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Note

On Ramsey numbers of uniform hypergraphs with given maximum degree

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Abstract

For every $\epsilon > 0$ and every positive integers Δ and r, there exists $C = C(\epsilon, \Delta, r)$ such that the Ramsey number, $R(H, H)$ of any r-uniform hypergraph H with maximum degree at most Δ is at most $C|V(H)|^{1+\epsilon}$.

Keywords: Ramsey numbers; Uniform hypergraphs; Maximum degree

1. Introduction

For r-uniform hypergraphs H_1 and H_2, the Ramsey number $R(H_1, H_2)$ is the minimum positive integer N such that in every 2-coloring of edges of the complete r-uniform hypergraph $K_N^{(r)}$, there is either a copy of H_1 with edges of the first color or a copy of H_2 with edges of the second color. The classical Ramsey number $r(k, l)$ is in our terminology $R(K_k^{(2)}, K_l^{(2)})$.

Say that a family $\mathcal{F}$ of r-uniform hypergraphs is $f(n)$-Ramsey if $R(G, G) \leqslant f(n)$ for every positive integer n and every $G \in \mathcal{F}$ with $|V(G)| = n$.

Burr and Erdős [2] conjectured that for every Δ and d,

(a) *the family of graphs with maximum degree at most Δ is Cn-Ramsey, where $C = C(\Delta)$;*
(b) *the family $\mathcal{D}_d$ of d-degenerate graphs is Dn-Ramsey, where $D = D(d)$.*

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Recall that a graph is *d-degenerate* if every of its induced subgraphs has a vertex of degree (in this subgraph) at most d. Equivalently, a graph G is d-degenerate if for some linear ordering of the vertex set of G every vertex of G is adjacent to at most d vertices of G that precede it in the ordering.

Chvátal, Rödl, Szemerédi and Trotter [4] proved the first conjecture. The second conjecture is open. In recent years, some subfamilies of the family $\mathcal{D}_d$ were shown to be Dn-Ramsey by Alon [1], Chen and Schelp [3], and Rödl and Thomas [10]. In [8], the authors recently proved that $\mathcal{D}_d$ is n^2-Ramsey, and in [7] they established an $n^{1+o(1)}$ bound for a subfamily of $\mathcal{D}_d$. This approach was improved by Kostochka and Sudakov [9], who showed that for every positive integer d, the family $\mathcal{D}_d$ is $n^{1+o(1)}$-Ramsey. In particular, in [9], the following Turán-type result was proved for bipartite graphs.

Theorem 1. [9] *Let $0 < c \leqslant 1$ be a constant and let d, N and n be positive integers satisfying*

$$d \leqslant \frac{1}{64}\ln n \quad \text{and} \quad N \geqslant n\left(\frac{2e}{c}\right)^{2d^{1/3}\ln^{2/3} n}. \tag{1}$$

Then every bipartite graph $G = (V_1, V_2; E)$ with $|V_1| = |V_2| = N$ and $|E| = cN^2$ contains every d-degenerate bipartite graph of order n.

Frequently Turán-type results have implications for Ramsey-type problems. For example, Theorem 1 implies that for N satisfying the conditions of the theorem and for each coloring of the edges of $K_{N,N}$ with $\lfloor 1/c \rfloor$ colors, the monochromatic subgraph with most edges contains every d-degenerate bipartite graph of order n.

In this paper, we discuss analogues of the above Burr–Erdős conjectures for uniform hypergraphs. Similarly to graphs, we say that a hypergraph is *d-degenerate* if every of its induced subgraphs has a vertex of degree (in this subgraph) at most d.

Our first result is the following extension to m-uniform m-partite hypergraphs of a weaker version of Theorem 1.

Theorem 2. *For α, $0 < \alpha \leqslant 1/2$, and integers $d \geqslant 2$, $l \geqslant 2$, let n and N be such that for $c = \ln\frac{1}{\alpha} + 2(d-1)(l-1)$ we have*

$$\ln n > \big(\max\{2, c\}\big)^l$$

and

$$N > ne^{c(\ln n)^{(l-1)/l}}.$$

Let G be an l-uniform l-partite hypergraph with partite sets $V_1, \dots, V_l$ each of cardinality N and at least $2\alpha N^l$ edges. If an l-uniform l-partite hypergraph H contains at most n edges, and the degrees of all of the partite sets of H except one are at most d, then G contains H.

We also show that for each $l \geqslant 3$, the statement of the above theorem does not hold without degree restrictions on H even if H is 1-degenerate.

