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An Ore-type analogue of the Sauer-Spencer Theorem

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Abstract. Two graphs G_1 and G_2 of order n *pack* if there exist injective mappings of their vertex sets into $[n]$, such that the images of the edge sets do not intersect. Sauer and Spencer proved that if $\Delta(G_1)\Delta(G_2) < 0.5n$, then G_1 and G_2 pack.

In this note, we study an Ore-type analogue of the Sauer–Spencer Theorem. Let $\theta(G) = \max\{d(u) + d(v) : uv \in E(G)\}$. We show that if $\theta(G_1)\Delta(G_2) < n$, then G_1 and G_2 pack. We also characterize the pairs (G_1, G_2) of n-vertex graphs satisfying $\theta(G_1)\Delta(G_2) = n$ that do not pack.

Key words. Graph packing, Ore-type conditions.

1. Introduction

Two n-vertex graphs G_1 and G_2 are said to *pack* if there exist injective mappings of their vertex sets onto $[n] = \{1, \dots, n\}$ such that the images of their edge sets are disjoint. In other words, G_1 and G_2 *pack* if G_1 is isomorphic to a subgraph of the complement of G_2.

A number of graph theory problems can be naturally stated in terms of packing. For example, the fact that an n-vertex graph G is hamiltonian is equivalent to the fact that the complement, $\overline{G}$, of G packs with the cycle C_n.

Active study of extremal problems on packings of graphs was started in the 1970s by Sauer and Spencer [10], Bollobás and Eldridge [1, 2], and Catlin [3]. In particular, Sauer and Spencer [10] proved the following fact.

Theorem 1. (Sauer and Spencer) *Let G_1 and G_2 be n-vertex graphs with maximum degrees Δ_1 and Δ_2, respectively. If $2\Delta_1\Delta_2 < n$, then G_1 and G_2 pack.*

Kaul and Kostochka [6] characterized the pairs (G_1, G_2) of n-vertex graphs with $2\Delta(G_1)\Delta(G_2) = n$ that do not pack.

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Theorem 2. (Kaul and Kostochka) *Let G_1 and G_2 be n-vertex graphs with maximum degrees Δ_1 and Δ_2, respectively. If $2\Delta_1\Delta_2 \le n$, then G_1 and G_2 do not pack if and only if one of G_1 and G_2 is a perfect matching, and the other either is $K_{n/2,n/2}$ with $n/2$ odd or contains $K_{n/2+1}$.*

Dirac [4] found sufficient conditions for a simple graph to be hamiltonian in terms of the minimum degree. In the language of packing, Dirac's Theorem says that if G is an n-vertex graph and $\Delta(G) \le 0.5n - 1$, then G packs with the cycle C_n. Thus the Sauer–Spencer Theorem is of the same nature as Dirac's Theorem, but in a more general situation.

The Ore's extension [9] of Dirac's Theorem sounds in the language of packing as follows.

Theorem 3. (Ore) *Let G be an n-vertex graph such that for each edge uv of G, the sum $d(u) + d(v)$ of the degrees of its ends is at most $n - 2$. Then G packs with the cycle C_n.*

Several Ore-type results on packing appear in [5, 8, 11].

The goal of this note is to prove an Ore-type analogue of Theorems 1 and 2.

Let

$$\theta(G) = \max\{d(u) + d(v) : uv \in E(G)\}.$$

Note that $\theta(G) = \Delta(L(G)) + 2$, where $L(G)$ is the line graph of G. Our result is:

Theorem 4. *If two n-vertex graphs G_1 and G_2 satisfy*

$$\theta(G_1)\Delta(G_2) \le n, \tag{1}$$

then G_1 and G_2 pack, with the following exceptions:

(I) *G_1 is a perfect matching and G_2 either is $K_{n/2,n/2}$ with $n/2$ odd or contains $K_{n/2+1}$;*
(II) *G_2 is a perfect matching, and G_1 either is $K_{r,n-r}$ with r odd or contains $K_{n/2+1}$.*

Note that using $\theta(G_1)$ instead of $2\Delta(G_1)$ yields more exceptional pairs of graphs, but not many.

We believe that the following fact is also true, although we were not able to prove it.

