Graphs and Combinatorics (2008) 24:291–301
Digital Object Identifier (DOI) 10.1007/s00373-008-0795-7

Graphs and Combinatorics

Hadwiger Number and the Cartesian Product of Graphs

L. Sunil Chandran[1], Alexandr Kostochka[2,3] ⋆, J. Krishnam Raju[4]

[1] Department of Computer Science and Automation, Indian Institute of Science, Bangalore 560012. e-mail: sunil@csa.iisc.ernet.in, sunil.cl@gmail.com
[2] University of Illinois at Urbana-Champaign, IL 61801, USA.
[3] Institute of Mathematics, Novosibirsk 630090, Russia. e-mail: kostochk@math.uiuc.edu
[4] David R. Cheriton School of Computer Science, University of Waterloo. e-mail: krjampan@cs.uwaterloo.ca

Abstract. The Hadwiger number $\eta(G)$ of a graph G is the largest integer n for which the complete graph K_n on n vertices is a minor of G. Hadwiger conjectured that for every graph G, $\eta(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G. In this paper, we study the Hadwiger number of the Cartesian product $G \,\square\, H$ of graphs.

As the main result of this paper, we prove that $\eta(G_1 \,\square\, G_2) \geq h\sqrt{l}\,(1 - o(1))$ for any two graphs G_1 and G_2 with $\eta(G_1) = h$ and $\eta(G_2) = l$. We show that the above lower bound is asymptotically best possible when $h \geq l$. This asymptotically settles a question of Z. Miller (1978).

As consequences of our main result, we show the following:

1. Let G be a connected graph. Let $G = G_1 \,\square\, G_2 \,\square\, ... \,\square\, G_k$ be the (unique) prime factorization of G. Then G satisfies Hadwiger's conjecture if $k \geq 2 \log\log \chi(G) + c'$, where c' is a constant. This improves the $2 \log \chi(G) + 3$ bound in [2].
2. Let G_1 and G_2 be two graphs such that $\chi(G_1) \geq \chi(G_2) \geq c \log^{1.5}(\chi(G_1))$, where c is a constant. Then $G_1 \,\square\, G_2$ satisfies Hadwiger's conjecture.
3. Hadwiger's conjecture is true for G^d (Cartesian product of G taken d times) for every graph G and every $d \geq 2$. This settles a question by Chandran and Sivadasan [2]. (They had shown that the Hadiwger's conjecture is true for G^d if $d \geq 3$).

Key words. Hadwiger Number, Hadwiger's Conjecture, Graph Cartesian product, Minor, Chromatic number.

1. Introduction

1.1. General Definitions and Notation

In this paper we only consider undirected simple graphs, i.e., graphs without multiple edges and loops. For a graph G, we use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set.

⋆ Research of this author is supported in part by NSF grant DMS-0650784 and grant 06-01-00694 of the Russian Foundation for Basic Research.

A *k-coloring* of a graph $G(V, E)$ is a function $f : V \rightarrow \{1, 2, \ldots, k\}$. A k-coloring f is *proper* if for all edges (x, y) in G, $f(x) \neq f(y)$. A graph is k-colorable if it has a proper k-coloring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colorable.

Let S_1 and S_2 be non-empty disjoint subsets of $V(G)$. We say that S_1 and S_2 are *adjacent* in G if there exists an edge $(u, v) \in E(G)$ such that $u \in S_1$ and $v \in S_2$. The edge (u, v) is said to *connect* S_1 and S_2.

Contraction of an edge $e = (x, y)$ is the replacement of the vertices x and y with a new vertex z, whose incident edges are the edges other than e that were incident to x or y. The resulting graph denoted by $G.e$ may be a multigraph, but since we are only interested in simple graphs, we discard any parallel edges.

A *minor* M of $G(V, E)$ is a graph obtained from G by a sequence of contractions of edges and deletions of edges and vertices. We call M a minor of G and write $M \preceq G$.

It is not difficult to verify that M is a minor of G if and only if for each vertex $x \in V(M)$, there exists a set $V_x \subseteq V(G)$ such that (1) every V_x induces a connected subgraph of G, (2) all V_x are disjoint, and (3) for each $(x, y) \in E(M)$, V_x is adjacent to V_y in G.