Our main result is

Theorem 3. *For every fixed Δ and r, for arbitrary r-uniform hypergraphs H_1 and H_2 with maximum degree at most Δ on m vertices,*

$$R(H_1, H_2) \leqslant m^{1+o(1)}.$$

We suspect that for every fixed Δ and r the class of r-uniform hypergraphs with maximum degree at most Δ is Dn-Ramsey for some $D = D(\Delta, r)$ but were not able to prove this.

We also show that the r-uniform analogue of the second Burr–Erdős Conjecture fails for $r \geqslant 4$ and d-degenerate hypergraphs (even for 1-degenerate hypergraphs) if we use the above definition of d-degenerate hypergraphs.

The structure of the paper is as follows. In the next section we present examples that establish some lower bounds. In Section 3 we prove Theorem 2. The idea of the proof for a l-uniform l-partite hypergraph G is to find an $(l-1)$-uniform $(l-1)$-partite hypergraph G' with "many" edges such that for each d-tuple $\{e_1, \dots, e_d\}$ of the edges of G', there are "many" vertices v in G such that each of $e_1 + v, \dots, e_d + v$ is an edge of G. In this way, we gradually reduce the size of the hyperedges with which we work. In Section 4 we prepare for the proof of Theorem 3. We prove that every r-uniform hypergraph H with maximum degree d and n edges is a "part" of a $(d(r-1)+1)$-uniform $(d(r-1)+1)$-partite hypergraph with maximum degree d and n edges. We also recall a couple of known results. In the final section we finish the proof of Theorem 3 by constructing for given r-uniform hypergraphs G and H auxiliary $(d(r-1)+1)$-uniform $(d(r-1)+1)$-partite hypergraphs and applying Theorem 2 to them.

2. Examples

2.1. Lower bound for partite Ramsey numbers

First, we give an example of a 3-uniform 3-partite 1-degenerate hypergraph H with n edges for which the conclusion of Theorem 2 does not hold. The same example also shows that, in fact, the partite Ramsey numbers of 1-degenerate k-graphs, for $k \geqslant 3$, grow exponentially. More precisely, first we construct a 1-degenerate 3-uniform 3-partite hypergraph H with $3t + 3t^2$ vertices such that, whenever $N \leqslant 2^{\lfloor t/2 \rfloor}$, a complete 3-uniform 3-partite hypergraph $K^{(3)}_{N,N,N}$ admits a 2-coloring with no monochromatic copy of H. Consider the following example. Let $n = 3t^2$ and F be the complete 3-partite graph with partite sets W_1, W_2, W_3 of cardinality t. Let $H = H_n$ be the 3-uniform 3-partite hypergraph obtained from F by adding to each edge e a new vertex $w_e \notin W_1 \cup W_2 \cup W_3$ (all w_e are different). Then H is a 1-degenerate hypergraph with n edges and $n + 3t$ vertices.

Let $N = 2^{\lfloor t/2 \rfloor}$. Let $|V_1| = |V_2| = |V_3| = N$ and $\mathbb{C} = C_{\text{red}} \cup C_{\text{blue}}$ be a random 2-coloring of the edges of $K_{N,N}$ with partite sets V_1 and V_2, where for each pair $(a, b) \in V_1 \times V_2$, C_{red} contains (a, b) with probability $1/2$ independently of all other choices. Standard arguments show [13, Chapter 12] that with positive probability, $\mathbb{C}$ will be such that

(∗) *neither C_{red} nor C_{blue} contains a complete bipartite subgraph $K_{t,t}$ with partite sets of size t.*

Hence there is a coloring $C_{\text{red}} \cup C_{\text{blue}}$ of $K_{N,N}$ possessing (∗). Let G_{red} be the 3-uniform 3-partite hypergraph with partite sets V_1, V_2, and V_3 such that $E(G_{\text{red}}) = \{(a, b, c) \mid (a, b) \in C_{\text{red}}$ and $c \in V_3\}$. Define G_{blue} analogously. Note that G_{red} is far from a random subhypergraph of $K^{(3)}_{N,N,N}$: each pair in $V_1 \times V_2$ is contained either in N edges of G_{red} or in none.

