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Decomposing a planar graph with girth 9 into a forest and a matching

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Abstract

W. He et al. showed that a planar graph of girth 11 can be decomposed into a forest and a matching. D. Kleitman et al. proved the same statement for planar graphs of girth 10. We further improve the bound on girth to 9.

1. Introduction

He, Hou, Lih, Shao, Wang and Zhu [4] proved a family of results on decompositions (i.e., partitions of the edges) of planar graphs with specified girth conditions into a forest and another graph whose maximum degree is not too high. They used these results to derive upper bounds on the game chromatic number and the game coloring number of planar graphs with girth conditions. Balogh et al. [2] proved that a planar graph can be decomposed into three forests so that one of the forests has maximum degree at most 8. They further conjectured that a planar graph can be decomposed into two forests and a third graph with maximum degree at most 4. Gonçalves [3] proved this conjecture. Improving a bound in [4], Kleitman [5] proved that a planar graph with girth 6 can be decomposed into a forest and a subgraph with maximum degree at most

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2. This is an exact result, and our paper was inspired by Kleitman's talk on this result at EXCILL Conference in November, 2006.

In particular, He et al. [4] proved that a planar graph with girth 11 or more can be decomposed into a forest and a matching. Kleitman et al. [1] proved the same statement for planar graphs with girth at least 10. Our main result here strengthens these results.

Theorem 1. *Every planar graph with girth at least* 9 *can be decomposed into a forest and a matching.*

This implies that the game chromatic number and the game coloring number of every planar graph with girth at least 9 is at most 5.

By an *FM-coloring* of a graph we mean a partition of its edges into a forest colored with F and a matching colored with M. Given a graph G and a cycle C in G, an *FM*-coloring of $G - E(C)$ is called *a good coloring of G w.r.t. C* (or just *a good coloring* whenever G and C are clear from the context) if it has the following properties (i)–(iii):

(i) the edges colored F form a forest and those colored M form a matching;

(ii) all edges not in C incident with vertices of C are colored with F;

(iii) there is no path joining two vertices of C whose all edges are colored F and do not belong to C.

Instead of Theorem 1, it was easier for us to prove a stronger fact:

Theorem 2. *For every planar graph G with girth at least* 9 *and any cycle C in G of length at most* 13*, there is a good coloring of G w.r.t. C.*

Note that a good coloring of G w.r.t. C combined with any *FM*-coloring of C yields an *FM*-coloring of G. Hence, Theorem 1 follows from Theorem 2, since if a graph G has no cycles of length $l \in \{9, 10, 11, 12, 13\}$, we can add such a cycle C disjoint from G and apply Theorem 2 to the new graph.

In fact, our proof can be modified to yield a polynomial-time algorithm for finding *FM*-colorings in planar graphs with girth at least 9.

The question whether the result of Theorem 1 holds for planar graphs of girth 8 remains open, and is an interesting challenge. D.J. Kleitman (private communication) suggests that it does.

The structure of the paper is as follows. In the next section we derive some elementary properties of a hypothetical minimal counterexample G to Theorem 2. In Section 3 we prove that this G cannot contain faces of some special kinds. We finish the proof with a discharging argument in Section 4.

2. Properties of minimal counterexamples

Let G be a counterexample to Theorem 2 with the fewest vertices, and let C_0 be a cycle in G of length at most 13 such that there is no good coloring of G w.r.t. C_0.

In this section we prove five elementary properties of G.

Claim 3. *G is connected.*

Proof. If G_1 and G_2 are two distinct components of G, then identifying a vertex of G_1 with a vertex of G_2 creates a planar graph G' of girth at least 9 with fewer vertices. By the minimality of G, graph G_1 has a good coloring w.r.t. C_0, which yields a good coloring of G w.r.t. C_0. □

Claim 4. *G has no vertex with degree at most one.*

Proof. By Claim 3, if v is such a vertex, then $d(v) = 1$. Furthermore, any good coloring of $G - v$ yields a good coloring of G when we color the edge at v with F. □

Claim 5. *If u and v are adjacent vertices of degree two in G, then both u and v are on C_0.*

Proof. Let u and v be two adjacent 2-vertices not both in C_0. Then neither of them is in C_0. Let $G' = G - \{u, v\}$. Then a good coloring of G' augmented by coloring edge uv with M and the other two deleted edges with F is a good coloring of G; a contradiction. □

Claim 6. *G has no separating cycle of length at most* 13.

