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Packing d-degenerate graphs

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Abstract

We study packings of graphs with given maximal degree. We shall prove that the (hitherto unproved) Bollobás–Eldridge–Catlin Conjecture holds in a considerably stronger form if one of the graphs is d-degenerate for d not too large: if $d, \Delta_1, \Delta_2 \geqslant 1$ and $n > \max\{40\Delta_1 \ln \Delta_2, 40d\Delta_2\}$ then a d-degenerate graph of maximal degree Δ_1 and a graph of order n and maximal degree Δ_2 pack. We use this result to show that, for d fixed and n large enough, one can pack $\frac{n}{1500d^2}$ arbitrary d-degenerate n-vertex graphs of maximal degree at most $\frac{n}{1000d \ln n}$.

Keywords: Graph packing; d-Degenerate graphs; Maximum degree

1. Introduction

Let us recall one of the basic notions of graph theory, that of *packing*. Two graphs of the same order, G_1 and G_2, are said to *pack*, if G_1 is a subgraph of the complement $\overline{G}_2$ of G_2, or, equivalently, G_2 is a subgraph of the complement $\overline{G}_1$ of G_1. The study of packings of graphs was started in the 1970s by Sauer and Spencer [12] and Bollobás and Eldridge [5].

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In particular, Sauer and Spencer [12] proved the following result. Here, and in what follows, we shall write Δ_i for the maximal degree of a graph G_i. Also, our graphs G_i will have order n. Nevertheless, we shall frequently emphasize this convention.

Theorem 1. *Suppose that G_1 and G_2 are graphs of order n such that $2\Delta(G_1)\Delta(G_2) < n$. Then G_1 and G_2 pack.*

The main conjecture in the area is the following Bollobás–Eldridge–Catlin (BEC) Conjecture (see [3–5,8]).

Conjecture 1. *If G_1 and G_2 are graphs with n vertices, maximal degrees Δ_1 and Δ_2, respectively, and $(\Delta_1 + 1)(\Delta_2 + 1) \leqslant n + 1$, then G_1 and G_2 pack.*

If true, the BEC Conjecture is a considerable extension of the Hajnal–Szemerédi Theorem [10] on equitable colorings, which itself is an extension of the Corrádi–Hajnal Theorem on equitable 3-colorings. Indeed, the Hajnal–Szemerédi Theorem is the special case of the BEC Conjecture when G_2 is a disjoint union of cliques of the same size [10]. The conjecture has also been proved when either $\Delta_1 \leqslant 2$ [1,2], or $\Delta_1 = 3$ and n is huge [9].[3]

Although, the conjecture is sharp, as we shall show, when one of the two graphs is sparse, to be precise, d-degenerate for a small d, then much weaker conditions on Δ_1 and Δ_2 imply the existence of a packing. Recall that a graph G is *d-degenerate* if every subgraph of it has a vertex of degree at most d. Our main result is the following.

Theorem 2. *Let $d \geqslant 2$. Let G_1 be a d-degenerate graph of order n and maximal degree Δ_1 and G_2 a graph of order n and maximal degree at most Δ_2. If*

$$40\Delta_1 \ln \Delta_2 < n \tag{1}$$

and

$$40d\Delta_2 < n \tag{2}$$

then there is a packing of G_1 and G_2.

Both restrictions (1) and (2) are weakest up to a constant factor. The examples of Bollobás and Eldridge [3–5] of n-vertex graphs G_1 and G_2 with $(\Delta_1 + 1)(\Delta_2 + 1) = n + 2$, that do not pack show that (2) is best possible up to a constant factor. Examples in [7] show that (1) cannot be significantly weakened either. More precisely, in [7] we proved the following fact.

Theorem 3. *Let k be a positive integer and q a prime power. Then, for every $n \geqslant q\frac{q^{k+1}-1}{q-1}$, there are graphs $G_1(n,k)$ and $G_2(n,q,k)$ of order n that do not pack and have the following properties*:

(a) *$G_1(n,k)$ is a forest with $n - k$ edges and maximal degree at most n/k;*
(b) *$G_2(n,q,k)$ is a $\frac{q^k-1}{q-1}$-degenerate graph of maximal degree at most $2n/q$.*

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[3] One of the referees informed us that the conjecture is also proved in the case when one of the graphs is bipartite and has small maximum degree.

Thus if $q = 3$, $k \geqslant 3$ and $n = \frac{3}{2}(3^{k+1} - 1)$, then the graphs $G_1 = G_1(n,k)$ and $G_2 = G_2(n,3,k)$ of Theorem 3 satisfy $\Delta(G_1)\ln\Delta(G_2) \leqslant \frac{n}{k}\ln n < \frac{n}{k}(1 + (k+1)\ln 3) < 2n$. Note that the graph G_1 is 1-degenerate. The idea of the proof of Theorem 2 is a refinement of that used in [11] for a somewhat similar result on equitable coloring, a partial case of the packing problem.