Suppose that there exists an embedding f of H into G_{red}. By the symmetry of H, we may assume that W_i maps into V_i for $i = 1, 2, 3$. Since every pair $(w_1, w_2) \in W_1 \times W_2$ is contained in an edge of H, the set $f(W_1) \times f(W_2)$ should induce a complete bipartite subgraph in C_{red}. This contradicts (∗). If there exists an embedding f of H into G_{blue}, the argument is the same. The example also shows that the condition of H having a maximum degree d in Theorem 2 cannot be replaced by H being 1-degenerate. Indeed, for any coloring of $K^{(3)}_{N,N,N}$ by red and blue either G_{red} or G_{blue} contains $0.5N^3$ edges. Consequently for $\alpha = 1/4$, $l = 3$ and n sufficiently large one of the colors contains $2\alpha N^3 = 0.5N^3$ edges and yet, the corresponding graph does not contain a copy of H.

This construction easily generalizes to every $k \geqslant 3$ as follows. Let $n = \binom{k}{2}t^2$ and F be the complete k-partite graph with partite sets $W_1, \dots, W_k$ of cardinality t. Let $H = H_n^k$ be the k-uniform k-partite hypergraph obtained from F by adding $k-2$ new vertices $w_{e,1}, \dots, w_{e,k-2} \notin W_1 \cup \dots \cup W_k$ to each edge e (all $w_{e,i}$ are different). Then H is a 1-degenerate hypergraph with n edges and $(k-2)n + kt$ vertices. Moreover, each edge of H has $k-2$ vertices of degree one, so the average degree of H is less than $k/(k-2)$.

As above, let $N = 2^{\lfloor t/2 \rfloor}$ and $\mathbb{C} = C_{\text{red}} \cup C_{\text{blue}}$ be a 2-coloring of $K_{N,N}$ possessing $(*)$. Let G_{red} be the k-uniform k-partite hypergraph with partite sets $V_1, V_2, \dots, V_k$ such that $E(G_{\text{red}}) = \{(a_1, \dots, a_k) \mid (a_1, a_2) \in C_{\text{red}} \text{ and } a_i \in V_i \text{ for } i = 3, \dots, k\}$, and let G_{blue} be defined analogously. The same argument as for 3-uniform hypergraphs shows that H is not a subgraph of either G_{red} or G_{blue}.

2.2. *Lower bound for k-uniform 1-degenerate hypergraphs with $k \geqslant 4$*

We do not know how to construct sequences of 1-degenerate 3-uniform hypergraphs with exponentially growing Ramsey numbers, but can construct such k-uniform hypergraphs for each $k \geqslant 4$. We describe here such 4-uniform hypergraphs.

Let $n = \binom{t}{3}$ and $K_t^{(3)}$ be the complete 3-uniform hypergraph with vertex set W of cardinality t. Let H be the 4-uniform hypergraph obtained from $K_t^{(3)}$ by adding to each edge e a new vertex $w_e \notin W$ (all w_e different). Then H is a 1-degenerate 4-uniform hypergraph with n edges and $n + t$ vertices.

We will be using the well-known fact that the logarithms of the number of (labeled) bipartite graphs and of triangle-free graphs are essentially the same. More precisely, we use the fact from [5,6] that for every $\epsilon > 0$ there exists $t(\epsilon)$ such that the number of triangle-free graphs on t vertices, for $t \geqslant t(\epsilon)$, is less than $2^{\frac{t^2}{4}(1+\epsilon)}$.

Set $N = \lfloor 2^{\frac{t}{4}(1-\epsilon)} \rfloor$, where ϵ and t satisfy $t \geqslant t(\epsilon)$, and let $\mathbb{C} = C_{\text{red}} \cup C_{\text{blue}}$ be a random 2-coloring of $K_N^{(2)}$ where each pair is in C_{red} (C_{blue}) with probability 1/2, independently of all other choices. We now show that with positive probability, $\mathbb{C}$ will be such that

$(**)$ *each subset $T \subset V = V(K_N^{(2)})$, with $|T| = t$, contains both red and blue triangles.*

Let T be a t-element subset of V and $\mathbb{R}_T$ ($\mathbb{B}_T$) be a random variable counting the number of red (blue) triangles in T. Set $\mathbb{X}_T = \min\{\mathbb{R}_T, \mathbb{B}_T\}$. Since the number of triangle-free graphs on t vertices is less than $2^{\frac{t^2}{4}(1+\epsilon)}$ we infer that

$$\Pr(\mathbb{X}_T = 0) \leqslant \Pr(\mathbb{B}_T = 0) + \Pr(\mathbb{R}_T = 0) \leqslant \frac{2 \cdot 2^{\frac{t^2}{4}(1+\epsilon)}}{2^{\binom{t}{2}}} = 2^{-\frac{t^2}{4}(1-\epsilon)+t/2+1}.$$

