)$. This implies that $T_B(b,c,d) \cap T_B(b',c',d')$ is given by

$$
\begin{aligned}
&P^-(r,(1,d)) \cup \tilde{T}_h^-((1,d)) \cup \bigcup_{\boldsymbol{y}_1\in X} P^-\big(((1,d),\boldsymbol{y}_1),((1,d),\boldsymbol{y}_1,\pi_{(b,c,d)}(\boldsymbol{y}_1))\big)\\
&\quad\cup \tilde{T}_h\big(((1,d),\boldsymbol{v}_1,\pi_{(b,c,d)}(\boldsymbol{v}_1))\big) \cup \tilde{T}_h\big(((1,d),\boldsymbol{v}_2,\pi_{(b,c,d)}(\boldsymbol{v}_2))\big)\\
&\quad\cup \bigcup_{\boldsymbol{y}_2\in X} P^-\big(((1,d),\boldsymbol{v}_1,\pi_{(b,c,d)}(\boldsymbol{v}_1),\boldsymbol{y}_2),((1,d),\boldsymbol{v}_1,\pi_{(b,c,d)}(\boldsymbol{v}_1),\boldsymbol{y}_2,\kappa_{(b,c,d)}(\boldsymbol{v}_1,\boldsymbol{y}_2))\big)\\
&\quad\cup \bigcup_{\boldsymbol{y}_2\in X} P^-\big(((1,d),\boldsymbol{v}_2,\pi_{(b,c,d)}(\boldsymbol{v}_2),\boldsymbol{y}_2),((1,d),\boldsymbol{v}_2,\pi_{(b,c,d)}(\boldsymbol{v}_2),\boldsymbol{y}_2,\kappa_{(b,c,d)}(\boldsymbol{v}_2,\boldsymbol{y}_2))\big).
\end{aligned}
$$

A moment's thought shows that the above intersection is smaller than the one in the previous case and thus that (ii) holds.

For any n satisfying $n \le (q-2)(q-1)^2 = \left(2^h-2\right)\left(2^h-1\right)^2$, the above construction and Section 2.1 yield a $(3,3,t)$-representation of any $G \subset K_{n,n}$, where $t = 7(2^h+1)(h+1)-6$.

Given n, we find an upper bound on h using the fact that

$$\left(2^{h-1}-2\right)\left(2^{h-1}-1\right)^2 < n$$

because, otherwise, we could use $t = 7(2^{h-1}+1)((h-1)+1)-6$. Thus, we have $n > \left(2^{h-1}-2\right)^3$ from which we deduce that $2^h < 2n^{1/3}+4$. A short calculation shows that

$$t_{\text{bip}}(n) \le t = 7(2^h+1)(h+1)-6 \le 7(2n^{1/3}+4)\left(\frac{\log_2 n}{3}+2\right)-6.$$

4. Lower bound

In this section, we prove Theorem 5. For any $(3,3,t)$-representation of $K_{n,n}$, let T be the host tree, and $A_1,\ldots,A_n$, respectively $B_1,\ldots,B_n$, denote the vertex sets of subtrees corresponding to vertices in each partite set of $K_{n,n}$. Thus we know that $|A_i \cap A_j| < t$, $|B_i \cap B_j| < t$ for all $1 \le i < j \le n$, and $|A_i \cap B_j| \ge t$ for all $1 \le i, j \le n$.

Claim 12. *Either* $|A_i \cap A_j| \ge 1$ *for all* $1 \le i < j \le n$ *or* $|B_i \cap B_j| \ge 1$ *for all* $1 \le i < j \le n$.

Proof. Suppose that $A_i \cap A_j = \emptyset$ for some $1 \le i < j \le n$. Let P be vertices of the unique shortest path (in T) between A_i and A_j, that is $|P| > 0$. Since every B_k must contain a vertex from both A_i and A_j and $T[B_k]$ is a tree, B_k must also contain P for all $k = 1,\ldots,n$. □

Due to symmetry, we may assume that $|A_i \cap A_j| \ge 1$ for all $1 \le i < j \le n$. Recall that any family of subtrees of a tree has the *Helly property*, i.e., if the members of the family are pairwise intersecting, then there is a vertex common to the whole family (cf. Chapter 1 in [1]). Thus we have

Claim 13. $\left|\bigcap_{j=1}^n A_j\right| \ge 1$.

Now we prove that every B_i has a non-empty intersection with $\bigcap_{j=1}^{n} A_j$.

Claim 14. *For every $i \in [n]$ there is a vertex $v_i \in B_i \cap \bigcap_{j=1}^{n} A_j$.*

Proof. Suppose $B_i \cap \bigcap_{j=1}^{n} A_j = \emptyset$ and let v_i be the closest point of B_i to $A = \bigcap_{j=1}^{n} A_j$ in T. Note that since every two points in T are connected by a unique path and v_i is the closest point of B_i to $A = \bigcap_{j=1}^{n} A_j$, each path between B_i and A must contain v_i.