The Hadwiger number $\eta(G)$ is the largest integer h such that the complete graph on h vertices K_h is a minor of G. Since every graph on at most h vertices is a minor of K_h, it is easy to see that $\eta(G)$ is the largest integer such that any graph on at most $\eta(G)$ vertices is a minor of G. Hadwiger [6] conjectured that *for every graph G, $\eta(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G.*

In other words, Hadwiger's conjecture states that if $\eta(G) \leq k$, then G is k-colorable. It is known to hold for small k. Graphs of Hadwiger number at most 2 are the forests. By a theorem of Dirac [5], the graphs with Hadwiger number at most 3 are the series-parallel graphs. Graphs with Hadwiger number at most 4 are characterized by Wagner [19]. The case $k = 4$ of the conjecture implies the Four Color Theorem because any planar graph has no K_5 minor. On the other hand, Hadwiger's conjecture for the case $k = 4$ follows from the Four Color Theorem and a structure theorem of Wagner [19]. Hadwiger's conjecture for $k = 5$ was settled by Robertson et al. [15]. The case $k = 6$ onwards is still open.

Since Hadwiger's conjecture in the general case is still open, researchers have shown interest to derive lower bounds for Hadwiger number in terms of the chromatic number. Mader [13] showed (improving an earlier result of Wagner [20]) that for any graph G, $\eta(G) \geq \frac{\chi(G)}{16 \log(\chi(G))}$. Later, Kostochka [10] and Thomason [18] independently showed that there exists a constant c such that for any graph G, $\eta(G) \geq \frac{\chi(G)}{c \cdot \sqrt{\log(\chi(G))}}$.

It is also known that Hadwiger's conjecture is true for almost all graphs on n vertices.

Improving on previous results by other authors, Kühn and Osthus [12] showed that if the girth (i.e., the length of a shortest cycle) is at least g for some odd g and the minimum degree δ is at least 3, then $\eta(G) \geq \frac{c(\delta)^{(g+1)/4}}{\sqrt{\log \delta}}$. As a consequence of this result, Kühn and Osthus[12] showed that Hadwiger's conjecture is true for graphs of sufficiently large chromatic number with no 4-cycles.

1.2. The Cartesian Product of Graphs

Let G_1 and G_2 be two undirected graphs, where the vertex set of G_1 is $\{0, 1, \ldots, n_1 - 1\}$ and the vertex set of G_2 is $\{0, 1, \ldots, n_2 - 1\}$. The *Cartesian product*, $G_1 \square G_2$, of G_1 and G_2 is the graph with the vertex set $V = \{0, 1, \ldots, n_1 - 1\} \times \{0, 1, \ldots, n_2 - 1\}$ whose edge set is defined as follows. There is an edge between vertices $\langle i, j \rangle$ and $\langle i', j' \rangle$ of $G_1 \square G_2$ if and only if, either $j = j'$ and $(i, i') \in E(G_1)$, or $i = i'$ and $(j, j') \in E(G_2)$.

In other words, the Cartesian product can be viewed in the following way: let the vertices of $G_1 \square G_2$ be partitioned into n_2 classes $W_1, \ldots, W_{n_2}$, where $W_j = \{\langle 1, j \rangle, \ldots, \langle n_1, j \rangle\}$ induces a graph that is isomorphic to G_1, where the vertex $\langle i, j \rangle$ corresponds to vertex i of G_1. If edge (j, j') belongs to G_2 then the edges between classes W_j and $W_{j'}$ form a matching such that the corresponding vertices, i.e., $\langle i, j \rangle$ and $\langle i, j' \rangle$, are matched. If edge (j, j') is not present in G_2, then there is no edge between W_j and $W_{j'}$.

It is easy to verify that the Cartesian product is a commutative and associative operation on graphs. Due to the associativity, the product of graphs $G_1, \ldots, G_k$ can be simply written as $G_1 \square \ldots \square G_k$ and has the following interpretation. If the vertex set of graph G_i is $V_i = \{1, \ldots, n_i\}$, then $G_1 \square \ldots \square G_k$ has the vertex set $V = V_1 \times V_2 \times \ldots \times V_k$. There is an edge between vertex $\langle i_1, \ldots, i_k \rangle$ and vertex $\langle i'_1, \ldots, i'_k \rangle$ of V if and only if there is a position t, $1 \leq t \leq k$, such that $i_1 = i'_1, i_2 = i'_2, \ldots, i_{t-1} = i'_{t-1}, i_{t+1} = i'_{t+1}, \ldots, i_k = i'_k$, and the edge (i_t, i'_t) belongs to graph G_t.

We denote the product of graph G taken k times as G^k. It is easy to verify that if G has n vertices and m edges, then G^k has n^k vertices and $mk \cdot n^{k-1}$ edges.

Well known examples of Cartesian products of graphs are the d-dimensional hypercube Q_d, which is isomorphic to K_2^d, and a d-dimensional grid, which is isomorphic to P_n^d, where P_n is a simple path on n vertices.

