

# On Directed Triangles in Digraphs\*

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## Abstract

Using a recent result of Chudnovsky, Seymour, and Sullivan, we slightly improve two bounds related to the Caccetta-Haggkvist Conjecture. Namely, we show that if  $\alpha \geq 0.35312$ , then each  $n$ -vertex digraph  $D$  with minimum outdegree at least  $\alpha n$  has a directed 3-cycle. If  $\beta \geq 0.34564$ , then every  $n$ -vertex digraph  $D$  in which the outdegree and the indegree of each vertex is at least  $\beta n$  has a directed 3-cycle.

## 1 Introduction

In this note we follow the notation of [5]. For a vertex  $u$  in a digraph  $D = (V, E)$ , let  $N^+(u) = \{v \in V : (u, v) \in E\}$  and  $N^-(u) = \{v \in V : (v, u) \in E\}$ . Every digraph in this note has no parallel or antiparallel edges.

Caccetta and Häggkvist [2] conjectured that each  $n$ -vertex digraph with minimum outdegree at least  $d$  contains a directed cycle of length at most  $\lceil n/d \rceil$ . The following important case of the conjecture is still open: *Each  $n$ -vertex digraph with minimum outdegree at least  $n/3$  contains a directed triangle.* Caccetta and Häggkvist [2] proved the following weakening of the conjecture.

**Theorem 1.** [2] *If  $\alpha \geq (3 - \sqrt{5})/2 \sim 0.38196\dots$ , then each  $n$ -vertex digraph  $D$  with minimum outdegree at least  $\alpha n$  has a directed 3-cycle.*

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Then Bondy [1] relaxed the restriction on  $\alpha$  in Theorem 1 to  $\alpha \geq (2\sqrt{6} - 3)/5 \sim 0.37979$  and Shen [5] relaxed it to  $\alpha \geq 3 - \sqrt{7} \sim 0.354248$ .

De Graaf, Schrijver, and Seymour [4] considered the corresponding problem for digraphs in which both the outdegrees and indegrees are bounded from below. They proved that every  $n$ -vertex digraph in which the outdegree and the indegree of each vertex is at least  $0.34878n$  has a directed 3-cycle. Shen's bound [5] on  $\alpha$  implies an improvement of the de Graaf–Schrijver–Seymour bound to  $0.347785n$ . Here we use a recent result of Chudnovsky, Seymour, and Sullivan [3] to somewhat improve these results as follows.

**Theorem 2.** *If  $\alpha \geq 0.35312$ , then each  $n$ -vertex digraph  $D$  with minimum outdegree at least  $\alpha n$  has a directed 3-cycle.*

**Theorem 3.** *If  $\beta \geq 0.34564$ , then each  $n$ -vertex digraph  $D$  in which both minimum outdegree and minimum indegree is at least  $\beta n$  has a directed 3-cycle.*

In the next section, we cite the Chudnovsky–Seymour–Sullivan result and a conjecture of theirs, and derive a useful consequence. In Section 3, we outline Shen's proof of his bound on  $\alpha$  in [5]. In Sections 4 and 5 we prove Theorem 2. In Section 6 we outline a part of the proof in [4] and prove Theorem 3.

## 2 A result on dense digraphs

Chudnovsky, Seymour, and Sullivan [3] proved the following fact.

**Lemma 4.** *If a digraph  $D$  is obtained from a tournament by deleting  $k$  edges and has no directed triangles, then one can delete from  $D$  an additional  $k$  edges so that the resulting digraph  $D'$  is acyclic.*

We use this fact for the following lemma.

**Lemma 5.** *If a digraph  $D$  is obtained from a tournament by deleting  $k$  edges and has no directed triangles, then it has a vertex with outdegree less than  $\sqrt{2k}$  (and a vertex with indegree less than  $\sqrt{2k}$ ).*

**Proof.** Let  $m = \lceil \sqrt{2k} \rceil$ . By Lemma 4,  $D$  contains an acyclic digraph  $D'$  with at least  $|E(D)| - k$  edges. Arrange the vertices of  $D'$  in an order  $u_1, u_2, \dots, u_q$  so that there are no backward edges. If  $D$  has no vertices with outdegree less than  $m$ , then for each  $i = 0, 1, \dots, m$ , the set  $E(D) - E(D')$  contains at least  $m - i$  edges starting at vertex  $u_{q-i}$ . Hence

$$k \geq 1 + 2 + \dots + m = \binom{m+1}{2} > \frac{m^2}{2} \geq k,$$

a contradiction. □

In fact, Chudnovsky, Seymour, and Sullivan [6, Conjecture 6.27] conjectured the following improvement of Lemma 4.