Consequently, the probability that a random coloring $\mathbb{C} = C_{\text{red}} \cup C_{\text{blue}}$ fails to have property $(**)$ can be estimated from above by

$$\sum_{T\in\binom{V}{t}} \Pr(\mathbb{X}_T=0) \leqslant \binom{N}{t} 2^{-\frac{t^2}{4}+\epsilon\frac{t^2}{4}+\frac{t}{2}+1} \leqslant \frac{N^t}{t!} 2^{-\frac{t^2}{4}+\epsilon\frac{t^2}{4}+\frac{t}{2}+1} \leqslant \frac{2^{t\frac{t}{4}(1-\epsilon)}}{2^{\frac{t^2}{4}-\epsilon\frac{t^2}{4}}} \frac{2^{\frac{t}{2}+1}}{t!} = o(1).$$

This means that there exists a coloring $\mathbb{C} = C_{\text{red}} \cup C_{\text{blue}}$ with property $(**)$. Define the 4-uniform hypergraph G_{red} on vertex set V as follows. A quadruple $Q \subset V$ is an edge of G_{red} if Q contains a triangle of C_{red}. Set $G_{\text{blue}} = K_N^{(4)} - G_{\text{red}}$, and consider an embedding f of H in $K_N^{(4)}$. We will show that $f(H)$ is neither red nor blue (i.e. is not a subset of either G_{red} or G_{blue}). Let $T = f(W)$. Then $|T| = t$ and so by $(**)$ T induces both red and blue triangles in $\mathbb{C}$. Let T_{red} be a triple of T that induces a red triangle in $\mathbb{C}$ and T_{blue} be a triple that induces a blue triangle. Since every quadruple of V containing T_{red} is in G_{red}, not all edges of $f(H)$ are in G_{blue}. On the other hand, any quadruple of V containing T_{blue} cannot, at the same time, contain a triple inducing a red triangle in $\mathbb{C}$. Consequently, such a quadruple cannot be in G_{red}. Thus $f(H)$ is not monochromatic. Finally, we note that since $n = \frac{t^3}{6}(1-o(1))$, N is exponential in $n^{1/3}$.

There is a very similar construction for $k \geqslant 5$, just define that a k-tuple $Q \subset V$ is an edge of G is Q contains at least $k(k-1)/4$ edges of B. We omit the details.

2.3. *Lower bound for* 3*-uniform hypergraphs*

We were unable to find the similar lower bounds for 1-degenerate 3-uniform hypergraphs. In fact we are not completely convinced that this is possible. Here we give a weaker result which for d fixed and n sufficiently large implies that there is a d-degenerate hypergraph H with Ramsey number greater than $n^{d^{1/4}}$.

The hypergraph H is the disjoint union of a 3-uniform clique $K_s^{(3)}$ and the 3-uniform "hedgehog," $F_m^{(3)}$, defined below.

Let $F_m^{(3)}$ be the 3-uniform hypergraph of $m + \binom{m}{2}$ vertices

$$\{v_1, v_2, \dots, v_m\} \cup \{u_{ij} \mid 1 \leqslant i < j \leqslant m\}$$

and $\binom{m}{2}$ edges $\{\{v_i, v_j, u_{ij}\} \mid 1 \leqslant i < j \leqslant m\}$. We prove that for s sufficiently large there exists an integer $N \geqslant m^{\sqrt{s}}$ and a red–blue coloring of the triples of $K_N^{(3)}$ with neither red $K_s^{(3)}$ nor blue $F_m^{(3)}$.

Given an integer s, we set t to be the largest integer such that the Ramsey number $r(3,t) \leqslant s$. We will use the result of Shearer [11] which implies that

$$t \geqslant \bigl(1-o(1)\bigr)\sqrt{(s/2)\ln s}.$$

Similarly, for integers t and m, let $r(t,m)$ be the (graph) Ramsey number.

Set $N = r(t,m) - 1$. It is proved in [12] that

$$N \geqslant c_1 \left(\frac{m}{\ln m}\right)^{(t+1)/2}. \tag{2}$$

By the definition of $r(t,m)$ there exists a graph G with vertex set V, $|V| = N$, containing no $K_t^{(2)}$, the complement of which has no $K_m^{(2)}$, where $m = c_2 N^{2/(t+1)} \ln N$.