Unique Prime Factorization(UPF) of graphs: A graph P is *prime* with respect to the Cartesian product if and only if P has at least two vertices and it is not isomorphic to the product of two non-identity graphs, where an identity graph is the graph on a single vertex and having no edge. It is well-known that every connected undirected graph G with at least two vertices has a UPF with respect to Cartesian product in the sense that if G is not prime then it can be expressed in a unique way as a product of prime graphs ([7]). If G can be expressed as the product $G_1 \square G_2 \square \ldots \square G_k$, where each G_i is prime, then we say that the product dimension of G is k. The UPF of a given connected graph G can be found in $O(m \log(n))$ time, where m and n are the number of edges and number of vertices of G respectively [1].

Imrich and Klavžar have published a book [7] dedicated exclusively to the study of graph products. Readers who are interested to get an introduction to the wealth of profound and beautiful results on graph products are referred to this book.

We will use the following result by Sabidussi [17] (which was rediscovered several times).

Lemma 1. $\chi(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

1.3. Our Results

The question of studying the Hadwiger number with respect to the Cartesian product was suggested by Miller in the open problems section of a 1978 paper [14]. He mentioned a couple of special cases (such as $\eta(C_n \square K_2) = 4$ and $\eta(T \square K_n) = n+1$, where C_n and T denote a cycle and a tree respectively) and left the general case open. In this paper, we answer this question asymptotically. We give the following results.

Result 1. Let G_1 and G_2 be two graphs with $\eta(G_1) = k_1$ and $\eta(G_2) = k_2$. Then $\eta(G_1 \square G_2) \geq k_1\sqrt{k_2}\,(1 - o(1))$. (Since the Cartesian product is commutative, we can assume that $k_1 \geq k_2$). We demonstrate that this lower bound is asymptotically best possible.

We also show that in general, $\eta(G_1 \square G_2)$ does not have any upper bound that depends only on $\eta(G_1)$ and $\eta(G_2)$ by demonstrating sequences of graphs $\{G_1(n)\}_{n=1}^{\infty}$ and $\{G_2(n)\}_{n=1}^{\infty}$ such that $\eta(G_1(n))$ and $\eta(G_2(n))$ are bounded, whereas $\eta(G_1(n) \square G_2(n)) \geq n$.

Remark. Note that if the average degrees of G_1 and G_2 are d_1 and d_2 respectively, then the average degree of $G_1 \square G_2$ is $d_1 + d_2$. In comparison, by Result 1, the Hadwiger number of $G_1 \square G_2$ grows much faster.

Hadwiger's conjecture for Cartesian products of graphs was studied in [2]. It was shown there that if the product dimension (number of factors in the unique prime factorization of G) is k, then Hadwiger's conjecture is true for G if $k \geq 2\log\chi(G)+3$. As a consequence of Result 1, we are able to improve this bound. We show the following.

Result 2. Let the (unique) prime factorization of G be $G = G_1 \square G_2 \square \cdots \square G_k$. Then Hadwiger's conjecture is true for G if $k \geq 2\log(\log(\chi(G))) + c'$, where c' is a constant.

Another consequence of Result 1 is that if G_1 and G_2 are two graphs such that $\chi(G_2)$ is not "too low" compared to $\chi(G_1)$, then Hadwiger's conjecture is true for $G_1 \square G_2$. More precisely:

Result 3. If $\chi(G_2) \geq c\log^{1.5}(\chi(G_1))$, where c is a constant, then Hadwiger's conjecture is true for $G_1 \square G_2$.

It is easy to see that Result 3 implies the following: Let G_1 and G_2 be two graphs such that $\chi(G_1) = \chi(G_2)$. (For example, as in the case $G_1 = G_2$). Then $G_1 \square G_2$ satisfies Hadwiger's conjecture if $\chi(G_1) = \chi(G_2) = t$ is sufficiently large. (t has to be sufficiently large, because of the constant c involved in Result 3). For this special case, namely $\chi(G_1) = \chi(G_2)$, we give a different proof (which does not depend on Result 1), to show that Hadwiger's conjecture is true for $G_1 \square G_2$. This proof does not require that $\chi(G_1)$ be sufficiently large.

Result 4. Let G_1 and G_2 be any graphs with $\chi(G_1) = \chi(G_2)$. Then Hadwiger's conjecture is true for $G_1 \square G_2$.

It was shown in [2] that Hadwiger's conjecture is true for G^d, where $d \geq 3$, for any graph G. As a consequence of Result 4, we are able to sharpen this result.

Result 5. For any graph G and every $d \geq 2$, Hadwiger's Conjecture is true for G^d.

