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Even cycles in hypergraphs

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Abstract

A cycle in a hypergraph is an alternating cyclic sequence $A_0, v_0, A_1, v_1, \ldots, A_{k-1}, v_{k-1}, A_0$ of distinct edges A_i and vertices v_i such that $v_i \in A_i \cap A_{i+1}$ for all i modulo k. In this paper, we determine the maximum number of edges in hypergraphs on n vertices containing no even cycles.

Keywords: Hypergraph; Hypertree; Cycle

1. Introduction

A *hypergraph* on a set X is a family $\mathcal{A}$ of labelled (but not necessarily distinct) subsets of X. In what follows these subsets of X are to have size at least two. According to Berge [1], a *cycle* in $\mathcal{A}$ is an alternating cyclic sequence $A_0, v_0, A_1, v_1, \ldots, A_{k-1}, v_{k-1}, A_0$ of distinct edges A_i of $\mathcal{A}$ and distinct vertices v_i of $\mathcal{A}$ such that $v_i \in A_i \cap A_{i+1}$ for all i modulo k. In this definition, we allow the members A_i of $\mathcal{A}$ to be equal as sets, but insist that they are distinct as members of $\mathcal{A}$. A cycle with k edges is referred to as a k-*cycle* or a *cycle of length* k.

Let us begin our discussion with the following well-known result: every maximal acyclic graph is a tree. We say that a hypergraph $\mathcal{A}$ is *acyclic* if it contains no cycle, and *connected*

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if for every non-empty subset e of X, $\mathcal{A} \cup \{e\}$ contains a cycle $\mathcal{C}$ with $e \in \mathcal{C}$. A *hypertree* is a connected acyclic hypergraph. The following holds (see [5]):

Theorem 1.1. *If $\mathcal{A}$ is an acyclic hypergraph on X, then $\sum_{A\in\mathcal{A}}(|A|-1) \leqslant |X|-1$, with equality if and only if $\mathcal{A}$ is a hypertree.*

In this paper, we are interested in the maximum size of a hypergraph containing no even cycle—in other words, no cycle of even size. It is straightforward to prove that any graph with no even cycle has at most $\lfloor \frac{3}{2}(n-1) \rfloor$ edges, and equality holds if and only if all blocks in the graph, except possibly one, are triangles. The extension to hypergraphs is somewhat more difficult to establish. Throughout the introduction, we assume $\mathcal{A}$ is a hypergraph on a set X. The *lower rank* of a hypergraph $\mathcal{A}$ is the size of a smallest element of $\mathcal{A}$, namely $\min\{|A| : A \in \mathcal{A}\}$. Gyárfás et al. [3,4] settled the extremal question for odd cycles by proving the following theorem:

Theorem 1.2. *If $\mathcal{A}$ is a hypergraph of lower rank at least three, containing no odd cycle, then $\sum_{A\in\mathcal{A}}(|A|-1) \leqslant 2|X|-2$, with equality if and only if $\mathcal{A}$ is the hypergraph on X consisting of two identical uniform hypertrees on X.*

In other words, the extremal object $\mathcal{A}$ in Theorem 1.2 is a hypertree in which every edge is doubled—otherwise known as a *doubled hypertree*. In fact, the authors of [4] proved a stronger statement. They proved that if $\sum_{A\in\mathcal{A}}(|A|-1) > 2|X|-2$, then $\mathcal{A}$ contains a cycle $A_0, v_0, A_1, v_1, \ldots, A_{k-1}, v_{k-1}, A_0$ where some A_i contains at least three vertices in $\{v_0, , \ldots, v_{k-1}\}$. The main result of this paper is the solution of the extremal problem for even cycles:

Theorem 1.3. *Let $k \geqslant 2$, and let $\mathcal{A}$ be a hypergraph of lower rank at least k, containing no even cycle. Then $\sum_{A\in\mathcal{A}}(|A|-1) \leqslant \lfloor \frac{k}{k-1}(|X|-1) \rfloor - 1$.*

We will give a construction showing that this bound is sharp. Note that the relations in Theorems 1.1 and 1.2 do not depend on the lower rank, while the relation in Theorem 1.3 does. The following bound is easily implied by Theorem 1.3:

Corollary 1.4. *Let $\mathcal{A}$ be a hypergraph on at least three vertices, containing no even cycle. Then $\sum_{A\in\mathcal{A}}(|A|-\frac{3}{2}) \leqslant \frac{3}{2}|X|-3$.*

This paper is organized as follows: in Section 2 we give a standard reduction of hypergraph problems to their bipartite incidence graphs (the same reduction was used in [3,4]), in Section 3 we give a construction showing that Theorem 1.3 is sharp, and in Section 4 we prove Theorem 1.3. The following notation will be used throughout:

Notation: We consider a graph G on X as a 2-uniform hypergraph, and $|G|$ denotes the number of edges of G. We write $V(G)$ for the (non-empty) *vertex set* of G. For $v \in V(G)$, we let $N(v) = \{w \in V(G) - v : \{v, w\} \in G\}$. If w is a set of vertices of G, we denote $N(W) = \bigcup_{v\in W} N(v)$ and write $G - W$ for the graph on $V(G) - W$ consisting of all edges of G that are disjoint from w. We write $\Gamma_G(W)$ for the *neighborhood* of w, i.e. $N(W) - W$.

Let $d_G(W) = |\Gamma_G(W)|$ denote the *degree* of w. For a subgraph H of G, by the *neighborhood* $\Gamma_G(H)$ of H we mean $\Gamma_G(V(H))$. The *distance* between two vertices u, v of a graph G, denoted $\text{dist}_G(u, v)$, is the length of a shortest path in G between u and v. A *component* of G is a maximal connected subgraph of G. A *cut vertex* of G is a vertex v such that $G - \{v\}$ has more components than G, and a block of G is a maximal subgraph H of G such that H has no cutvertices. A *cut set* of G is a set S of vertices of G such that $G - S$ has more components than G. A *pendant vertex* is an $x \in X$ such that $d_G(x) = 1$, and a *pendant block* in G is a block of G containing at most one cutvertex of G.

2. Bipartite incidence graphs

We write $G(A, B)$ to indicate that $G = G(A, B)$ is a bipartite graph with parts A and B. We will prove Theorem 1.3 by appealing to the natural point-set incidence bipartite graph associated with a hypergraph $\mathcal{A}$ on X: this is the bipartite graph $G = G(\mathcal{A}, X)$ in which $x \in X$ is adjacent to $A \in \mathcal{A}$ if $x \in A$. Conversely, we may associate to a bipartite graph $G(A, B)$ the hypergraph $\mathcal{A}$ on B consisting of the family of neighborhoods of vertices in A. Therefore $G(\mathcal{A}, X)$ and $\mathcal{A}$ are equivalent representations of the same object. It is clear that $\mathcal{A}$ contains a cycle of length k modulo two if and only if $G(\mathcal{A}, X)$ contains a cycle of length $2k$ modulo four. In this context, Theorem 1.3 may be stated in the following form:

Theorem 2.1. *Let $k \geqslant 2$, and let $G = G(A, B)$ be a bipartite graph containing no cycle of length zero modulo four, and in which every vertex of A has degree at least k. Then $|G| \leqslant |A| + \lfloor \frac{k}{k-1}(|B| - 1) \rfloor - 1$.*

We will prove Theorem 1.3 by proving Theorem 2.1. Before doing so, we give a construction showing that Theorem 1.3 is sharp.

3. Construction

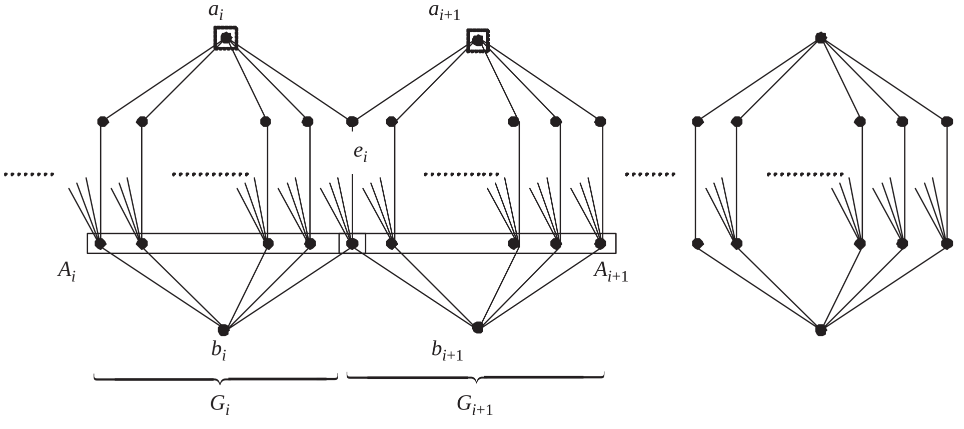
The following construction (see the illustration) shows that Theorems 2.1 and 1.3 cannot be improved for $k \geqslant 3$: let m and $k \geqslant 3$ be positive integers, define a bipartite graph $H' = H'(m, k) = \bigcup_{i=1}^{m} G_i$, where $G_i = G_i(A_i, B_i)$ consists of k internally disjoint paths of length three between two vertices $a_i \in A_i$ and $b_i \in B_i$, where