Note that Theorem 2 yields the following result concerning the BEC Conjecture.

Corollary 4. *Let G_1 be a d-degenerate graph of order n and maximal degree at most Δ_1, and G_2 a graph of order n with maximal degree at most Δ_2 such that $\Delta_1\Delta_2 < n$. If $\frac{\Delta_2}{\ln\Delta_2} \geqslant 40$ (i.e., $\Delta_2 \geqslant 215$) and $\Delta_1 \geqslant 40d$ then there is a packing of G_1 and G_2.*

As an immediate consequence of this corollary, note that the BEC Conjecture holds for two graphs of 'large' maximal degree provided one of them is planar, since every planar graph is 5-degenerate.

Corollary 5. *Let G_1 be a planar graph of order n with maximal degree at most Δ_1 and G_2 be a graph of order n with maximal degree at most Δ_2 such that $\Delta_1\Delta_2 < n$. If $\Delta_1 \geqslant 200$ and $\Delta_2 \geqslant 215$, then there is a packing of G_1 and G_2.*

Adapting the proof of Theorem 2 to control the maximal degree of the union of the two packed graphs, we prove the following result on simultaneous packings of many graphs.

Theorem 6. *Let n, d, Δ and q be positive integers such that $d \geqslant 2$, $q \leqslant \frac{n}{1500d^2}$, and*

$$1000d\Delta < \frac{n}{\ln n}. \tag{3}$$

Let $F_1, \dots, F_q$ be d-degenerate graphs of order n and maximal degree at most Δ. Then $F_1, \dots, F_q$ pack.

For a fixed d, Theorem 6 allows packing linearly many (in n) d-degenerate n-vertex graphs of moderate maximal degree. In fact, the phenomenon we come across here is similar to that observed by Bollobás and Guy [6] for equitable colorings: it is much easier to pack graphs if the number of vertices is significantly greater than the maximal degrees of the graphs to be packed.

The structure of the paper is as follows. In the next section, we prove an auxiliary partition lemma that allows us to apply some ideas of Kostochka, Nakprasit and Pemmaraju [11] to the general packing problem. In Section 3 we prove Theorem 2. In the last section we modify our proof of Theorem 2 in order to get restriction on the maximal degree of the packing of two graphs which almost immediately yields Theorem 6.

2. A partition lemma

Lemma 7. *Let G be a graph with maximal degree at most $\Delta \geqslant 90$ (so that $\Delta \geqslant 20\ln\Delta$) and set $m = \lceil\frac{\Delta}{\ln\Delta}\rceil$. Then for every $V' \subseteq V(G)$, there exists a partition $(V_1, \dots, V_m)$ of V' such that for each vertex v of G, the neighborhood $N(v)$ has the following properties:*

(a) *for each i, $|N(v) \cap V_i| \leqslant 5\ln\Delta$,*
(b) *for each i_1 and i_2, $|N(v) \cap (V_{i_1} \cup V_{i_2})| \leqslant 8.7\ln\Delta$, and*
(c) *for each i_1, i_2, and i_3, $|N(v) \cap (V_{i_1} \cup V_{i_2} \cup V_{i_3})| \leqslant 12.3\ln\Delta$.*

Proof. We color V' with m colors uniformly at random. Let $B(u,c)$ be the event that vertex u has more than $5\ln\Delta$ neighbors in V' colored with c and let $b(u,c)$ be the probability of $B(u,c)$. Then for $k_0 = 1 + \lfloor 5\ln\Delta \rfloor$, we have

$$b(u,c) \leqslant \binom{\Delta}{k_0} m^{-k_0} \leqslant \frac{1}{\sqrt{6k_0}} \left(\frac{e\Delta}{k_0 m}\right)^{k_0}.$$

Since $\ln\Delta > 4.4$ for $\Delta \geqslant 90$, we obtain

$$b(u,c) \leqslant \frac{1}{\sqrt{6k_0}} \left(\frac{e\Delta}{k_0 m}\right)^{k_0} < \frac{1}{\sqrt{132}} \left(\frac{e}{5}\right)^{5\ln\Delta} < \frac{1}{10} \Delta^{5(1+\ln 0.2)}.$$