Conjecture 1. If G_1 and G_2 are n-vertex graphs and $\theta(G_1)\theta(G_2) < 2n$, then G_1 and G_2 pack.

2. Proof of Theorem 4

Suppose that some n-vertex graphs satisfy (1), but do not pack. Then there is a *critical pair* (G_1, G_2) of n-vertex graphs satisfying (1), that is, a pair such that G_1 and G_2 do not pack, but for each $e_1 \in E(G_1)$, $G_1 - e_1$ and G_2 pack, and for each $e_2 \in E(G_2)$, G_1 and $G_2 - e_2$ pack. In this case, for each edge e in G_1 or G_2, we can 'pack' G_1 and G_2 so that e is the only conflicting edge. Such a packing is called an *e-packing*.

In order to prove Theorem 4, it is enough to show that critical pairs satisfy either (I) or (II). Indeed, assume that G_1' and G_2' are n-vertex graphs with $\theta(G_1')\Delta(G_2') \leq n$ that do not pack. Then there exist $G_1 \subseteq G_1'$ and $G_2 \subseteq G_2'$ such that (G_1, G_2) is a critical pair. By Theorem 4, G_1 and G_2 satisfy either (I) or (II). Thus $n \leq \theta(G_1)\Delta(G_2) \leq \theta(G_1')\Delta(G_2') \leq n$. If G_i is a perfect matching, or is $K_{r,n-r}$, or contains $K_{n/2+1}$, then so does G_i' and hence the pair (G_1', G_2') also satisfies either (I) or (II).

Without loss of generality, we may assume that $V_1 = [n]$ and the graph $G_1 = (V_1, E_1)$ is fixed. Each packing of G_2 with G_1 will be viewed as a bijection $f : V(G_2) \to [n]$.

The result of each bijection $f : V_2 \to [n]$ will be considered as the (multi)graph $G = G(f)$ with labelled (with 1 and 2) edges whose vertex set is $[n]$ and two vertices u_1 and u_2 are connected by an edge in E_1 (respectively, E_2) if $u_1u_2 \in E(G_1)$ (respectively, $f^{-1}(u_1)f^{-1}(u_2) \in E(G_2)$). Each vertex $u \in V(G)$ has two kinds of neighborhoods: $N_1(u) = \{v \in V(G) : uv \in E_1\}$ and $N_2(u) = \{v \in V(G) : uv \in E_2\}$. Let $N(u) = N_1(u) \cup N_2(u)$ and $d_1(u) = |N_1(u)|$ and $d_2(u) = |N_2(u)|$.

For each such mapping f, a $(u, v; i, j)$-*link* is a path of length two from $u \in V(G)$ to $v \in V(G)$ passing through some vertex $w \in V(G)$ such that $uw \in E_i$ and $wv \in E_j$, $i, j \in \{1, 2\}$. A *link* is a $(u, v; i, j)$-link for some $u, v \in V(G)$ and $i, j \in \{1, 2\}, i \neq j$.

The $(u_1, u_2, \ldots, u_k)$-*swap* replaces a mapping f with f' which differs from f only in that $(f')^{-1}(u_{i+1}) = f^{-1}(u_i)$, for each $i = 1, \ldots, k$, where indices sum modulo k.

The following lemma from [7] will be helpful. It allows us to transform an embedding of G_2 by making 'vertex swaps' that do not increase the number of conflicting edges. In the statement of the lemma, the indices sum modulo k.

Lemma 1 ([7]). *Let $u_1, \ldots, u_k$ be vertices of G. Suppose that there are no $(u_i, u_{i+1}; 2, 1)$-links for any i. If there are no $1 \leq i < j \leq k$ such that $u_iu_j \in E(G_2)$ and $u_{i+1}u_{j+1} \in E(G_1)$, then the $(u_1, \ldots, u_k)$-swap does not create new conflicting edges.*

Let $\delta_1 \neq 0$ be the smallest degree in G_1 among all non-isolated vertices. Let $xy \in E_1$ be such that $d_1(x) = \Delta_1$. Then by the definition of θ_1, we have

$$\theta_1 \geq d_1(x) + d_1(y) \geq \Delta_1 + \delta_1. \qquad (2)$$

If $\delta_1 = \Delta_1$, then $\theta(G_1) = 2\Delta_1$ and the statement follows from [6]. Thus, we may assume that $\delta_1 < \Delta_1$ and, in particular, $\Delta_1 \geq 2$.