**Conjecture 6.** *If a digraph  $D$  is obtained from a tournament by deleting  $k$  edges and has no directed triangles, then one can delete from  $D$  at most  $k/2$  additional edges so that the resulting digraph  $D'$  is acyclic.*

If true, this conjecture would imply the following strengthening of Lemma 5: *Each digraph  $D$  obtained from a tournament by deleting  $k$  edges, that has no directed triangles, has a vertex with outdegree less than  $\sqrt{k}$ .* This in turn would imply some improvements in the bounds of Theorems 2 and 3.

### 3 A sketch of Shen's proof

In this section, we outline the proof in [5]. Assume that there exists an  $n$ -vertex digraph  $D = (V, E)$  without directed triangles with  $\deg^+(u) = r = \lceil n\alpha \rceil$  for all  $u \in V(D)$ . We may assume that  $D$  has the fewest vertices among digraphs with this property.

For each arc  $(u, v) \in E$ , set

$$P(u, v) := N^+(v) \setminus N^+(u),$$

$$p(u, v) := |P(u, v)|, \text{ the number of induced directed 2-paths whose first edge is } (u, v);$$

$$Q(u, v) := N^-(u) \setminus N^-(v),$$

$$q(u, v) := |Q(u, v)|, \text{ the number of induced directed 2-paths whose last edge is } (u, v);$$

$$T(u, v) := N^+(u) \cap N^+(v),$$

$$t(u, v) := |T(u, v)|, \text{ the number of transitive triangles having edge } (u, v) \text{ as "base."}$$

Let  $t$  be the number of *transitive* triangles in  $D$ . Note that

$$t = \sum_{(u,v) \in E(D)} t(u, v). \tag{1}$$

It was proved in [5] that

$$n > 2r + \deg^-(v) + q(u, v) - \alpha t(u, v) - p(u, v) \tag{2}$$

for every  $(u, v) \in E(D)$ . The idea is the following: the sets  $N^+(v)$ ,  $N^-(v)$ , and  $Q(u, v)$  are disjoint. Moreover, every vertex in  $T(u, v)$  cannot have outneighbors in  $N^-(v) \cup Q(u, v)$ . By the minimality of  $D$ , some vertex  $w \in T(u, v)$  (if  $T(u, v)$  is non-empty) has fewer than  $\alpha t(u, v)$  outneighbors in  $T(u, v)$ . Hence  $w$  has at least  $r - p(u, v) - \alpha t(u, v)$  outneighbors outside of  $N^-(v) \cup Q(u, v)$ . This yields (2).

Summing inequalities (2) over all edges in  $D$  and observing that

$$\sum_{(u,v) \in E(D)} (2r - n) = rn(2r - n),$$

$$\sum_{(u,v) \in E(D)} \deg^-(v) = \sum_{v \in V(D)} (\deg^-(v))^2 \geq r^2n, \tag{3}$$

$$\sum_{(u,v) \in E(D)} q(u,v) = \sum_{(u,v) \in E(D)} p(u,v), \quad (4)$$

by (1), Shen concludes that

$$\alpha t > rn(3r - n). \quad (5)$$

Noting that  $t \leq n \binom{r}{2}$ , Shen derives the inequality  $\alpha^2 - 6\alpha + 2 > 0$  and concludes that  $\alpha < 3 - \sqrt{7}$ .

## 4 Preliminaries

In this and the next sections, we will follow Shen's scheme and use Lemma 5 to prove Theorem 2.

So, let  $\alpha \geq 0.35312$  and let  $D$  be the smallest counterexample to Theorem 2. Below we use notation from the previous section.