Another author who studied the minors of the Cartesian product of graphs is Kotlov [11]. He showed that for every bipartite graph G, the strong product ([7]) $G \boxtimes K_2$ is a minor of $G \square C_4$. As a consequence of this he showed that $\eta(K_2^d) \geq 2^{\frac{d+1}{2}}$.

2. Hadwiger Number for $G_1 \square G_2$

2.1. Lower Bound on $\eta(G_1 \square G_2)$

The following Lemma is not difficult to prove.

Lemma 2. [2] *If $M_1 \preceq G_1$ and $M_2 \preceq G_2$, then $M_1 \square M_2 \preceq G_1 \square G_2$.*

Let G_1 and G_2 be two graphs such that $\eta(G_1) = h$ and $\eta(G_2) = l$, with $h \geq l$. In this section we show that $\eta(G_1 \square G_2) \geq h\sqrt{l}\,(1 - o(1))$. Since by Lemma 2, $K_h \square K_l \preceq G_1 \square G_2$ it is sufficient to prove that $\eta(K_h \square K_l) \geq h\sqrt{l}\,(1 - o(1))$.

Definition 1. An *affine plane $\mathcal{A}$ of order m* is a family $\{A_{q,t} \ : \ q = 1, \ldots, m+1, \ t = 1, \ldots, m\}$ of *m-elements subsets of an m^2-element set A such that*

$$|A_{q,t} \cap A_{q',t'}| = \begin{cases} 1 \text{ if } q' \neq q; \\ 0 \text{ if } q' = q \text{ and } t' \neq t. \end{cases}$$

The sets A_{it} are *the lines of $\mathcal{A}$*. By definition, for each q, the sets $A_{q,1}, A_{q,2}, \ldots, A_{q,m}$ are disjoint and form a partition of A. These sets are viewed as *parallel lines*.

The following fact is widely known (see, e.g., [16]):

Lemma 3. *If m is a prime power, then there exists an affine plane of order m.*

It is also known that the set of prime numbers is quite dense. The following is a weakening of the result of Iwaniec and Pintz [9].

Lemma 4. *For every sufficiently large positive x, the interval $[x, x + x^{0.6}]$ contains a prime number.*

Corollary 1. *For every sufficiently large positive x, the interval $[x - 6x^{0.9}, x]$ contains a number of the form $(p(p + 1))^2$, where p is some prime.*

Theorem 1. $\eta(K_h \square K_l) \geq h\sqrt{l}(1 - o(1))$.

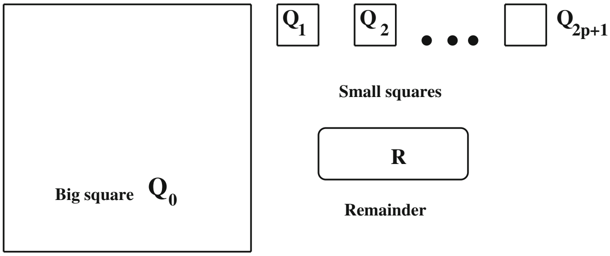

Fig. 1.

Proof. Consider $G = K_h \Box K_l$, where $h \geq l$ and l is large. Let p be the maximum prime such that $l \geq (p(p+1))^2$. We view $K_h \Box K_l$ as a set of h copies of K_l. Suppose that $s(p-1)(2p+1)/2 \leq h < (s+1)(p-1)(2p+1)/2$. We neglect some $h - s(p-1)(2p+1)/2$ copies of K_l and partition the remaining $s(p-1)(2p+1)/2$ copies into s large groups of the same size, and each of these groups into $(p-1)/2$ groups of size $2p+1$. In other words, we consider $\mathcal{S} = \{K_l(i, j, m) : i = 1, \ldots, s, \; j = 1, \ldots, (p-1)/2, \; m = 1, \ldots, 2p+1\}$, where each $K_l(i, j, m)$ is a copy of K_l. For $i = 1, \ldots, s$, let $\mathcal{S}_i = \{K_l(i, j, m) : j = 1, \ldots, (p-1)/2; \; m = 1, \ldots, 2p+1\}$.

In $\mathcal{S}_1$, we will find $p^2(p-1)(2p+1)/2$ disjoint sets $M(1, j, m, t)$ ($j = 1, \ldots, (p-1)/2$, $m = 1, \ldots, 2p+1$, $t = 1, \ldots, p^2$) of size $(p+1)^2$. These sets will have the property that

(a) the subgraph $G(M(1, j, m, t))$ induced by $M(1, j, m, t)$ is connected;
(b) for any two quadruples $(1, j, m, t)$ and $(1, j', m', t')$, there is a vertex v in K_l such that each of $M(1, j, m, t)$ and $M(1, j', m', t')$ contains a copy of v (in different copies of K_l, since our sets are disjoint).