Proof. Suppose C is a separating cycle of length at most 13 (coinciding with C_0 if C_0 is separating). By the symmetries between C_0 and C and between the interior and the exterior of C, we may assume that no vertex of C_0 is (strictly) inside C. Let G' and G'' be the graphs obtained from G by deleting all vertices inside and outside of C, respectively. By definition, each of G' and G'' has fewer vertices than G. Hence G' has a good coloring φ' w.r.t. C_0 and G'' has a good coloring φ'' w.r.t. C (C and C_0 may coincide). By pasting φ' and φ'', we get a good coloring of G w.r.t. C_0. □

By Claim 6, from now on we may assume that C_0 is the boundary cycle of the outer face, f_∞, of G.

Claim 7. *G has no cut vertex.*

Proof. Assume the contrary. Let B be a pendant block of G that does not contain C_0, and let y be the only cut vertex in B. Then $G' = G - (B - y)$ has a good coloring φ' w.r.t. C_0.

Let xy be an edge in B. We construct graph G'' from B by adding to B the path $P = (x, v_1, \ldots, v_7, y)$, where $v_1, \ldots, v_7$ are all new vertices. Let C be the cycle formed by P and edge xy. Since at least eight vertices of C_0 do not belong to B and hence to G'', G'' has fewer vertices than G. Thus G'' has a good coloring φ'' w.r.t. C. Let φ be the edge coloring of B obtained from φ'' restricted to $E(B)$ by coloring xy with F. By the definition of a good coloring w.r.t. C, φ is an *FM*-coloring of B and every edge incident with y is colored F. Now pasting φ' and φ yields a desired good coloring of G w.r.t. C_0. □

3. On short faces in G

In this section we prove the non-existence of some "short" faces in G disjoint from C_0. This is an important step in the proof of the non-existence of our counterexample G.

If a face shares an edge with C_0, then it is called an *L-face*, otherwise it is an *N-face*. An N-face is an *N^*-face* if it has no common vertices with C_0. A vertex v of degree 2 is an *L-vertex* if v is incident with an L-face and $v \notin C_0$.

In a (partial or full) good coloring of G w.r.t. C_0, a vertex is called *anchored* if there is a path from that vertex to C_0 using only edges colored F. Two vertices are *related* if they either belong to the same F-component, or are both anchored. (One may view all anchored vertices as belonging to the same virtual F-component containing C_0.) Observe that while extending a good partial coloring of G to a good coloring of G, we should neither join related vertices by F-paths nor create adjacent M-edges.

Claim 8. *G has no N^*-face of length* 10 *with degree sequence* $(x, 2, 3, 2, 3, 2, 3, 2, 3, 2)$.

Proof. Suppose that G contains an N^*-face f with the boundary cycle $C = (v_1, v_2, \ldots, v_{10})$, whose degree sequence is $(x, 2, 3, 2, 3, 2, 3, 2, 3, 2)$. By Claim 5, $x = d(v_1) \geq 3$. For $i \in \{1, \ldots, 10\}$, let v_i' be one of the neighbors of v_i in $G - C$ whenever $d(v_i) \geq 3$. Let G' be obtained from G by adding the edge $v_1 v_5'$ and deleting all vertices of C except v_1.

First we show that G' has no cycle of length at most 8. Indeed, otherwise G has a path, P, of length at most 7 from v_1 to v_5', and this path together with the path $v_1 v_2 v_3 v_4 v_5 v_5'$ constitutes a cycle, S, of length at most 12 in G. Observe that S is separating; for instance, it separates v_3' (which cannot lie in P since S has no chords) from v_6. This contradicts Claim 6.

Since C is an N^*-face, $V(C_0) \subseteq V(G')$. By the minimality of G, G' has a good coloring φ' w.r.t. C_0. We will extend φ to a good coloring of G w.r.t. C_0.