**Lemma 7.** *If  $|V(D)| = n$ , then  $t > 0.476r^2n$ .*

**Proof.** If  $t \leq 0.476r^2n$ , then by (5)

$$0.476r^2n\alpha > rn(3r - n).$$

Dividing by  $r^2n$  and rearranging we get

$$0.476\alpha + \frac{n}{r} > 3.$$

Since  $\frac{n}{r} \leq \frac{1}{\alpha}$  and  $\alpha > 0$  we have

$$0.476\alpha^2 - 3\alpha + 1 > 0.$$

This means that  $\alpha < 0.35312$ , a contradiction.  $\square$

**Lemma 8.** *For every  $v \in V(D)$ ,  $|N^-(v)| < 1.186r$ .*

**Proof.** Suppose that  $|N^-(v)| \geq 1.186r$ . By the minimality of  $D$ , some vertex  $w \in N^+(v)$  has fewer than  $\alpha r$  outneighbors in  $N^+(v)$ . Since  $N^+(w)$  and  $N^-(v)$  are disjoint,

$$n > |N^-(v)| + 2r - \alpha r \geq r(3.186 - \alpha).$$

Hence  $\alpha^2 - 3.186\alpha + 1 > 0$  and therefore,  $\alpha < 1.593 - \sqrt{1.593^2 - 1} < 0.353$ , a contradiction.  $\square$

For each  $(u,v) \in E(D)$ , let  $f(u,v)$  be the number of missing edges in  $N^+(u) \cap N^+(v)$ . Similarly, for each  $u \in V(D)$ , let

$$f(u) = \binom{r}{2} - |E(D(N^+(u)))| \quad \text{and} \quad t(u) = |E(D(N^+(u)))|.$$

Clearly,  $f(u)$  is the number of missing edges in  $N^+(u)$  and  $t(u)$  is the number of transitive triangles in  $D$  with source vertex  $u$ . By definition,  $t(u) + f(u) = \binom{r}{2}$  for each  $u \in V(D)$ , and  $t = \sum_{u \in V(D)} t(u)$ . Let  $f = \sum_{u \in V(D)} f(u)$  and  $\gamma = \frac{f}{r^2 n}$ . Then

$$t = \binom{r}{2} n - f = \binom{r}{2} n - \gamma r^2 n \leq (0.5 - \gamma) r^2 n,$$

and by Lemma 7,

$$\gamma \leq 0.5 - \frac{t}{r^2 n} < 0.5 - 0.476 = 0.024. \quad (6)$$

**Lemma 9.**

$$\sum_{(u,v) \in E(D)} f(u,v) < \frac{1.172}{2} r f = 0.586r \sum_{u \in V(D)} f(u).$$

**Proof.** Let  $\overline{E}(D)$  denote the set of *non-edges* of  $D$ , that is, the pairs  $xy \in \binom{V(D)}{2}$  such that neither  $(x, y)$  nor  $(y, x)$  is an edge in  $D$ . Note that  $\sum_{u \in V(D)} f(u) = \sum_{xy \in \overline{E}(D)} |N^-(x) \cap N^-(y)|$  and that  $\sum_{(u,v) \in E(D)} f(u,v) = \sum_{xy \in \overline{E}(D)} |E(D(N^-(x) \cap N^-(y)))|$ . Therefore, the statement of the lemma holds if for every  $xy \in \overline{E}(D)$ ,

$$|E(D(N^-(x) \cap N^-(y)))| < 0.586r |N^-(x) \cap N^-(y)|. \quad (7)$$

Let  $|N^-(x) \cap N^-(y)| = q$ . Since  $|E(D(N^-(x) \cap N^-(y)))| \leq \binom{q}{2} = \frac{q-1}{2}q$ , we see that (7) is clearly true when  $q < r$ . Therefore we assume that  $q \geq r$ . Let  $k$  denote the number of edges missing from  $D(N^-(x) \cap N^-(y))$ . Note that any acyclic digraph on  $q$  vertices, with maximum outdegree at most  $r$ , has at most  $\binom{r}{2} + r(q-r) = \binom{q}{2} - \binom{q-r}{2}$  edges. Since  $D(N^-(x) \cap N^-(y))$  itself contains no directed triangle and has maximum outdegree at most  $r$ , by Lemma 4 it contains an acyclic subgraph with at least  $\binom{q}{2} - 2k$  edges. Therefore

$$\binom{q}{2} - 2k \leq \binom{q}{2} - \binom{q-r}{2},$$

implying that  $k \geq \frac{1}{2} \binom{q-r}{2}$ . Therefore we find  $|E(D(N^-(x) \cap N^-(y)))| \leq \binom{q}{2} - \frac{1}{2} \binom{q-r}{2}$ . To verify (7) then, we simply need to check that for  $q \geq r$  we have

$$\binom{q}{2} - \frac{1}{2} \binom{q-r}{2} < 0.586rq.$$

Suppose the contrary. Then

$$\begin{aligned} \binom{q}{2} - \frac{1}{2} \binom{q-r}{2} &\geq 0.586rq \\ 2q(q-1) - (q-r)(q-r-1) &\geq 2.344rq \\ q^2 + (2r-1-2.344r)q - r(r+1) &\geq 0 \\ q^2 - 0.344rq - r^2 &> 0. \end{aligned}$$