Thus, as every A_j has non-empty intersections with B_i and A and $T[A_j]$ is a tree, we obtain $v_i \in A_j, \forall j$, hence $v_i \in A$ and $B_i \cap A \neq \emptyset$. □

Observe that since $|A_i \cap A_j| < t$ for every $i < j$, we have $|\bigcap_{j=1}^{n} A_j| < t$. In view of the previous claim, there must be a vertex $v \in \bigcap_{j=1}^{n} A_j$ such that

$$|\{i : v \in B_i\}| \geq n/t.$$

Set $I = \{i : v \in B_i\}$ and imagine T as a rooted tree with root v. Let $i \in I$ and $j \in [n]$. Each intersection $A_j \cap B_i$, $j \in [n]$, contains a subtree with t vertices rooted in v (because $|A_j \cap B_i| \geq t$). No two intersections $A_j \cap B_i$ and $A_{j'} \cap B_{i'}$ can be the same since $|A_j \cap A_{j'}| < t$ and $|B_i \cap B_{i'}| < t$ for all $i \neq i'$, $j \neq j'$. Therefore,

$$n \times \frac{n}{t} \leq \#\text{ subtrees of } T \text{ of size } t \text{ rooted at } V. \tag{9}$$

It is a well-known fact (cf. [16], page 220) that the number of rooted subtrees of size t of a binary tree is given by the Catalan number $C_t = \binom{2t}{t}/(t+1)$. Since we allow the root v to have three neighbors (denoted by v_1, v_2, v_3), we must adjust the counting: if we remove the edge vv_3, any subtree of size k rooted in v splits into two trees — one rooted in v of size k (where $k \in \{1, \ldots, t\}$) and the other rooted in v_3 of size $t - k$. Notice that the new trees are rooted subtrees of the binary tree, and, therefore, we get

$$\#\text{ subtrees of } T \text{ of size } t \text{ rooted at } v \leq \sum_{k=1}^{t} C_k C_{t-k} < C_{t+1}. \tag{10}$$

The last inequality follows from the fact that $\sum_{k=0}^{t} C_k C_{t-k} = C_{t+1}$.

Combining (9) and (10) yields

$$\frac{n^2}{t} \leq C_{t+1} = \frac{1}{t+2}\binom{2t+2}{t+1}.$$

Since $\binom{2m}{m} \leq 2^{2m}/\sqrt{2m}$, we obtain

$$n^2 \leq \frac{t}{t+2} \times \frac{2^{2t+2}}{\sqrt{2t+2}} < 2^{2t}.$$

Hence, $t > \log_2 n$. □

5. Monotonicity of tree representations

Here we prove Proposition 7.

Proof. Suppose $G \in [\Delta, d, t]$ with vertices $\{v_1, v_2, \ldots, v_n\}$. Let H be the host tree in a (Δ, d, t)-representation of S and $v_i \mapsto S_i$ for every $i \in [n]$.

Let v be any leaf of H. We construct a $(\Delta, \min\{\Delta, d+1\}, t+1)$-representation, $\{S'_i : i \in n\}$, of G, from the subtrees $\{S_i : i \in n\}$ and host tree H.

The $(\Delta, \min\{\Delta, d+1\}, t+1)$-representation will have host tree H' which is H with an additional vertex v' and edge vv'. Notice that H' has the same maximum degree as H. Consider v' to be the root of H'.

Given any subtree S of H, we can consider it to be a subtree of H'. Let $r(S)$ be the vertex in the smallest level, $\ell(r(S))$, of H'. Then for every subtree S of H, $r(S)$ has a unique parent in H'. We call this parent $p(S)$. Notice that if S and T are two subtrees of H then

$$S \cap T \neq \emptyset \quad \text{if and only if} \quad r(S) \in V(T) \text{ or } r(T) \in V(S). \tag{11}$$

For each $i \in [n]$, set $S'_i = S_i + p(S_i)$. Observe that the maximum degree of S'_i is $\min\{\Delta, d+1\}$. We claim that

$$|S_i \cap S_j| \leq |S'_i \cap S'_j| \leq |S_i \cap S_j| + 1 \tag{12}$$

for all $i \neq j$.

The first inequality in (12) is obvious.

Suppose for some $i \neq j$, $|S'_i \cap S'_j| \geq |S_i \cap S_j| + 2$. It is clear that $p(S_i) \neq p(S_j)$, since if $p(S_i) = p(S_j)$ then $|S'_i \cap S'_j| = |S_i \cap S_j| + 1$. Since we gained two vertices, it must be true that $p(S_i) \in V(S_j)$ and $p(S_j) \in V(S_i)$. But $p(S_i) \in V(S_j)$ implies $\ell(r(S_j)) < \ell(r(S_i))$ and $p(S_j) \in V(S_i)$ implies $\ell(r(S_i)) < \ell(r(S_j))$. We have reached a contradiction. Therefore, (12) holds.

By (12), if $|S_i \cap S_j| \leq t-1$ then $|S'_i \cap S'_j| \leq t$.

If $|S_i \cap S_j| = t$, then by (11) either $r(S_i) \in V(S_j)$ or $r(S_j) \in V(S_i)$. If $r(S_i) = r(S_j)$, then $p(S_i) = p(S_j)$ and we have $|S'_i \cap S'_j| = t+1$. Otherwise, without loss of generality, assume that $r(S_i) \in V(S_j)$ and $\ell(r(S_j)) < \ell(r(S_i))$. Then $p(S_i) \in V(S_j)$ and thus, $|S'_i \cap S'_j| \geq t+1$. □

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Appendix. In search of the smallest n such that $t(K_{n,n}) < n$

Careful analysis of the proof of Theorem 3 reveals that our construction gives $t(K_{n,n}) < n$ for $n > n_0 = 589$.

We recall that Theorem 3 yields a $(3, 3, t)$-representation of $K_{n,n}$ for $(2^{h-1}-1)(2^{h-1}-2)^2 < n \leq (2^h-1)(2^h-2)^2$ and $t = t(K_{n,n}) = 7(2^h+1)(h+1) - 6$. It is an easy exercise to verify that $t \leq (2^{h-1}-1)(2^{h-1}-2)^2$ for $h \geq 5$, from which $t(K_{n,n}) < n$ for $n > 3150$ follows.

For $h = 4$, we get a $(3, 3, 589)$-representation of $K_{n,n}$ for every $590 \leq n \leq 3150$. Hence we can set $n_0 = 589$.

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