- for $i = 1, \ldots, m - 1$, there is an edge e_i, disjoint from $\{a_i, a_{i+1}, b_i, b_{i+1}\}$, with $G_i \cap G_{i+1} = \{e_i\}$,
- for $|i - j| \geqslant 2$, $V(G_i) \cap V(G_j) = \emptyset$.

We now add pendant vertices in H', adjacent to $A = \bigcup A_i$ in such a way that every vertex of A has degree exactly k, to obtain the bipartite graph $H_{m,k} = H_{m,k}(A, B)$. We claim that $H_{m,k}$ contains no cycle of length zero modulo four. If $m = 1$, then this obvious. Suppose that $H_{m-1,k}$ contains no cycle of length zero modulo four and C is such a cycle in $H_{m,k}$. Then C has some vertices outside of G_1 and some vertices in $G_1 - e_1$. Therefore, C contains both ends of e_1 and can be split into two paths connecting them. By the choice of m, each of these paths has length one modulo four. Hence C has length two modulo four, a contradiction to the choice of m. Furthermore, $|B| = (k-1)^2m + k$, $|A| = mk + 1$, and $|H_{m,k}| = mk^2 + k$. It follows that

$$|A| + \left\lfloor \frac{k}{k-1}(|B|-1) \right\rfloor = mk + 1 + \frac{k}{k-1}[(k-1)^2m + k - 1]$$
$$= mk^2 + k + 1 = |H_{m,k}| + 1.$$

This completes the construction. An illustration is provided below.



4. Proof of Theorem 2.1

To prove Theorem 2.1 we require two simple lemmas.

Lemma 4.1. *Let $t \geqslant 2$, and let $P_1, P_2, \ldots, P_t$ be internally disjoint paths with the same pair of endpoints in a bipartite graph G. If G contains no cycle of length zero modulo four, then $|P_1| = |P_2| = \cdots = |P_t| = 1$ modulo four or $|P_1| = |P_2| = \cdots = |P_t| = 3$ modulo four, or $t = 2$ and P_1 and P_2 have different even lengths modulo four.*

An *ear* in a graph H is an inclusion maximal path whose all internal vertices have degree two. A ear is *non-trivial* if it has internal vertices (i.e. it has at least three vertices). In general, we represent paths by sequences of vertices, for example $(a_1, a_2, a_3, \ldots, a_k)$ is a path with endvertices a_1 and a_k.

Lemma 4.2. *Let H be a simple graph, not containing a subdivision of K_4, and with minimum degree at least two. If H is not a cycle, then it has at least two* (*not necessarily disjoint*) *non-trivial ears.*

Proof. It is known that if H contains K_4 as a minor, then H contains a subdivision of K_4. The following fact is also known (see, for example, [6, p. 218]): every simple graph on at least two vertices with at most one vertex of degree less than three contains a subdivision of K_4. Therefore H has at least one non-trivial ear, say, $Y = (v_0, v_1, \ldots, v_m)$. When we contract all vertices $v_1, \ldots, v_{m-1}$ into a new vertex v^*, the resulting graph H^* still satisfies the conditions of our claim: H^* is simple, contains no subdivision of K_4, and has no pendant vertices. Thus, using the fact above, H^* has a vertex of degree two distinct from v^*, and that vertex must belong to a ear in H distinct from Y. □

Proof of Theorem 2.1. Let $k \geqslant 2$, and let $G(A, B)$ be a counterexample to Theorem 2.1 with fewest edges. Then $G = G(A, B)$ has size at least $\phi_k(A, B) = |A| + \lfloor \frac{k}{k-1}(|B| - 1) \rfloor$ and no cycle of length zero modulo four. If $|A| = 1$, then $|G| = |B| < \phi_k(A, B)$, a contradiction. So $|A| > 1$. We proceed by a series of claims.