If we manage this, then copying these sets for every $i = 2, \ldots, s$, by (b), we will create $s \; p^2(p-1)(2p+1)/2 = p^2 h(1 - o(1))$ disjoint sets $M(i, j, m, t)$ that satisfy

(a') the subgraph $G(M(i, j, m, t))$ induced by $M(i, j, m, t)$ is connected;
(b') for any two quadruples (i, j, m, t) and (i', j', m', t'), there is a vertex v in K_l such that each of $M(i, j, m, t)$ and $M(i', j', m', t')$ contains a copy of v

So, we go after (a) and (b).

To achieve this, we view the set of vertices of each K_l as the disjoint union of a "big" square Q_0 of size $p^2 \times p^2$ with $2p+1$ "small" squares Q_k, $k = 1, \ldots, 2p+1$ of size $p \times p$ and the remainder R (of size $l - p^4 - (2p+1)p^2 = l - (p(p+1))^2$) (see Fig. 1).

By Lemma 3, there exists an affine plane of order p^2. We consider the lines $A_{q,t}$ of this plane as subsets of the big square Q_0. For $q = 1, \ldots, (p-1)(2p+1)/2$, we view the family $\{A_{q,1}, A_{q,2}, \ldots, A_{q,p^2}\}$ as a partition of the copy $Q_0(q)$ of the big square Q_0. If $q = (j-1)(2p+1) + m$, then the set $A_{q,t}$ will be the main part of the future set $M(1, j, m, t)$. The whole line $A_{q,t}$ lies in one copy of K_l and so $G(A_{q,t})$ is connected. By the definition of the affine plane, if $q' = (j'-1)(2p+1) + m'$ and $q' \neq q$, then the projections of the sets $A_{q,t}$ and $A_{q',t'}$ on Q_0 intersect and for them (b) holds. Thus, we need only to take care of sets $M(1, j, m, t)$ and $M(1, j', m', t')$ when $j = j'$

and $m = m'$. So, our goal is to add $2p + 1$ vertices to each of $A_{(j-1)(2p+1)+m,t}$ to provide (b) for $M(1, j, m, t)$ and $M(1, j, m, t')$.

Fix j and m. For every $t = 1, \ldots, p^2$, the set $M(1, j, m, t)$ will be obtained from $A_{(j-1)(2p+1)+m,t}$ by adding a $(2p+1)$-element subset of $\cup_{r=1}^{2p+1} Q_m(j, r)$, where $Q_m(j, r)$ is the copy of Q_m that is contained in $K_l(1, j, r)$. Every $t = 1, \ldots, p^2$ can be written in the form $t = (a_1 - 1)p + a_2$, where $1 \leq a_1, a_2 \leq p$. So, we include into $M(1, j, m, t)$ the entry (a_1, a_2) of the square $Q_m(j, m)$. We call this vertex $F(a_1, a_2, j, m)$. Since $F(a_1, a_2, j, m)$ is in the same copy of K_l as $A_{(j-1)(2p+1)+m,t}$, it is adjacent to every vertex in this set. Let $C_a(j, r, m)$ and $R_b(j, r, m)$ denote the ath column and the bth row of the square $Q_m(j, r)$, respectively. If $t = (a_1 - 1)p + a_2$, then our set $M(1, j, m, t)$ will consist of $A_{(j-1)(2p+1)+m,t}$, the vertex $F(a_1, a_2, j, m)$, the row $R_{a_1}(j, m + a_2, m)$ and the column $C_{a_2}(j, m - a_1, m)$, where the values $m + a_2$ and $m - a_1$ are calculated modulo $2p + 1$. Since $F(a_1, a_2, j, m)$ is adjacent to the a_2-s entry of the row $R_{a_1}(j, m + a_2, m)$ and to the a_1-s entry of the column $C_{a_2}(j, m - a_1, m)$, condition (a) holds. Since the projection on Q_m of $R_{a_1}(j, m + a_2, m) \cup C_{a_2}(j, m - a_1, m)$ is a cross, (b) also holds.