Similarly, let $B(u,c_1,c_2)$ be the event that vertex u has more than $8.7\ln\Delta$ neighbors in V' colored with c_1 or c_2. Let $b(u,c_1,c_2)$ be the probability of $B(u,c_1,c_2)$ and $k_1 = 1 + \lfloor 8.7\ln\Delta \rfloor$. Then as above

$$\begin{aligned} b(u,c_1,c_2) &\leqslant \binom{\Delta}{k_1}\left(\frac{2}{m}\right)^{k_1} \leqslant \frac{1}{\sqrt{6k_1}}\left(\frac{2e\Delta}{k_1 m}\right)^{k_1} \leqslant \frac{1}{\sqrt{228}}\left(\frac{20e}{87}\right)^{8.7\ln\Delta} \\ &\leqslant \frac{1}{15}\Delta^{8.7(1+\ln(20/87))}. \end{aligned}$$

Similarly, let $B(u,c_1,c_2,c_3)$ be the event that vertex u has more than $12.3\ln\Delta$ neighbors in V' colored with c_1, c_2, or c_3. Let $b(u,c_1,c_2,c_3)$ be the probability of $B(u,c_1,c_2,c_3)$ and $k_2 = 1 + \lfloor 12.3\ln\Delta \rfloor$. Then as above

$$\begin{aligned} b(u,c_1,c_2,c_3) &\leqslant \binom{\Delta}{k_2}\left(\frac{3}{m}\right)^{k_2} \leqslant \frac{1}{\sqrt{6k_2}}\left(\frac{3e\Delta}{k_2 m}\right)^{k_2} \leqslant \frac{1}{\sqrt{320}}\left(\frac{10e}{41}\right)^{12.3\ln\Delta} \\ &\leqslant \frac{1}{17}\Delta^{12.3(1+\ln(10/41))}. \end{aligned}$$

Set

$$B(u) = \bigcup_{c=1}^{m}\left(B(u,c) \cup \bigcup_{c_1=c+1}^{m} B(u,c,c_1) \cup \bigcup_{c_1=c+1}^{m}\bigcup_{c_2=c_1+1}^{m} B(u,c,c_1,c_2)\right)$$

and write $b(u)$ for the probability of $B(u)$. Then, of course,

$$b(u) \leqslant m\frac{1}{9}\Delta^{5(1+\ln 0.20)} + \binom{m}{2}\frac{1}{11}\Delta^{8.7(1+\ln(20/87))} + \binom{m}{3}\frac{1}{13}\Delta^{12.3(1+\ln(10/41))}.$$

Since $5\ln 0.20 < -8$, $8.7\ln\frac{20}{87} < -12.7$, $12.3\ln\frac{10}{41} < -17.3$, and $m \leqslant \frac{2\cdot\Delta}{\ln\Delta}$, we have

$$\begin{aligned} b(u) &\leqslant \frac{\Delta^{6+5\ln 0.20}}{5\ln\Delta} + \frac{4\Delta^{10.7+8.7\ln\frac{20}{87}}}{30\ln^2\Delta} + \frac{4\Delta^{15.3+12.3\ln\frac{10}{41}}}{50\ln^3\Delta} \\ &< \frac{\Delta^{-2}}{5\ln\Delta} + \frac{4\Delta^{-2}}{30\ln^2\Delta} + \frac{4\Delta^{-2}}{50\ln^3\Delta} < \frac{1}{10\Delta^2}. \end{aligned} \tag{4}$$

The event $B(u)$ does not depend on the collection of all events $B(v)$ such that v has no common neighbors with u. Hence the maximal degree of the dependency graph of the events $B(u)$ for $u \in V(G)$ is at most $\Delta(\Delta-1)$. Thus, by the Lovász Local Lemma, it suffices to check that $eb(u)\Delta^2 \leqslant 1$ which holds by (4). This proves the lemma. □

3. Packing two graphs

In this section we prove Theorem 2. First, we introduce some notions.

Let $v_1, v_2, \ldots, v_n$ be an enumeration of the vertices of a graph G. For $1 \leqslant i \leqslant n$, let $G(i)$ be the subgraph of G induced by the vertices $v_i, v_{i+1}, \ldots, v_n$; thus $G(1) = G$ and $G(n)$ consists of the single vertex v_n. We call $v_1, v_2, \ldots, v_n$ a *greedy enumeration* of the vertices or, somewhat loosely, a *greedy order* on G, if $d_{G(i)}(v_i) = \Delta(G(i))$ for every i, $1 \leqslant i \leqslant n$, i.e., the vertex v_i has maximal degree in $G(i)$. Similarly, the enumeration and order are *degenerate* if $d_{G(i)}(v_i) = \delta(G(i))$ for every i, $1 \leqslant i \leqslant n$, i.e., the vertex v_i has minimal degree in $G(i)$. Note that if $v_1, v_2, \ldots, v_n$ is a greedy order on G then $v_i, v_{i+1}, \ldots, v_n$ is a greedy order on $G(i)$, and an analogous assertion holds for the degenerate order. Another simple observation is that $v_1, v_2, \ldots, v_n$ is a greedy order on G if and only if it is a degenerate order for the complement $\overline{G}$. Needless to say, a graph may have numerous greedy orders and degenerate orders.