Claim 1. *The graph $G - \{b\}$ is connected for all $b \in B$. If $d_G(b) = 1$ for some $b \in B$, then the unique neighbor a of b has degree k in G.*

Proof. Assume that some $b \in B$ is a cut vertex. Let $G_1 = G_1(A_1, B_1)$ and $G_2 = G_2(A_2, B_2)$ be two connected subgraphs of G with at least two vertices each, having only the vertex b in common and whose union is G. As G_1 and G_2 are both subgraphs of G, neither G_1 nor G_2 has a cycle of length zero modulo four. By the minimality of G, $|G_i| < \phi_k(A_i, B_i)$, and

$$|G| = |G_1| + |G_2| < \phi_k(A_1, B_1) + \phi_k(A_2, B_2) \leqslant \phi_k(A, B).$$

This is a contradiction. Finally, if the last part of the claim were false for some $b \in B$, then $G - \{b\}$ would be a smaller counterexample than G, a contradiction. This proves Claim 1. □

We have shown that B contains no cutvertex. Next we make a claim concerning cutsets of order two in B. Edges e, f in a graph G are said to be *parallel* if they join the same pair of vertices. An edge of G is a *parallel edge* if there exists another edge of G to which it is parallel.

Claim 2. *If two vertices $b_1, b_2 \in B$ form a cut set in G, then*
(a) *$G - \{b_1, b_2\}$ has exactly two components,*
(b) *the distance in G between b_1 and b_2 is two,*
(c) *one of the components of $G - \{b_1, b_2\}$ is a star consisting of a central vertex $a_0 \in A$ with exactly $k - 2$ leaves in B, all of which are pendant vertices of G.*

Proof. Assume that $\{b_1, b_2\} \subset B$ is a cut set in G. Let $H_1, \ldots, H_m$ be the components of $G - \{b_1, b_2\}$. By Claim 1, each of H_i contains a b_1, b_2-path, P_i, for $i = 1, \ldots, m$. Since G has no cycle of length zero modulo four, and $|P_i|$ is even for all i, the lengths of $P_1, \ldots, P_m$ are distinct modulo four. Thus $m = 2$ and in the remainder of this claim we may assume that $H_1 = H_1(A_1, B_1)$ and $H_2 = H_2(A_2, B_2)$ are the two components of $G - \{b_1, b_2\}$ such that $|P_1| = 0 \pmod 4$ and $|P_2| = 2 \pmod 4$. In particular, this proves (a).

To prove (b) and (c) suppose, for a contradiction, that H_2 contains more than one vertex in A and hence at least k vertices in total. Let $H_1' = H_1'(C_1, D_1)$ be obtained from $G - H_2$ by adding a vertex a_0 adjacent to b_1 and b_2 and then $k - 2$ pendant vertices $b_1', \ldots b_{k-2}'$ adjacent only to a_0. Now H_1' contains no cycle of length zero modulo four (otherwise, G

would also have such a cycle by the definition of H_2). By our assumption, $|H_1'| < |G|$, and therefore, by the minimality of G, $|H_1'| \leqslant \phi_k(C_1, D_1) - 1$. Now let $H_2'(C_2, D_2) = G - H_1$. Then we have $|H_2'| \leqslant \phi_k(C_2, D_2) - 1$. Thus,

$$\begin{aligned}|G| &= |H_1'| + |H_2'| - k \leqslant \phi_k(C_1, D_1) + \phi_k(C_2, D_2) - 2 - k \\ &= |C_1| + |C_2| - 2 - k + \left\lfloor \frac{k(|D_1| - 1)}{k-1} \right\rfloor + \left\lfloor \frac{k(|D_2| - 1)}{k-1} \right\rfloor \\ &\leqslant |C_1| + |C_2| - 2 - k + \left\lfloor \frac{k}{k-1}(|D_1| + |D_2| - 2) \right\rfloor.\end{aligned}$$

Recall that $|C_1| + |C_2| = |A| + 1$ and $|D_1| + |D_2| = |B| + k$. It follows that

$$\begin{aligned}|G| &\leqslant (|A| + 1) - 2 - k + \left\lfloor \frac{k}{k-1}(|B| + k - 2) \right\rfloor \\ &= |A| - 1 - k + \left\lfloor \frac{k}{k-1}(|B| - 1) \right\rfloor + k = \phi_k(A, B) - 1.\end{aligned}$$

This contradiction completes the proof of Claim 2. □

Let G_1 be obtained from G by deleting from G all vertices of degree one in G (recall that all these vertices are in B, since every vertex of A has degree at least two). Suppose the parts of G_1 are A_1 and B_1.