This finishes the construction. It implies that the Hadwiger number of $G = K_h \Box K_l$, with $h \geq l$ is at least

$$\left\lfloor \frac{h}{(p-1)(2p+1)/2} \right\rfloor p^2(p-1)(2p+1)/2 = (h - O(p^2))p^2.$$

By Corollary 1, $p^2 = (1 - o(1))\sqrt{l}$. Hence the result. □

The following theorem is an immediate consequence of Theorem 1

Theorem 2. *Let G_1, G_2 be any two graphs with $\eta(G_1) = h$, $\eta(G_2) = l$ and $\eta(G_1) \geq \eta(G_2)$. Then $\eta(G_1 \Box G_2) = \eta(G_2 \Box G_1) \geq h\sqrt{l}\,(1 - o(1))$.*

2.2. Tightness of the Lower Bound and the Nonexistence of an Upper Bound

Let K_n and K_m be the complete graphs on n and m vertices respectively ($n \geq m$) and let h be the maximum number such that $K_h \preceq K_n \Box K_m$. Let the sets $V_0, \ldots, V_{h-1} \subseteq V(K_n \Box K_m)$ be the pre-images of vertices of K_h in $K_n \Box K_m$. Thus, the vertex sets $V_0, \ldots, V_{h-1}$ are pairwise disjoint and pairwise adjacent. Moreover, V_i $(0 \leq i \leq h-1)$ induces a connected subgraph in $K_n \Box K_m$.

Without loss of generality, let $n_0 = |V_0| = \min_{0 \leq i \leq h-1} |V_i|$. Thus $n_0 \leq \frac{nm}{h}$. For $S \subseteq V(K_m \Box K_n)$, let $N(S) = \bigcup_{u \in S} N(u) - S$. (Here $N(u)$ denotes the set of the neighbors of u in $K_n \Box K_m$.) Since K_h is a complete graph minor of $K_m \Box K_n$, we have:

$$|N(V_0)| \geq h - 1 \tag{1}$$

Since V_0 induces a connected graph in $K_n \Box K_m$, the vertices of V_0 can be ordered $v_1, \ldots, v_{n_0}$ so that for $2 \leq j \leq n_0$, v_j is adjacent to at least one of the vertices in $\{v_1, \ldots, v_{j-1}\}$. Let us define a sequence of sets $\emptyset = X_0, X_1, \ldots, X_{n_0} = V_0$ by setting $X_j = X_{j-1} \cup \{v_j\}$, for $1 \leq j \leq n_0$. Clearly, $|N(X_1)| = n + m - 2$. We claim that

$|N(X_j)| \leq |N(X_{j-1})| + n - 2$, for $2 \leq j \leq n_0$. To see this, recall that v_j is adjacent to at least one vertex $v_k \in X_{j-1}$. Clearly, out of the $n + m - 2$ neighbors of v_j, at least $m - 2$ are neighbors of v_k also, and thus are already in $N(X_{j-1})$. Now, accounting for v_j and v_k we have $|N(X_j)| \leq |N(X_{j-1})| + n - 2$, as required. Thus we get $|N(V_0)| = |N(X_{n_0})| \leq n + m - 2 + (n_0 - 1)(n - 2) \leq n + m - 2 + (\frac{nm}{2} - 1)(n - 2)$. Combining this with Inequality (1) we get:

$$n + m - 2 + \left(\frac{nm}{h} - 1\right)(n - 2) \geq h - 1$$

It is easy to verify that for $h > n\sqrt{m} + m$, the above inequality will not be satisfied. So we infer that $h \leq n\sqrt{m} + m$. (Recalling $n \geq m$, the upper bound tends to $n\sqrt{m}$ asymptotically.)

After proving our lower bound it is natural to ask the following question. Let G_1 and G_2 be two arbitrary graphs with $\eta(G_1) = k_1$ and $\eta(G_2) = k_2$. Does there exists a function $f : N \times N \rightarrow N$ such that $\eta(G_1 \square G_2) \leq f(k_1, k_2)$? As examples below show, in general such a function cannot exist.

Let $R_n = P_n \square P_n$ be the $n \times n$ grid graph and let $D_n = R_n \square K_2$ be the $n \times n$ double-grid graph. Since R_n is a planar graph, we have $\eta(R_n) \leq 4$. Thus, D_n is the Cartesian product of graphs with Hadwiger numbers two and at most four. But, as it was proved in [2], the Hadwiger number of D_n is at least n. (A sketch of their proof is as follows. Let G_1 and G_2 be the two grids R_n comprising $D_n = R_n \square K_2$. Observe that there is an edge between any "row" of G_1 and any "column" of G_2. Contracting all the rows of G_1 and all the columns of G_2 we get a complete bipartite graph $K_{n,n}$ whose Hadwiger number is at least $n + 1$.)

Another (even more impressive) example immediately follows from a result of Ivančo, who proved in [8] that $\eta(K_{1,n} \square K_{1,n}) = 2 + n$.