If $2\Delta_1\Delta_2 < n$, then we are done by the Sauer–Spencer result. Thus we assume that $2\Delta_1\Delta_2 \geqslant n$ which together with (1) yields $\Delta_2/\ln\Delta_2 > 20$. Hence, we can apply Lemma 7 to $G = G_2$ with $V' = V(G_2)$.

Let $m = \lceil \frac{\Delta_2}{\ln\Delta_2} \rceil$. Let $(V_1, \ldots, V_m)$ be a partition of $V(G_2)$ satisfying Lemma 7. We may assume that $|V_i| \geqslant |V_{i+1}|$ for all i. Define $V_i' = V_1 \cup V_2 \cup \cdots \cup V_i$. Then $|V_i'| \geqslant in/m$ for each $i \leqslant m$.

We now choose disjoint subsets of $V(G_1)$ to be sets $W_1, W_2, \ldots, W_m$. For notational convenience, set $A_0 = B_0 = W_0 = \emptyset$. For each $i = 1, 2, \ldots, m$, we construct sets A_i and B_i and set $W_i = A_i \cup B_i$.

We let $A_i' = \bigcup_{j=0}^{i} A_j$, $B_i' = \bigcup_{j=0}^{i} B_j$ and $W_i' = \bigcup_{j=0}^{i} W_j$.

Let $a = \lfloor \frac{7n}{25m} \rfloor$. Arrange the vertices of $G_1 - W_{i-1}'$ in a greedy order and let A_i be the set of the first a vertices in this order. Select B_i from the set of vertices in $G_1 - W_{i-1}' - A_i$ as follows. Initially, set $B_i = \emptyset$ and if there is a vertex $w \in V(G_1) - W_{i-1}' - A_i - B_i$ that has at least $4d$ neighbors in $A_i \cup B_i \cup W_{i-1}'$, add that vertex w to B_i. Repeat this process until every vertex $w \in V(G_1) - W_{i-1}' - A_i - B_i$ has fewer than $4d$ neighbors in $W_{i-1}' \cup A_i \cup B_i$.

We claim that by repeatedly adding these vertices, we have $|B_i'| < \frac{ia}{3}$. Let $e(H)$ denote the number of edges in a graph H. It follows from our construction that, for each $i = 0, 1, \ldots, m$, we have $e(G_1[W_i']) \geqslant 4d|B_i'|$. On the other hand, $G_1[W_i']$ is a d-degenerate graph and has $|A_i'| + |B_i'|$ vertices; consequently, $e(G_1[W_i']) < (|A_i'| + |B_i'|)d$. It follows that $3|B_i'| < |A_i'| = ia$. This completes the construction of A_i and B_i and we simply set $W_i = A_i \cup B_i$. Note that $|W_i'| < \frac{4ia}{3}$.

Now, we start packing. We consider the vertices of G_2 fixed and will place the vertices of G_1 one by one on the vertices of G_2. Furthermore, in the first m steps, every placement is final, but in the final step we allow one replacement while we accommodate a vertex. For convenience, we call the edges of G_1 *red* and those of G_2 *blue*.

Step 1: We pack W_1 in V_1. Order the vertices of W_1 in a reverse degenerate order and place them consecutively in this order.

Each vertex w in W_1 at the moment of embedding has at most d embedded red neighbors in W_1. Each of these red neighbors, w_j, is placed on a vertex v_j of G_2 that has at most $5\ln\Delta_2$ blue neighbors in V_1 by Lemma 7(a). Hence w has less than $5d\ln\Delta_2$ red–blue 'neighbors' in V_1 preventing packing, and at most $|W_1| - 1 < \frac{4a}{3} - 1$ vertices in V_1 already occupied by vertices of W_1. But by (2) and the fact that $\frac{1.05\Delta_2}{\ln\Delta_2} \geqslant m$,

$$5di\ln\Delta_2 + \frac{4a}{3} < \left(\frac{5n \cdot 1.05}{40m} + \frac{4}{3} \cdot \frac{7n}{25m}\right) < \frac{n}{m}(0.2 + 0.38) < \frac{n}{m} \leqslant |V_1|.$$

This shows that there is enough room in V_1 to accommodate w so that no red edge is parallel to a blue edge.