If no edge incident with v_1 in $G' - v_1 v_5'$ has color M, then it suffices to color the edges $v_1 v_2, v_3 v_4, v_5 v_6, v_7 v_8, v_9 v_{10}$ with M and all other uncolored edges on or incident with C with F. Assume now that $v_1 v_1'$ is colored with M. Then $v_1 v_5'$ has color F in G'. We color $v_3 v_4, v_5 v_6, v_7 v_8, v_9 v_{10}$ by M and the other uncolored edges by F; this coloring is denoted by $\varphi[G]$.

Suppose $\varphi[G]$ fails to be good. Note that the only vertex of C that may have more than one neighbor outside C is v_1. Thus if $\varphi[G]$ is not good, then the F-colored path $P_{13} = v_1 v_2 v_3 v_3'$ belongs either to an F-cycle, or to an F-path joining two vertices of C_0; that is, v_1 and v_3' are related in $G - v_2$, and also in $G' - v_1 v_5'$. In this case, since $v_1 v_5'$ is colored with F, vertices v_3' and v_5' are not related in $G' - v_1 v_5'$. Therefore, the coloring $\varphi^*[G]$ obtained from $\varphi[G]$ by swapping colors on the edges $v_2 v_3$ and $v_3 v_4$ is good. □

Claim 9. *G has no N^*-face of length 9 with degree sequence* $(3, 3, 2, x, 2, 3, 2, 3, 2)$, $(3, 3, 2, 3, 2, x, 2, 3, 2)$, *or* $(x, 3, 2, 3, 2, 3, 2, 3, 2)$.

Proof. Suppose that G contains an N^*-face $C = (v_1, v_2, \ldots, v_9)$ with one of the above degree sequences. W.l.o.g., we may assume that $d(v_3) = d(v_5) = d(v_7) = d(v_9) = 2$. Let v_i' be one of the neighbors of v_i in $G - C$ when $d(v_i) \geq 3$.

Case 1. $d(v) \leq 3$ whenever $v \in C - v_4$. Let G' be the graph obtained from G by identifying v_4 with v_1' and removing all vertices in $C - \{v_4\}$. The girth of G' is still at least 9, since otherwise there would be a separating cycle of length at most 12 using the path $v_1' v_1 v_2 v_3 v_4$. Since C is an N^*-face, $V(C_0) \subseteq V(G')$. Thus, G' has a good coloring φ w.r.t. C_0. We will extend it to $G - E(C_0)$. Recall that in doing so, we should not connect related vertices by F-paths.

First suppose that all edges incident with v_4 in G are colored with F. Note that by the construction of G', vertices v_1' and v_4 are not related. Color $v_2 v_3$, $v_4 v_5$, $v_6 v_7$, and $v_8 v_9$ with M, and all other uncolored edges with F. This coloring is good unless v_1' is related to v_2', in which case v_2' is not related to v_4, and it suffices to recolor edges $v_1 v_2$ and $v_2 v_3$.

Now assume that $\varphi(v_4 v_4') = M$. Let φ_1 be obtained from φ by coloring edges $v_2 v_3$, $v_5 v_6$, $v_7 v_8$, and $v_9 v_1$ with M, and all other uncolored edges with F. Then φ_1 fails to be good only if v_1' and v_2' are related. Let φ_2 be obtained from φ_1 by recoloring edges $v_1 v_1'$ and $v_1 v_9$. Then φ_2 is not good only if v_1' is related to v_8'. Suppose this is the case; then recoloring $v_7 v_8$ and $v_8 v_9$ does not work only if v_8' is related to v_6'. By transitivity, v_6' is then related to v_1' and thus cannot be related to v_4, so that recoloring $v_5 v_6$ and $v_6 v_7$ produces a good coloring of G.

Case 2. $d(v) \leq 3$ whenever $v \in C - \{v_1, v_6\}$. Let G' be the graph obtained from G by identifying v_1 with v_6 and removing the vertices in $C - \{v_1, v_2, v_6\}$. Then the girth is still at least 9, since otherwise there would be a separating cycle in G of length at most 12 using the path $v_1v_9v_8v_7v_6$. Again, G' has a good coloring φ w.r.t. C_0, and we will extend it to $G - E(C_0)$.