2.3. Consequences of Theorem 2: Hadwiger's Conjecture for Graph Products

2.3.1. In Terms of Chromatic Number. Theorem 2 naturally leads to the following question: Let G_1 and G_2 be any two graphs with $\chi(G_1) = k_1$ and $\chi(G_2) = k_2$, where $k_1 \geq k_2$. Let $f(k_1)$ be the smallest m such that if $k_2 \geq m$, then Hadwiger's conjecture holds for $G_1 \square G_2$. Hadwiger's conjecture states that $f(k_1) = 1$. Since Hadwiger's conjecture in the most general case seems to be hard to prove, it is interesting to obtain reasonable bounds on $f(k_1)$. We need the following result, proved independently by Kostochka [10] and Thomason [18].

Lemma 5. *There exists a constant c_2 such that for each graph G, $\eta(G) \geq \frac{c_2\chi(G)}{\sqrt{\log \chi(G)}}$.*

Theorem 2 yields the following result.

Theorem 3. *Let G_1 and G_2 be any two graphs. There exists a constant c' such that if $\chi(G_1) \geq \chi(G_2) \geq c' \log^{1.5}(\chi(G_1))$, then Hadwiger's conjecture is true for $G_1 \square G_2$.*

Proof. Let $k_1 = \chi(G_1)$ and $k_2 = \chi(G_2)$. Applying Lemma 5 and Theorem 2 and noting that $\sqrt{\sqrt{\log(k_2)}} \leq (\sqrt{\log(k_1)})^{0.5}$, we have

$$\eta(G_1 \square G_2) \geq c_1 {c_2}^{1.5} \frac{k_1\sqrt{k_2}}{(\sqrt{\log k_1})^{1.5}}.$$

Now taking $c' = \frac{1}{(c_1 {c_2}^{1.5})^2}$, ($c_1$ and c_2 are the constants that correspond to Theorem 2[1] and Lemma 5 respectively) and recalling that $k_2 \geq c' \log^{1.5}(k_1)$, we get $\eta(G_1 \square G_2) \geq k_1 = \chi(G_1 \square G_2)$. The latter equality follows from Lemma 1. □

2.3.2. In Terms of Product Dimension. Recall that the product dimension of a connected graph G is the number of prime factors in its (unique) prime factorization. It was shown in [2] that if the product dimension of G is at least $2 \log \chi(G) + 3$, then Hadwiger's conjecture is satisfied for G. Using theorem 2, we can bring this bound to $2 \log \log \chi(G) + c'$, where c' is a constant. The following Lemma proved in [2] gives a lower bound for the Hadwiger number of the d-dimensional Hypercube, Q_d.

Lemma 6. $\eta(Q_k) \geq 2^{\lfloor (k-1)/2 \rfloor} \geq 2^{(k-2)/2}$

Theorem 4. *There exists a constant c', such that for every connected graph G with product dimension k, if $k \geq 2 \log \log \chi(G) + c'$, then Hadwiger's conjecture holds for G.*

Proof. Let $c' = 4 \log \frac{1}{c_1 c_2} + 3$, where c_1 and c_2 are the constants[2] by Theorem 2 and Lemma 5 respectively. We may assume that $\chi(G_1) \geq \chi(G_i)$, for all $i > 1$. By Lemma 1, $\chi(G) = \max\{\chi(G_1), \chi(G_2), \ldots, \chi(G_k)\} = \chi(G_1)$.

Let $X = G_2 \square G_3 \square \cdots \square G_k$. Since G is connected, each G_i is also connected. Moreover, G_i has at least two vertices (and hence at least one edge) since G_i is prime. It follows that the $(k-1)$-dimensional hypercube is a minor of X. Thus by Lemma 6, $\eta(X) \geq \eta(Q_{k-1}) \geq 2^{(k-3)/2} \geq 2^{\log \log \chi(G) + 2 \log \frac{1}{c_1 c_2}}$.

Applying Theorem 2 to $G_1 \square X$, we get

$$\eta(G) = \eta(G_1 \square X) \geq c_1 \eta(G_1) \sqrt{\eta(X)}$$

Recalling that (by Lemma 5), $\eta(G_1) \geq \frac{c_2 \chi(G_1)}{\sqrt{\log \chi(G_1)}}$, we get $\eta(G) \geq \chi(G)$. □

3. Hadwiger's Conjecture for $G_1 \square G_2$ when $\chi(G_1) = \chi(G_2)$

Theorem 3 implies that $G_1 \square G_2$ satisfies Hadwiger's conjecture if $\chi(G_1) = \chi(G_2) = t$ is sufficiently large. (t has to be sufficiently large, because of the constant c' involved

[1] By Theorem 2 we have $\eta(G_1 \square G_2) \geq c_1 h \sqrt{l}$, where c_1 is a constant.