Choose $u \in V(G)$ such that $d_1(u) = \delta_1$. For a vertex $u^* \in N_G(u)$, consider a uu^*-packing.

Let $A = N_2(N_1(u)) - N(u) - u$, $B = N_1(N_2(u)) - N(u) - u$, $N_1 = N_1(u) - u^*$, and $N_2 = N_2(u) - u^*$. We will show that $V(G) = \{u, u^*\} \cup A \cup B \cup N_1 \cup N_2$ is a partition of $V(G)$ and a bit more.

Lemma 2. (i) *$V(G) = \{u, u^*\} \cup A \cup B \cup N_1 \cup N_2$ is a partition of $V(G)$; furthermore, every vertex $v \neq u, u^*$ can be reached from u by a unique link, and there are no links from u to u^*;*

(ii) *$\theta_1 = \delta_1 + \Delta_1$ and $n = (\delta_1 + \Delta_1)\Delta_2$;*

(iii) *For every $x \in V(G)$ with $d_1(x) = \Delta_1$, $d_1(y) = \delta_1$ for each $y \in N_1(x)$;*

(iv) *$d_1(u^*) = \Delta_1$ and for every $v \in N_2(u)$, $d_1(v) = \Delta_1$;*

(v) *For every vertex u' with $d_1(u') = \delta_1$ and $u'' \in N_1(u')$, we have $d_1(u'') = \Delta_1$.*

Proof. If a vertex v is not reachable from u by a link, then after the (u, v)-swap, no new conflicting edges appear by Lemma 1, while the conflicting edge uu^* disappears. Thus each vertex $v \neq u, u^*$ is reachable from u by a link. Hence, $V(G) = \{u, u^*\} \cup A \cup B \cup N_1 \cup N_2$.

Since there are at most $\delta_1\Delta_2$ $(1, 2)$-links and at most $d_2(u)\Delta_1$ $(2, 1)$-links starting from u, and all vertices except u^* are reached by a link and two of those links go back to u, we derive from (1) that

$$n - 2 \leq \delta_1\Delta_2 + d_2(u)\Delta_1 - 2 \leq (\delta_1 + \Delta_1)\Delta_2 - 2 \leq \theta_1\Delta_2 - 2 \leq n - 2. \tag{3}$$

Thus equalities hold in (2) and (3). Therefore, (i)-(iv) are proved.

To see (v), we assume that there exists u' with $d_1(u') = \delta_1$ and $u'' \in N(u')$ such that $d_1(u'') \neq \Delta_1$ or $d_2(u'') \neq \Delta_2$. Then considering a $u'u''$-packing, we have a contradiction to (iv) with our new u and u^* being u' and u'', respectively.

Statement (ii) of the lemma implies that if $\theta_1\Delta_2 < n$, then G_1 and G_2 pack. Thus the possible exceptions only occur when $\theta_1\Delta_2 = n$.

Now we consider the links from u to N_1 and N_2. By (iv) and (v) of Lemma 2, for every vertex $v \in N_1 \cup N_2 \cup \{u^*\}$, $d_1(v) = \Delta_1$. Thus by (iii) of Lemma 2, $N_1 \cup N_2 \cup \{u^*\}$ induces an independent set in G_1. Therefore, N_1 cannot be reached by a link from u via a vertex in N_2. N_1 cannot be reached via u^* either, otherwise, we would have a $(u, u^*; 1, 2)$-link. Thus each vertex in N_1 must be reached via a vertex in N_1. Since there is exactly one link from u to each vertex $v \in N_1$, N_1 induces a perfect matching in G_2.