Subcase 2.1. $d(v_1) = 3$. Recall that in the partial coloring of G induced by φ, vertices v_1' and v_6 are not related since v_1 was identified with v_6 in G'. If v_1' and v_2' are not related in $G - v_1 - v_2$, then we uncolor $v_1'v_1$, v_1v_2, and v_2v_2'; afterwards, color v_2v_3, v_4v_5, v_7v_8, and v_9v_1 with M and all the other uncolored edges with F. So assume that v_1' and v_2' are related in $G - v_1 - v_2$. This means that either $\varphi(v_1v_1') = M$, or $\varphi(v_1v_2) = M$, or $\varphi(v_2v_2') = M$.

If $\varphi(v_1'v_1) = M$, then we uncolor v_2v_2' and color v_2v_3, v_4v_5, v_6v_7, and v_8v_9 with M and the other uncolored edges with F. So suppose $\varphi(v_1'v_1) = F$.

Suppose now that $\varphi(v_2v_2') = M$. If v_1' and v_4' are not related, then we color v_4v_5, v_7v_8, and v_9v_1 with M and the other uncolored edges with F. Otherwise, v_4' and v_6 are not related, so switching the colors of v_3v_4 and v_4v_5 blocks the F-path from v_1 to v_4 and yields an appropriate coloring.

Finally, suppose $\varphi(v_1v_2) = M$. Then v_6 has no incident M-edges, since it was identified with v_1 in G'. If v_8' is not related to v_1', then we can color v_3v_4, v_5v_6, and v_7v_8 with M and the other uncolored edges with F. Otherwise, v_8' is not related to v_6, and we just switch the colors of v_7v_8 and v_8v_9 in the last coloring.

Subcase 2.2. $d(v_6) = 3$. If $\varphi(v_6v_6') = M$, then it suffices to recolor v_6v_6' with F, color edges v_3v_4, v_5v_6, v_7v_8, and v_9v_1 with M, and the other uncolored edges with F. So suppose $\varphi(v_6v_6') = F$.

Note that in the partial coloring of G induced by φ, vertices v_1 and v_6' are not related. This implies that v_8' is not related either to v_1 or to v_6'. In the former case we are done by recoloring v_9v_1 with F in the previous coloring. In the latter, we are done by exchanging colors of edges v_7v_8 and v_8v_9 in the last coloring. □

4. Discharging

Now we employ a discharging argument to show that no planar graph of girth at least 9 can satisfy all Claims 3–9. That will finish the proof of Theorem 2.

Let $d(y)$ denote the degree of a vertex y or the size of a face y. Let the *initial charge* of a vertex v be $\mu(v) = 2d(v) - 6$, the initial charge of a face $f \neq f_\infty$ be $\mu(f) = d(f) - 6$, and let $\mu(f_\infty) = d(f_\infty) + 5.5$.

Since G is connected, Euler's formula yields

$$2 \sum_{v \in V(G)} (d(v) - 3) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

Hence,

$$2 \sum_{v \in V(G)} (d(v) - 3) + \sum_{f \in F(G), f \neq f_\infty} (d(f) - 6) + d(f_\infty) + 5.5 = -0.5,$$

and therefore

$$\sum_{y \in V(G) \cup F(G)} \mu(y) < 0. \tag{1}$$

The vertices and faces of G discharge their initial charge by the following rules:

Rule 1. Every N-face gives 1 to each incident vertex of degree 2.

Rule 2. Every L-face gives 1 to each of its L-vertices, and gives its remaining charge (positive or negative) to f_∞.

Rule 3. Every vertex v of degree at least 4 distributes its positive charge equally to the incident faces if $v \notin C_0$; otherwise, v gives 1 to each incident N-face.

Rule 4. f_∞ gives 2 to each incident vertex of degree 2.

In the rest of the proof we show that the *final charge* $\mu^*(y)$ is nonnegative for each $y \in V(G) \cup F(G)$, which contradicts (1), since the total charge does not change.