[2] $c_1, c_2 \leq 1$. So $\frac{1}{c_1 c_2} \geq 1$

in Theorem 3). In this section we give a different proof for this special case. We show that irrespective of the value of t $(= \chi(G_1))$, $G_1 \,\square\, G_2$ satisfies Hadwiger's conjecture if $\chi(G_1) = \chi(G_2)$.

A graph G is said to be k-*critical* if $\chi(H) < \chi(G) = k$ for every proper subgraph H of G. Every k-chromatic graph contains a k-critical subgraph.

Theorem 5. *If $\chi(G) = \chi(H)$, then Hadwiger's conjecture is true for $G \,\square\, H$.*

Proof. Let $\chi(G) = \chi(H) = n$. Let G' and H' be n-critical subgraphs of G and H respectively. Then $\delta(G') \geq n - 1$ (see [21]) and hence $K_{1,n-1} \preceq G'$. Similarly $K_{1,n-1} \preceq H'$. Thus we have $\eta(G \,\square\, H) \geq \eta(G' \,\square\, H') \geq \eta(K_{1,n-1} \,\square\, K_{1,n-1})$. By Ivančo's [8] result, we have $\eta(K_{1,n-1} \,\square\, K_{1,n-1}) = 2 + (n - 1) > n$. This together with Lemma 1, gives $\eta(G \,\square\, H) > n = \chi(G \,\square\, H)$, proving the Theorem. □

It was shown in [2] that if a graph G is isomorphic to F^d, for some graph F and $d \geq 3$ then Hadwiger's conjecture is true for G. The following improvement is an immediate consequence of Theorem 5 and Lemma 1.

Theorem 6. *Let a graph G be isomorphic to F^d for some graph F and for $d \geq 2$. Then Hadwiger's conjecture is true for G.*

Acknowledgements. We thank the referees for helpful comments.

References

1. Aurenhammer, F., Hagauer, J., Imrich, W.: Cartesian graph factorization at logarithmic cost per edge. Computational Complexity **2**, 331–349 (1992)
2. Chandran, L.S., Sivadasan, N.: On the Hardwiger's conjecture for graph products. Discrete Mathematics **307**, 266–273 (2007)
3. Diestel, R.: Graph Theory. Springer Verlag, New York (2000)
4. Diestel, R., Rempel, C.: Dense minors in graphs of large girth. Combinatorica **25**, 111–116 (2005)
5. Dirac, G.A.: In abstrakten Graphen vorhandene vollständige 4–Graphen und ihre Unterteilungen. Math. Nachr. **22**, 61–85 (1960)
6. Hadwiger, H.: Über eine Klassifikation der Streckenkomplexe, Vierteljscr. Naturforsch. Gessellsch. Zürich **88**, 132–142 (1943)
7. Imrich, W., Klavžar, S.: Product Graphs: Structure and Recognition. Wiley, New York (2000)
8. Ivančo, J.: Some results on the Hadwiger number of graphs. Math. Slovaca **38**, 221–226 (1988)
9. Iwaniec, H., Pintz, J.: Primes in short intervals. Monatsh. Math. **98**, 115–143 (1984)
10. Kostochka, A.V.: The minimum hadwiger number of graphs with a given mean degree of vertices, Metody Diskret. Analiz. **38**, 37–58 (1982) (In Russian)
11. Kotlov, A.: Minors and strong products. European Journal of Combinatorics **22**, 511–512 (2001)
12. Kühn, D., Osthus, D.: Minors in graphs of large girth. Random Structures and Algorithms **22**, 213–225 (2003)
13. Mader, W.: Homomorphiesätze für Graphen. Math. Annalen **178**, 154–168 (1968)

14. Miller, Z.: Contractions of graphs: A theorem of Ore and an extremal problem. Discrete Mathematics **21**, 261–273 (1978)
15. Robertson, N., Seymour, P.D., Thomas, R.: Hadwiger's conjecture for K6-free graphs. Combinatorica **13**, 279–361 (1993)
16. Ryser, H.J.: Combinatorial mathematics, The Carus Mathematical Monographs. **14** Wiley, New York (1963)
17. Sabidussi, G.: Graphs with given group and given graph theoretic properties. Canad. J. Math. **9**, 515–525 (1957)
18. Thomason, A.G.: An extremal function for contractions of graphs. Math. Proc. Camb. Phil. Soc. **95**, 261–265 (1984)
19. Wagner, K.: Über eine Eigenschaft der Ebenen Komplexe. Math. Annalen **114**, 570–590 (1937)
20. Wagner, K.: Beweis einer Abschwächung der Hadwiger–Vermutung. Math. Annalen **153**, 139–141 (1964)
21. West, D.B.: Introduction to Graph Theory. Prentice Hall India, NewDelhi (2003)

Received: July 17, 2007
Final Version received: March 20, 2008