We use the edges of G to color the triples $\binom{V}{3}$ of V as follows. Color $xyz \in \binom{V}{3}$ red if at least one of the pairs xy, yz, or xz is an edge of G. Color all other edges blue. Clearly, this coloring contains no blue copy of $F_m^{(3)}$. Suppose there is a red copy of $K_s^{(3)}$ where s is the

Ramsey number $r(3,t)$. This means that the complement of G, restricted to the vertex set of this $K_s^{(3)}$ would be triangle-free and consequently, due to the choice of s, G would contain $K_t^{(2)}$. Therefore the coloring contains neither red $K_s^{(3)}$ nor blue $F_m^{(3)}$. Now $N \geqslant c_1(m/\ln m)^{(t+1)/2}$, which is greater than $m^{2\sqrt{s}}$ for s sufficiently large.

Let H be the disjoint union of 1-degenerate $F_m^{(3)}$ and $\binom{s-1}{2}$-degenerate $K_s^{(3)}$. Then H has $n = m + \binom{m}{2} + s$ vertices and is d-degenerate for $d = \binom{s-1}{2} < s^2$. Since $n < m^2$ for m sufficiently large, it follows that H is a d-degenerate n-vertex hypergraph with Ramsey number greater than $N \geqslant m^{2\sqrt{s}} > n^{\sqrt{s}} = n^{d^{1/4}}$.

3. Turán problem for *l*-uniform *l*-partite hypergraphs

Lemma 1. *Let $0 < \alpha \leqslant 1/2$ be a real number and let d, k, s, N, and n be positive integers with $s \geqslant 2$. Let G be a k-uniform k-partite hypergraph with partite sets $V_1, \ldots, V_k$ each of size N having at least $2\alpha N^k$ edges. If*

$$\alpha N/n > N^{(k-1)(d-1)/s}, \tag{3}$$

then there exists a $(k-1)$-uniform $(k-1)$-partite hypergraph G' with partite sets $V_1, \ldots, V_{k-1}$ with more than $2\alpha^s N^{k-1}$ edges such that for each d edges $e_1, \ldots, e_d$ of G' there are at least n vertices $v \in V_k$ with $e_1 + v, \ldots, e_d + v \in E(G)$.

Proof. Let $W = V_1 \times \cdots \times V_{k-1}$ and for each $\mathbf{w} \in W$ let $N_G(\mathbf{w})$ denote the set of vertices $v \in V_k$ such that $\mathbf{w} + v$ is an edge in G. For a set $X \subseteq W$, by $N_G(X)$ we denote the set $\bigcap_{\mathbf{w}\in X} N_G(\mathbf{w})$.

Let $v_1, \ldots, v_s$ be a sequence of s not necessarily distinct vertices of V_k chosen at random uniformly and independently and let $S = \{v_1, \ldots, v_s\}$. Let $U = U_S = \{\mathbf{w} \in W \mid S \subseteq N_G(\mathbf{w})\}$. Then the size of U is a random variable. Using Jensen's inequality we have

$$\begin{aligned}\mathbf{E}\big(|U|\big) &= \sum_{\mathbf{w}\in W} \mathbf{Pr}(\mathbf{w} \in U) = \sum_{\mathbf{w}\in W} \left(\frac{|N_G(\mathbf{w})|}{N}\right)^s = \frac{\sum_{\mathbf{w}\in W}(|N_G(\mathbf{w})|)^s}{N^s} \\ &\geqslant \frac{N^{k-1}(\frac{|E(G)|}{N^{k-1}})^s}{N^s} \geqslant N^{k-1-s}\left(\frac{2\alpha N^k}{N^{k-1}}\right)^s = (2\alpha)^s N^{k-1} \geqslant 4\alpha^s N^{k-1}.\end{aligned}$$

On the other hand, by the definition of S, for a fixed set $X \subseteq W$, $\Pr(X \subseteq U_S) = (|N_G(X)|/N)^s$. Denote by z the number of subsets X of W of size d with $|N_G(X)| < n$. The expected value of z is at most

$$\begin{aligned}\mathbf{E}(z) &= \sum_{\{X\subseteq W:\ |X|=d,\ |N_G(X)|<n\}} \mathbf{Pr}(X \subseteq U) \leqslant \binom{N^{k-1}}{d}\left(\frac{n}{N}\right)^s \leqslant N^{(k-1)d}\left(\frac{n}{N}\right)^s \\ &= N^{(k-1)d-s} n^s.\end{aligned}$$