For $v \in V(G) \setminus V(C_0)$, we have $\mu^*(v) = 0$: either by Rules 1 and 2 if $d(v) = 2$, or by Rule 3 if $d(v) \geq 3$. Suppose $v \in V(C_0)$ and $d(v) \geq 4$; then $\mu^*(v) \geq 2(d(v) - 3) - (d(v) - 3) > 0$ by Rule 3 again. Every $v \in V(C_0)$ with $d(v) = 3$ gives out nothing, so $\mu^*(v) = \mu(v) = 0$. Every $v \in V(C_0)$ with $d(v) = 2$ has $\mu^*(v) = 0$ by Rule 4.

If f is an L-face then $\mu^*(f) = 0$ by Rule 2. Suppose f is an N-face. By Claim 5, f is incident with at most $\lfloor d(f)/2 \rfloor$ vertices of degree 2. Thus $\mu^*(f) \geq d(f) - 6 - 1 \cdot \lfloor d(f)/2 \rfloor + w = \lceil d(f)/2 \rceil - 6 + w$, where w is the charge obtained from vertices of degree at least 4; this implies that $\mu^*(f) \geq 0$ if $d(f) \geq 11$. If $d(f) = 10$ and $\mu^*(f) < 0$, then by Rules 1 and 2, f should be an N-face and have exactly five 2-vertices on its boundary. Furthermore, by Rule 3, f is an N^*-face incident with at least 4 vertices of degree 3. So, by Claim 5, f has degree sequence $(x, 2, 3, 2, 3, 2, 3, 2, 3, 2)$, a contradiction to Claim 8. If $d(f) = 9$ then a similar argument leads to a contradiction: If there were such an N-face f with negative charge, this face should be adjacent to four 2-vertices. Then by Rule 3, f would be an N^*-face with few large degree vertices, contrary to Claim 9.

Finally, we show that $\mu^*(f_\infty) \geq 0$. For each L-face f, let $C(f)$ denote the cycle bounding f and let $L = L(f)$ be the set of the common edges of $C(f)$ with C_0. The components of the subgraph of G spanned by the edges of $L(f)$ are paths. We call these paths *common segments of* $C(f)$ *and* C_0. If these segments are $X_1, \ldots, X_r$, then we say that $r(f) = r$ and denote $x_i = |E(X_i)|$ for $i = 1, \ldots, r$. The components of $C(f) - E(L(f))$ are also paths, called *segments of* $C(f)$ *distinct from* C_0. Clearly, the number of such segments is also $r(f)$. If these segments are $Y_1, \ldots, Y_r$, then let $y_i = |E(Y_i)|$ for $i = 1, \ldots, r(f)$. By definition, each L-face f has $r(f) \geq 1$, and $x_i \geq 1$ for each $1 \leq i \leq r(f)$.

By Claim 5, there are at most $\lfloor 0.5 y_i \rfloor$ vertices of degree 2 on each segment Y_i in $C(f)$. Since $\sum_{i=1}^{r} y_i + \sum_{i=1}^{r} x_i = d(f)$, the charge that f gives to f_∞ is at least

$$d(f) - 6 - \sum_{i=1}^{r} \lfloor 0.5 y_i \rfloor \geq d(f) - 6 - \sum_{i=1}^{r} 0.5 y_i = 0.5 d(f) - 6 + 0.5 \sum_{i=1}^{r} x_i .$$

Note that there are at least $\sum_f r(f)$ vertices of degree more than 2 on C_0; so

$$\mu^*(f_\infty) \geq (|C_0| + 5.5) - 2 \left(|C_0| - \sum_f r(f) \right) + \sum_f \left(0.5 d(f) - 6 + 0.5 \sum_{i=1}^{r(f)} x_i \right). \quad (2)$$

Since $\sum_f \sum_{i=1}^{r(f)} x_i = |C_0|$, we have $\mu^*(f_\infty) \geq 5.5 - 0.5|C_0| - \sum_f (6 - 0.5 d(f) - 2r(f))$. From $r(f) \geq 1$ and $d(f) \geq 9$, we obtain $0.5 d(f) + 2r(f) - 6 \geq 0.5$ for any L-face f. Recalling that there are at least two L-faces (by Claim 7) and that $|C_0| \leq 13$, we get $\mu^*(f_\infty) \geq 5.5 - 0.5|C_0| + 1 \geq 0$, as desired.

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