But this implies  $q > (0.344r + r\sqrt{4.118336})/2 > 1.1866r$ , contradicting Lemma 8.  $\square$

## 5 Proof of Theorem 2

Let  $(u, v) \in E(D)$ . By Lemma 5, some vertex  $w \in N^+(u) \cap N^+(v)$  has at most  $\sqrt{2f(u, v)}$  outneighbors in  $N^+(u) \cap N^+(v)$ . Other outneighbors of  $w$  are in  $V(D) \setminus (T(u, v) \cup Q(u, v) \cup N^-(v) \cup \{u\})$ . Thus, we have

$$n > 2r + \deg^-(v) + q(u, v) - p(u, v) - \sqrt{2f(u, v)}. \quad (8)$$

Summing over all  $(u, v) \in E(D)$ , we get

$$r \cdot n^2 > 2r^2n + \sum_{(u,v) \in E(D)} \deg^-(v) + \sum_{(u,v) \in E(D)} (q(u, v) - p(u, v)) - \sum_{(u,v) \in E(D)} \sqrt{2f(u, v)}.$$

Applying (3) and (4), we get

$$r \cdot n^2 > 3r^2n - \sum_{(u,v) \in E(D)} \sqrt{2f(u, v)} \geq 3r^2n - rn \sqrt{\frac{2 \sum_{(u,v) \in E(D)} f(u, v)}{rn}}. \quad (9)$$

By Lemma 9,

$$rn \sqrt{\frac{2 \sum_{(u,v) \in E(D)} f(u, v)}{rn}} \leq rn \sqrt{\frac{1.172r \cdot f}{rn}} = rn \sqrt{\frac{1.172\gamma r^2n}{n}} = r^2n \sqrt{1.172\gamma}.$$

Plugging this in (9) and dividing both sides by  $r^2n$ , we get

$$\frac{n}{r} > 3 - \sqrt{1.172\gamma}. \quad (10)$$

From this and (6), we have

$$\frac{r}{n} < \frac{1}{3 - \sqrt{1.172 \cdot 0.024}} \leq 0.35307,$$

a contradiction.

## 6 Digraphs with bounded indegrees and outdegrees

Let  $k = \lceil n\beta \rceil$  and assume that there exists an  $n$ -vertex digraph  $D = (V, E)$  without directed triangles with  $\deg^+(u) \geq k$  and  $\deg^-(u) \geq k$  for all  $u \in V(D)$ . We may assume that after deleting any edge, some vertex will have either indegree or outdegree less than  $k$ .

For each edge  $(u, v) \in E$ , set  $T^+(u, v) := N^+(u) \cap N^+(v)$ ,  $T^-(u, v) := N^-(u) \cap N^-(v)$ ,  $t^+(u, v) := |T^+(u, v)|$ ,  $t^-(u, v) := |T^-(u, v)|$ .

Let  $s = 1/\alpha$ , where  $\alpha$  is the smallest positive real such that for each  $n$  every  $n$ -vertex digraph with minimum outdegree greater than  $\alpha n$  has a directed triangle. By Theorem 2,  $\alpha \leq 0.35312$ .

The following properties of  $D$  are proved in [4].