This together with (3) yields

$$\mathbf{E}(z) < N^{(k-1)d-s}(\alpha N)^s N^{-(k-1)(d-1)} = \alpha^s N^{k-1}.$$

Therefore by linearity of expectation there exists a particular choice of $v_1, \ldots, v_s$ such that $|U| - z > 4\alpha^s N^{k-1} - \alpha^s N^{k-1} = 3\alpha^s N^{k-1}$. Fix these $v_1, \ldots, v_s$ and delete a $(k-1)$-tuple $\mathbf{w}$ from every subset X of U of size d with $|N_G(X)| < n$. This produces a set $U_1 \subseteq W$ of size greater than $3\alpha^s N^{k-1}$ with the property that for every $X \subseteq U_1$ with $|X| = d$, $|N_G(X)| \geqslant n$. Now, we define G' as the hypergraph on $V_1 \cup \cdots \cup V_{k-1}$ with the set of edges equal to U_1. □

By repeated application of Lemma 1, we obtain the following statement.

Lemma 2. *For α, $0 < \alpha \leqslant 1/2$, and integers $d \geqslant 2$, $l \geqslant 2$, let n and N be such that for some integer $s \geqslant 2$ and for all $i = 1, \dots, l$,*

$$\alpha^{s^{l-i}} N/n > N^{(i-1)(d-1)/s}. \tag{4}$$

Let G be an l-uniform l-partite hypergraph with partite sets $V_1, \dots, V_l$ each of cardinality N having at least $2\alpha N^l$ edges. Then there exists a sequence $G_1, \dots, G_l$ of hypergraphs such that $G_l = G$ and for each $i \in \{1, \dots, l-1\}$,

(i) *G_i is an i-uniform i-partite hypergraph with the partite sets $V_1, \dots, V_i$;*
(ii) *G_i has at least $2\alpha^{s^{l-i}} N^i$ edges;*
(iii) *for each d edges $e_1, \dots, e_d$ of G_i, there are at least n vertices $v \in V_{i+1}$ with $e_1 + v, \dots, e_d + v \in E(G_{i+1})$.*

Now we are ready to prove Theorem 2. For convenience, we state it again.

Theorem 2. *For α, $0 < \alpha \leqslant 1/2$, and integers $d \geqslant 2$, $l \geqslant 2$, let n and N be such that for $c = \ln \frac{1}{\alpha} + 2(d-1)(l-1)$ we have*

$$\ln n > \bigl(\max\{2, 3c\}\bigr)^l, \tag{5}$$

and

$$N > n e^{c(\ln n)^{(l-1)/l}} = n^{1+c/(\ln n)^{1/l}}. \tag{6}$$

Let G be an l-uniform l-partite hypergraph with partite sets $V_1, \dots, V_l$ each of cardinality N and at least $2\alpha N^l$ edges. If an l-uniform l-partite hypergraph H contains at most n edges, and the degrees of all of the partite sets of H except one are at most d, then G contains H.

Proof. Suppose that the partite sets of H are $Y_1, \dots, Y_l$ and that the degree of every vertex in $V(H) - Y_1$ is at most d.

Set $s = \lfloor (\ln n)^{1/l} \rfloor$ and note that $s \geqslant 2$ by (5). First we prove for this s the inequality

$$\alpha^{s^{l-1}} N^{1-(l-1)(d-1)/s} > n \tag{7}$$

which implies the validity of (4) for every $1 \leqslant i \leqslant l$. By (6) and the definition of c, the expression $N^{1-(l-1)(d-1)/s}$ is greater than

$$n^{(1+c/(\ln n)^{1/l})(1-(l-1)(d-1)/s)} = n^{1+\ln(1/\alpha)/(\ln n)^{1/l}+2(d-1)(l-1)/(\ln n)^{1/l}-(1+c/(\ln n)^{1/l})(l-1)(d-1)/s}. \tag{8}$$

Since

$$n^{\ln(1/\alpha)/(\ln n)^{1/l}} = (1/\alpha)^{(\ln n)^{(l-1)/l}} \leqslant (1/\alpha)^{s^{l-1}},$$

(8) yields that to prove (7), it is enough to check that

$$2(d-1)(l-1)/(\ln n)^{1/l} \geqslant \bigl(1 + c/(\ln n)^{1/l}\bigr)(l-1)(d-1)/s.$$

This inequality is equivalent to

$$2s \geqslant \bigl(1 + c/(\ln n)^{1/l}\bigr)(\ln n)^{1/l}. \tag{9}$$

Recall that $s \geqslant 2$ is the floor of $(\ln n)^{1/l}$ and that by (5), $1 + c/(\ln n)^{1/l} < 4/3$. Thus (9) holds and hence (7) holds. Therefore, G satisfies the conditions of Lemma 2.