Claim 3. *The minimum degree of G_1 is at least two.*

Proof. All remaining vertices in B did not change their degrees, so all vertices of B_1 have degree at least two in G_1. By Claim 1 and the fact that $|A| > 1$, there are no isolated vertices in G_1. Finally, if a vertex $a \in A_1$ is a pendant vertex in G_1, then it has $k - 1$ pendant neighbors in G, say $b_1, \ldots, b_{k-1}$. Let $G'(A', B') = G - \{a, b_1, \ldots, b_{k-1}\}$. By the minimality of G,

$$|G'| < \phi_k(A', B') = |A| - 1 + \left\lfloor \frac{k(|B| - 1 - (k-1))}{k-1} \right\rfloor < \phi_k(A, B)$$

and hence $|G| < \phi_k(A, B)$, a contradiction. □

Claim 4. *No two vertices in A_1 of degree two in G_1 have a common neighbor of degree two.*

Proof. Assume that vertices a_1 and a_2 in A_1 of degree two have a common neighbor b_0 also of degree two. By Claim 3, there is a neighbor $b_i \neq b_0$ of a_i in G_1 that has degree at least two in G_1, for $i \in \{1, 2\}$. Since G contains no cycle of length zero modulo four, $b_1 \neq b_2$. Then $\{b_1, b_2\}$ is a cut set in G. From Claims 2(b) and 2(c), we deduce that G_1 is a 6-cycle. It is easy to check that such a bipartite graph G satisfies the theorem. This completes the proof of Claim 4. □

4.1. Replacing ears with edges

Let F be a pendant block in G_1, and let G_2 be the multigraph obtained from F by replacing every non-trivial ear in F with an edge joining its endpoints. Note that F is not a cycle, by Claim 4, so F contains at least two vertices of degree three and G_2 has order at least two. Note also that G_2 has minimum degree at least three. We define an edge e of G_2 to be *i-complex* if e was obtained from a ear of length i in F.

Claim 5. *Each parallel edge in G_2 is* 3*-complex.*

Proof. Suppose e_1 and e_2 are parallel edges in G_2 connecting vertices w and u. Since F is 2-connected, G_2 is also 2-connected. As G_2 has order at least two, this means that there is another path P connecting w and u in G_2. Then, by Lemma 4.1, e_1 and e_2 correspond to paths P_1 and P_2 in G having the same odd length modulo four. By Claim 4, neither of P_1 and P_2 has length five or more. As G is simple, P_1 and P_2 cannot both have length one. Therefore both paths have length three. This proves Claim 5. □

4.2. Complex edges and reducible vertices

We define a new graph G_3 by replacing every set of pairwise parallel edges in G_2 with a single edge. Let A^* be the set of cutvertices of G_1 in F. Note that $|A^*| \leqslant 1$, since F is a pendant block in G_1. If $A^* = \{a^*\}$, and a^* is an internal vertex of a non-trivial ear in F, then we denote by e^* the edge in G_2 joining the endpoints of that ear, and let e^{**} be the single edge in G_3 corresponding to the edge e^*. In this case, let $E^* = \{e^*\}$ and $E^{**} = \{e^{**}\}$, otherwise let $E^* = E^{**} = \emptyset$. A vertex a of G_2 is said to be *reducible* if $a \in A$ and all but at most one of the edges of G_2 incident with a are not 3-complex and are not in E^*. So if a is not reducible, then a must be incident with e^* and with some other non-3-complex edge, or with at least two non-3-complex edges.