For every $v \in N_2$, there must be a $(u, v; 1, 2)$-link, since $N_1 \cup N_2 \cup \{u^*\}$ is independent in G_1. Thus either $u^*v \in E_2$, or $vv' \in E_2$ for some $v' \in N_1$. If the latter holds, then after the (u, v', v)-swap, we get a uv'-packing (see Figure 1) and there is a link between u and v' via u^*, a contradiction. So, $u^*v \in E_2$ for each $v \in N_2$. In other words,

$$N_2(u) - u^* = N_2(u^*) - u. \tag{4}$$

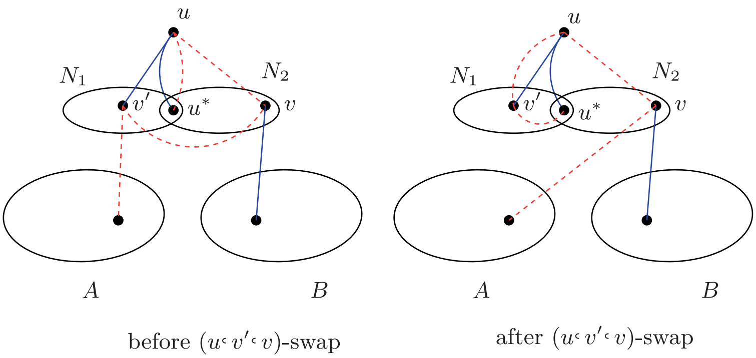

Fig. 1. (u, v', v)-swap

In particular, (4) implies that for each $v \in N_2$, the (u, v)-swap creates a uu^*-packing again. Since for this new 'packing', (4) should hold, we conclude that $N_2(v) - u^* = N_2(u^*) - v$. Therefore, $N_2 \cup \{u, u^*\}$ induces a K_{Δ_2+1} in G_2.

We now are ready to finish the proof.

Case 1. $N_1 \neq \emptyset$.

Since $N_1 \neq \emptyset$, there are $x, y \in N_1$ such that $xy \in E_2$. Consider the 'packing' after the (u, x)-swap. By Lemma 2, it is a uy-packing. Hence, by the previous paragraph, the conflicting edge is contained in a K_{Δ_2+1} in G_2. This means that the multi-edge xy in the original embedding is contained in a K_{Δ_2+1} in G_2. But by (i) of Lemma 2, $N_2(x) \cap N_2(y) = \emptyset$. This yields $\Delta_2 = 1$.

So G_2 is a matching (possibly not perfect) and $n = \Delta_1 + \delta_1$. By (iii) and (v) of Lemma 2, G_1 is bipartite and thus $G_1 = K_{\delta_1,\Delta_1}$. Then G_1 and G_2 pack unless G_2 is a perfect matching and both δ_1 and Δ_1 are odd. Thus we have (II) in the theorem.

Case 2. $N_1 = \emptyset$.

Then $\delta_1 = 1$ and we may assume that $\Delta_2 \geq 2$. We will show that G_1 and G_2 pack.

Note that in this case every component of G_1 is an isolated vertex or a K_{1,Δ_1}, every component of G_2 is either an isolated vertex or a K_{Δ_2+1}, and $n = (1+\Delta_1)\Delta_2$. Since n is divisible by $1 + \Delta_1$, we may assume that G_1 is the disjoint union of Δ_2 stars $S_1, \ldots, S_{\Delta_2}$, where $V(S_i) = \{v_{i,1}, \ldots, v_{i,\Delta_1}, v_i\}$ and S_i is centered at v_i, for $i = 1, \ldots, \Delta_2$. We can pack G_1 and G_2 if we can partition the vertex set of G_1 into three independents sets, two of which have size $\Delta_2 + 1$. We let $I_1 = \{v_1, v_2, \cdots, v_{\Delta_2-1}, v_{\Delta_2,1}, v_{\Delta_2,2}\}$. All edges of $G_1 - I_1$ are incident to v_{Δ_2}, and the number of non-neighbors of v_{Δ_2} in $G_1 - I_1$ is at least

$$n - (\Delta_2 + 1) - (\Delta_1 - 1) = (1 + \Delta_1)\Delta_2 - \Delta_2 - \Delta_1 = \Delta_1\Delta_2 - \Delta_1.$$

The last expression is at least Δ_2, since $\Delta_1 \geq 2$ and $\Delta_2 \geq 2$. Hence, we can choose Δ_2 non-neighbors of v_{Δ_2} in $G_1 - I_1$ to form together with v_{Δ_2} the set I_2. Then, by construction, $G_1 - I_1 - I_2$ is an independent set and thus G_1 and G_2 pack.

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