(i) *There exists a vertex  $v'$  with both indegree and outdegree equal to  $k$  (see Equation (4) on p. 280).*

(ii) *For all  $u, v, w \in V$ , if  $(u, v), (v, w), (u, w) \in E(D)$ , then*

$$t^-(u, v) + t^+(v, w) \geq 4k - n \quad (\text{see Equation (5) on p. 281}). \quad (11)$$

(iii) *For each edge  $(u, v) \in E$ ,*

$$t^-(u, v) \geq (3k - n)s = \frac{3k - n}{\alpha} \quad \text{and} \quad t^+(u, v) \geq (3k - n)s = \frac{3k - n}{\alpha} \quad (\text{see (6) on p. 281}). \quad (12)$$

(iv)  *$k^2 > 2(3k - n)(5k - n - 2(3k - n)s)s$  (see the equation between (14) and (16) on p. 282).*

In fact, the  $k^2$  on the left-hand side of the last inequality is simply the upper bound for the total number of edges,  $|E(D(N^-(v')))| + |E(D(N^+(v')))|$ , in the in-neighborhood and the out-neighborhood of  $v'$ . Thus, if the total number of edges in the in-neighborhood and the out-neighborhood of  $v'$  is  $(1 - \gamma)k^2$ , then instead of (iv) we can write

$$(1 - \gamma)k^2 > 2(3k - n)(5k - n - 2(3k - n)s)s. \quad (13)$$

Dividing both sides of (13) by  $k^2$  and rearranging, we get the following slight variation of Inequality (16) in [4]:

$$(4s^2 - 2s)(n/k)^2 - (24s^2 - 16s)(n/k) + (36s^2 - 30s + 1 - \gamma) > 0.$$

Note that there is a misprint in [4]: the last summand in (16) is  $(36s^2 - 20s + 1)$  instead of  $(36s^2 - 30s + 1)$ . Letting  $x = n/k$  and  $\lambda = 2s = 2/\alpha$ , we have

$$(\lambda^2 - \lambda)x^2 - 2(3\lambda^2 - 4\lambda)x + (9\lambda^2 - 15\lambda + 1 - \gamma) > 0. \quad (14)$$

The roots of (14) are

$$\begin{aligned} x_{1,2} &= \frac{3\lambda^2 - 4\lambda \pm \sqrt{(3\lambda^2 - 4\lambda)^2 - (\lambda^2 - \lambda)(9\lambda^2 - 15\lambda + 1 - \gamma)}}{\lambda^2 - \lambda} \\ &= \frac{3\lambda^2 - 4\lambda \pm \sqrt{\gamma\lambda^2 + (1 - \gamma)\lambda}}{\lambda^2 - \lambda} = 3 - \frac{1 \pm \sqrt{\gamma + (1 - \gamma)/\lambda}}{\lambda - 1}. \end{aligned}$$

Since  $x = n/k$  and we know from [4] that  $n/k > 2.85$ , we conclude that

$$x > 3 - \frac{1 - \sqrt{\gamma + (1 - \gamma)/\lambda}}{\lambda - 1}. \quad (15)$$

Let  $f_1$  be the number of non-edges in  $N^+(v')$  and  $f_2$  be the number of non-edges in  $N^-(v')$ . Then, by the definition of  $\gamma$ ,  $f_1 + f_2 + (1 - \gamma)k^2 = k^2 - k$ , and hence

$$\gamma k^2 > f_1 + f_2.$$

Comparing Lemma 5 with (iii), we have

$$\sqrt{2f_1} \geq (3k - n)s \quad \text{and} \quad \sqrt{2f_2} \geq (3k - n)s.$$

Hence

$$\gamma k^2 > f_1 + f_2 \geq (3k - n)^2 s^2 = k^2((3 - x)^2 s^2). \quad (16)$$

Assume now that  $\beta \geq 0.34564$ . Then  $x = n/\lceil \beta n \rceil \leq 1/\beta \leq 2.893184$ . By Theorem 2,  $s \geq 1/0.35312$ . Then by (16),

$$\gamma > \left( \frac{3 - 2.893184}{0.35312} \right)^2 \geq 0.302492^2 > 0.0915.$$

Since the right-hand side of (15) grows with  $\gamma$ , plugging  $\gamma = 0.0915$  and  $\lambda = 2s = 2/0.35312$  into (15) gives a lower bound on  $x$ , namely

$$\begin{aligned} x &> 3 - \frac{1 - \sqrt{0.0915 + (1 - 0.0915)0.35312/2}}{(2/0.35312) - 1} = 3 - \frac{1 - \sqrt{0.0915 + 0.9085 \cdot 0.17656}}{(2 - 0.35312)/0.35312} \\ &= 3 - 0.35312 \frac{1 - \sqrt{0.25190476}}{1.64688} \geq 3 - 0.35312 \frac{1 - 0.5019}{1.64688} > 2.89319, \end{aligned}$$

a contradiction to our assumption. This proves Theorem 3.

## References

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