Claim 6. *The multigraph G_2 contains no reducible vertices.*

Proof. Assume that $a_0 \in A$ is reducible in G_2 and its degree in G_2 is $r+1$. By the definition of reducible vertices, there exist ears $P_1, P_2, \ldots, P_r \subset F$, each of length three, of the form

$$P_i = (a_0, b_i, a_i, b_i'),$$

where none of the a_i is in A^*. Let b_0 be the remaining neighbor of a_0 in G_2 and L_0 be the set of pendant neighbors of a_0 in G. Let $|L_0| = s$. Now denote by L_i the set of pendant neighbors of a_i, $i = 1, \ldots, r$. By Claim 2(c), the component of $G - \{b_i, b_i'\}$ containing a_i must be a star with $k-2$ leaves, so $|L_i| = k-2$ for $i = 1, \ldots, r$. Consider the graph

$$G' = G - \{a_i \mid i = 0, 1 \ldots, r\} - \{b_j \mid j = 1 \ldots, r\} - \bigcup_{i=0}^{r} L_i$$

with parts A' and B'. Since the neighborhood of each $a \in A'$ is the same in G' as it is in G, and G' contains no cycle of length zero modulo four, $|G'| < \phi_k(A', B')$. Note that

$|A'| = |A| - r - 1$, $|B'| = |B| - r(k-1) - s$. Since every vertex a_i has degree k, for $i \in \{1, 2, \ldots, r\}$, we have

$$|G'| = |G| - \sum_{i=0}^{r} d(a_i) = |G| - rk - r - s.$$

Hence

$$\begin{aligned} |G| &< \phi_k(A', B') + rk + r + s \\ &\leqslant \phi_k(A, B) - r - 1 - \left\lfloor \frac{k(r(k-1)+s)}{k-1} \right\rfloor + rk + r + s \\ &\leqslant \phi_k(A, B) - \left\lfloor \frac{s}{k-1} \right\rfloor \leqslant \phi_k(A, B). \end{aligned}$$

This contradicts the fact that G was a counterexample to the theorem, and completes the proof of Claim 6. □

Claim 7. $|V(G_3)| > 2$.

Proof. If $|V(G_3)| = 2$, then G_2 is the union of a set of $l \geqslant 3$ paths $P_1, \ldots, P_l$ between two vertices, say v_1 and v_2. By Claim 5, the length of every P_i is three. Thus one of v_1 and v_2, say, v_1, is in A, and, moreover, v_1 is reducible. This contradicts Claim 6. □

Claim 8. *No vertex of A has degree two in G_3.*

Proof. Suppose that $a_0 \in A$ has degree two in G_3 and its neighbors are b_1 and b_1'. In G_2, the vertex a_0 has degree at least three, so we can assume $\{a_0, b_1\}$ corresponds to parallel edges in G_2. By Claim 5, these edges are 3-complex. Since a_0 is not reducible, one of these parallel edges is e^*, and $\{a_0, b_1'\}$ is not a parallel edge. Since e^* is 3-complex, $b_1 \in B$. Let b_2 be the first vertex on the path in G corresponding to the edge $\{a_0, b_1'\}$ in G_2. Note that $b_1' = b_2$ is possible if the path consists only of $\{a_0, b_1'\}$. In any case, $\{b_1, b_2\}$ is a cut set in G. Then Claim 3 implies that G_3 has only two vertices, namely a_0 and b_1. This contradicts Claim 7, and proves Claim 8. □

Now we are ready to prove Theorem 2.1. We consider two cases: (1) G_3 is a cycle and (2) G_3 is not a cycle. In Case (1), the degree of each vertex in G_2 is at least three, by definition of G_2, and each edge of G_3 is a parallel edge in G_2. By Claim 5, the ends of a parallel edge in G_2 belong to distinct parts of G. This violates Claim 8, and completes the proof in case (1).

Suppose Case (2) arises. By Claim 8, no vertex of A has degree two in G_3. If G_3 has no non-trivial ears, then G_3 contains a subdivision of K_4 (see, for example, [6, p. 218]). This means that G contains a subdivision H of K_4. Take two branching vertices v_1, v_2 of H (in other words, vertices of degree three in H) from the same part of G. By Lemma 4.1, H contains a cycle of length zero modulo four. Therefore G_3 has at least one non-trivial ear. By Claim 8, the internal vertices of this ear are not in A, so the ear contains a vertex $b_1 \in B$ of degree two in G_3, and if e^* is a parallel edge in G_2, then b_1 is not incident in G_3 with $e^{**} \in E^{**}$.