Consider a sequence $G_1, \dots, G_l$ of hypergraphs provided by this lemma. By the lemma, the number of edges (which are singletons) in G_1 is at least $2\alpha^{s^{l-1}} N > 2n$. We map the vertices of Y_1 into distinct edges (again, they are singletons) of G_1.

Suppose that we have already mapped the vertices in $Y_1 \cup \dots \cup Y_i$ into $V_1 \cup \dots \cup V_i$ so that the projection of each edge of H onto $Y_1 \cup \dots \cup Y_i$ is mapped into an edge of G_i. Let $Y_{i+1} = \{x_1, \dots, x_t\}$. We will map $x_1, \dots, x_t$ one by one. Suppose that we have already mapped $x_1, \dots, x_{j-1}$ and that x_j belongs to edges $e_1, \dots, e_r$. For each e_β, $\beta = 1, \dots, r$ consider the restriction $e'_\beta = e_\beta \cap \bigcup_{\alpha=1}^{i} Y_\alpha$ to $Y_1 \cup \dots \cup Y_i$. Let $f_1, \dots, f_r$ be the images of $e'_1, e'_2, \dots, e'_r$ in $V_1 \cup \dots \cup V_i$. By the definition of H, $r \leqslant d$. Therefore, by (iii) of Lemma 2, there are at least n possible vertices $v_j \in V_{i+1}$ to which x_j can be properly mapped, i.e., so that

$$f_\beta \cup \{v_j\} \in G_{i+1} \quad \text{for } \beta = 1, \dots, r. \tag{10}$$

At most $j - 1$ of these vertices are already spoiled by the images of $x_1, \dots, x_{j-1}$. Since $j \leqslant n$, we can find v_j with property (10). This proves the theorem. □

Remark. Clearly, the proof of the theorem can be modified to find many edge-disjoint copies of H in G. Also, our Turán type result implies corresponding l-partite Ramsey results for many colors.

4. Reductions

In this section, we discuss several facts that will be used in the next section to deduce bounds on Ramsey number of sparse r-uniform hypergraphs using Turán type bounds for l-uniform l-partite hypergraphs.

The following is a well-known folklore fact.

Lemma 3. *Let G be an l-uniform hypergraph with lN vertices and M edges. Then G contains an l-uniform l-partite subhypergraph G' with at least $M e^{-l}$ edges such that each partite set has exactly N vertices.*

Proof. There are $\frac{(lN)!}{(N!)^l}$ ways to partition the set of lN vertices into l (ordered) sets each of size N. Each edge e contains vertices from all l sets in $l!\frac{(l(N-1))!}{((N-1)!)^l}$ such partitions. Therefore, there is a partition such that the corresponding subhypergraph has at least

$$M\, l! \frac{(l(N-1))!}{((N-1)!)^l} \frac{(N!)^l}{(lN)!} \geqslant M \frac{l!}{l^l} > M e^{-l}$$

edges. □

Lemma 4. *For each l, r with $l > r$, there exist $c = c(l, r)$ and $N_0 = N_0(l, r)$ such that for each $N > N_0$ every edge 2-coloring of $K_N^{(r)}$ yields at least $c N^l$ monochromatic copies of $K_l^{(r)}$.*

Proof. Let $N_0 = R_r(l, l)$ be the minimum N with the property that every edge 2-coloring of $K_N^{(r)}$ yields a monochromatic copy of $K_l^{(r)}$. Let $N > N_0$ and consider an arbitrary edge 2-coloring f of $K_N^{(r)}$. By the definition of N_0, every N_0-element subset of $V(K_N^{(r)})$ contains a monochromatic copy of $K_l^{(r)}$. Since each such copy is contained in $\binom{N-l}{N_0-l}$ N_0-element subsets of $V(K_N^{(r)})$, the total number of such monochromatic subgraphs is at least

$$\frac{\binom{N}{N_0}}{\binom{N-l}{N_0-l}} = \frac{\binom{N}{l}}{\binom{N_0}{l}} \geqslant \frac{N^l}{N_0^l} = cN^l. \qquad \Box$$

We say that a hypergraph H is *the restriction of a hypergraph H' to V*, if $V(H) = V$ and $E(H) = \{e \cap V\colon\ e \in E(H')\}$.