Let v_1 and v_2 be the neighbors of b_1 in G_2. Then one of the edges $\{b_1, v_1\}$ and $\{b_1, v_2\}$, for instance $\{b_1, v_1\}$, is a parallel edge in G_2 and hence the edges parallel to $\{b_1, v_1\}$ are 3-complex. In particular, $v_1 \in A$. Let a_0 be the adjacent to b_1 vertex on the ear corresponding to $\{b_1, v_2\}$ in G_1. By Lemma 4.1, there are no two internally disjoint paths connecting v_1 and a_0 in $G_2 - \{b_1\}$. Let b_2 be the closest vertex (in G) to v_1, such that a_0 and v_1 are in different components of $G_1 - \{b_1, b_2\}$. By the choice of b_2, either it is adjacent to v_1, or there are two internally disjoint paths from v_1 to b_2 in $G - \{b_1\}$. In the latter case, we have three internally disjoint paths from b_2 to v_1 in G, and by Lemma 4.1, these paths must all have odd length. In both cases, $b_2 \in B$. Then $\{b_1, b_2\}$ forms a cut set in G so, by Claim 6, a_0 is a common neighbor of b_1 and b_2 and has degree two in G_1. Observe that $d_G(v_1) = k$, otherwise we could delete an ear from v_1 to b_1 reducing both $|G|$ and $\phi_k(A, B)$ by exactly $k + 1$. Let the ears in G_1 from v_1 to b_1 be denoted

$$P_i = (v_1, b_i', a_i', b_1), i = 1, \ldots, r.$$

Since $d_G(v_1) = k$, we have $r \leqslant k - 1$. Let $G' = G'(A', B')$ be obtained from G by deleting the vertices $a_1', \ldots, a_r', a_0$, all their pendant neighbors, and b_1. Then $|G| - |G'| = (r + 1)k$ and

$$\begin{aligned}\phi_k(A, B) - \phi_k(A', B') &= r + 1 + \left\lfloor \frac{k(|B| - 1)}{k - 1} \right\rfloor - \left\lfloor \frac{k(|B| - (k - 2)(r + 1) - 2)}{k - 1} \right\rfloor \\ &\geqslant r + 1 + \left\lfloor \frac{k(1 + (k - 2)(r + 1)}{k - 1} \right\rfloor \\ &= r + 1 + (1 + (k - 2)(r + 1)) + (r + 1) - \left\lceil \frac{r}{k - 1} \right\rceil \\ &\geqslant (r + 1)k.\end{aligned}$$

In the last line, we used $r \leqslant k - 1$ and the inequality $\lfloor a + b \rfloor \geqslant \lfloor a \rfloor + \lfloor b \rfloor$. This contradicts the minimality of G. The proof of Theorem 2.1 is now complete. □

We now turn to Corollary 1.4. The incidence graph version of it is:

Corollary 3.1. *Let G be a bipartite graph on $n \geqslant 4$ vertices, containing no cycle of length zero modulo four. Then $|G| \leqslant \lfloor 3n/2 \rfloor - 3$.*

Proof. Since $n \geqslant 4$, $\lfloor 3n/2 \rfloor - 2 \geqslant n$. Any bipartite graph with four or five vertices containing no even cycle of length zero modulo four has at most $n - 1$ edges, so the corollary is proved for $n = 4, 5$. Now suppose $n > 5$, and let G be a counterexample to the corollary with fewest vertices, and parts A and B such that $|A \cup B| = n > 5$. Clearly, we may assume that the minimum degree of G is at least two, and $|A| \geqslant |B|$. Then by Theorem 2.1 with $k = 2$,

$$|G| \leqslant \phi_2(A, B) - 1 = |A| + 2|B| - 3 \leqslant \lfloor 3n/2 \rfloor - 3,$$

a contradiction. □

This corollary is best possible for all $n \geqslant 4$. Indeed, for even n, let $G_n = G_n(A, B)$ consist of $n/2 - 1$ internally disjoint paths of length 3 between two fixed vertices. Then $|A| = |B|$, $|G_n| = 3(n/2 - 1) = \lfloor 3n/2 \rfloor - 3$, and G_n contains cycles only of length six. For an odd $n \geqslant 5$, G_n is obtained from G_{n+1} by deleting a vertex of degree two.

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Further reading

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