A set B of vertices of a hypergraph H is *strictly independent*, if no edge of H contains more than one vertex of B.

Lemma 5. *Let H be an r-uniform hypergraph with maximum degree at most d and let B be a strictly independent set in H. Let $l = d(r-1)+1$. Then there exists an l-uniform l-partite hypergraph H' with the same number of edges and $V(H') \supset V(H)$ such that*

(a) *the restriction of H' to $V(H)$ is H;*
(b) *the degree in H' of each vertex $w \in V(H') - V(H)$ is* 1;
(c) *all vertices of B are in the same partite set of H'.*

Proof. Let H_1 be the graph with $V(H_1) = V(H)$ such that $uv \in E(H_1)$ if and only if there is an edge of H containing both u and v. Note that B is an independent set in H_1 and the degree in H_1 of every other vertex is at most $d(r-1) = l-1$. It follows that there exists a proper vertex l-coloring f of H_1 such that all vertices of B are colored with the same color. This yields the partition of $V(H)$ into l color classes (some of them could be empty) of f.

Now, for every $e \in E(H)$ and every color class C with $C \cap e = \emptyset$, we create a new vertex $v_{e,C}$ of color C. The new hypergraph H' has vertex set $V(H) \cup \bigcup_{e \in E(H)} \{v_{e,C}\colon\ e \cap C = \emptyset\}$ and edge set $\{e \cup \{v_{e,C}\colon\ e \cap C = \emptyset\} \mid e \in E(H)\}$. $\Box$

5. Ramsey number

We prove Theorem 3 in the following slightly stronger form.

Theorem 4. *Let Δ and r be fixed. Let H_1 and H_2 be r-uniform hypergraphs on m vertices such that for $i = 1, 2$, H_i has a strictly independent vertex set B_i such that the degree of every $v \in V(H_i) - B_i$ in H_i is at most Δ. Then*

$$R(H_1, H_2) \leqslant m^{1+o(1)}.$$

Proof. Let $l = \Delta(r-1)+1$, c be the constant from Lemma 4, and $\alpha = c/2$. Let $n = \lfloor \Delta m/r \rfloor$, N_0 be as in Lemma 4, and N be the minimum positive integer satisfying (6) and the conditions $N \geqslant e^{2^l}$ and $N \geqslant N_0/l$. Observe that for fixed Δ and r, by (6) we have $N = n^{1+o(1)} = m^{1+o(1)}$.

Consider a red-blue coloring of the complete r-uniform hypergraph on lN vertices. Let G be the hypergraph consisting of all the red edges. By Lemma 4, either G or its complement contains at least $c(lN)^l$ monochromatic copies of $K_l^{(r)}$. We may assume that this is G. Consider the auxiliary l-uniform hypergraph $G^{(l)}$ whose edges are the l-tuples inducing complete subgraphs of G. By Lemma 3, $G^{(l)}$ contains an l-uniform l-partite subhypergraph $\tilde{G}^{(l)}$ with at least $ce^{-l}(lN)^l > cN^l$ edges such that each partite set has exactly N vertices.

Let B_1 be a strictly independent vertex set in H_1 such that the degree of every $v \in V(H_1) - B_1$ in H_1 is at most Δ. By Lemma 5, there exists an l-uniform l-partite hypergraph H_1' with the same number of edges as in H_1 and $V(H_1') \supset V(H_1)$ such that

(a) the restriction of H_1' to $V(H_1)$ is H_1,
(b) the degree in H_1' of each vertex $w \in V(H_1') - V(H_1)$ is 1, and
(c) all vertices of B_1 are in the same partite set of H_1'.

The number of edges in H_1' is at most $n = \lfloor \Delta m/r \rfloor$. Observe that we have chosen Δ, α, and N so that H_1' and $\tilde{G}^{(l)}$ satisfy the conditions of Theorem 2. Thus $\tilde{G}^{(l)}$ contains H_1'.

Let us check now that G contains H_1. Indeed, suppose we have an embedding f' of H_1' into $\tilde{G}^{(l)}$. Then f' induces a mapping f of $V(H_1)$ into $V(\tilde{G}^{(l)}) = V(G)$. Each edge $e \in E(H_1)$ is a part of an edge in H_1' and thus is mapped onto a part of an edge e_1 of $G^{(l)}$. By the definition of $G^{(l)}$, this part is an edge of G. □

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