Step i, $2 \leqslant i \leqslant m$*:* After we pack W'_{i-1} in V'_{i-1}, we continue to pack W_i in V'_i. Order the vertices of W_i in a reverse degenerate order and place them consecutively in this order.

Each vertex w in W_i at the moment of embedding has at most d embedded red neighbors in W_i and less than $4d$ red neighbors in W'_{i-1}. Each of these red neighbors, w_j, is placed on a vertex v_j of G_2 that has at most $4.4i \ln \Delta_2$ blue neighbors in V'_i by Lemma 7(b). Hence w has less than $22di \ln \Delta_2$ red–blue neighbors in V'_i preventing packing, and at most $|W'_i| - 1 < \frac{4ia}{3} - 1$ vertices in V'_i already occupied by vertices of W'_i. But by (2) and the fact that $\frac{1.05\Delta_2}{\ln \Delta_2} \geqslant m$,

$$22di \ln \Delta_2 + \frac{4ia}{3} < i\left(\frac{22n \cdot 1.05}{40m} + \frac{4}{3} \cdot \frac{7n}{25m}\right) < \frac{ni}{m}(0.58 + 0.38) < \frac{ni}{m} \leqslant \left|V'_i\right|.$$

Consequently, there is enough room in V'_i to accommodate w so that no red edge is parallel to a blue edge.

Final step: Put the vertices of $G' = G_1 - W'_m$ into a reverse degenerate order, and pack them into G_2 in this order without rearranging the vertices in W'_m. Suppose that it is the turn of vertex $w \in V(G_1)$ to be packed. First of all, there is some vertex $v \in V(G_2)$ not occupied by a vertex of G_1. As above, there are less than $4d$ red neighbors of w in W'_m (by construction), and at most d red neighbors in G' that are already packed. So w has less than $5d$ red neighbors that are packed previously. Each red neighbor of w has at most Δ_2 blue neighbors. Thus w has at most $5d\Delta_2$ red–blue neighbors preventing packing.

Let D_i denote the maximum degree of $G_1 - W'_{i-1}$. By the definition of A_i, it is the maximal number of neighbors in $G_1 - W'_{i-1}$ of a vertex in W_i. Suppose that v has exactly x_i neighbors in V_i, $i = 1, \dots, m$.

Write $\operatorname{br}(v)$ for the number of blue–red 'neighbors' of v in G' that arise because of $V'_i \cap N_{G_2}(v)$. Then $\operatorname{br}(v) \leqslant \sum_{j=1}^{i} x_j D_j$. By Lemma 7, for all $i \neq j \neq k \neq i$, we have

$$0 \leqslant x_i \leqslant 5 \ln \Delta_2, \quad x_i + x_j \leqslant 8.7 \ln \Delta_2, \quad \text{and} \quad x_i + x_j + x_k \leqslant 12.3 \ln \Delta_2. \tag{5}$$

Note that each vertex in A_i has at least D_{i+1} neighbors among the vertices that come later in the order. Hence, as G_1 is a d-degenerate graph, we have

$$dn > |A_1|D_2 + |A_2|D_3 + \cdots + |A_{m-1}|D_m = a(D_2 + \cdots + D_m).$$

It follows that $D_2 + \cdots + D_m < dn/a$. The maximum of the expression $\sum_{j=1}^{i} x_j D_j$ under conditions $D_2 + \cdots + D_m < dn/a$, $D_1 \geqslant \cdots \geqslant D_m$, and (5) is attained when $x_i \geqslant x_{i+1}$ for all i. Hence

$$\begin{aligned} \operatorname{br}(v) &< x_1\Delta_1 + (x_2 - x_3)D_2 + \frac{x_3 dn}{a} \leqslant (x_1 + x_2 - x_3)\Delta_1 + \frac{x_3 dn}{a} \\ &\leqslant (12.3 \ln \Delta_2 - 2x_3)\Delta_1 + \frac{x_3 dn}{a}. \end{aligned} \tag{6}$$

Let us define the set of *bad* vertices as the union of the set of vertices in G_2 where the vertices of W'_m are placed, the set of red–blue 'neighbors' of w, and the set of blue–red 'neighbors' of v. Here by the blue–red 'neighbors' we mean the vertices of G_1 already placed on vertices of G_2.

We have $|W'_m| \leqslant \frac{4ma}{3}$. Also, by (2), $\frac{7n}{25m} \geqslant \frac{7n \ln \Delta_2}{25 \cdot 1.05 \Delta_2} > 11.4 \ln \Delta_2 > 50$ and hence $1.02a > \frac{7n}{25m}$. Therefore, the total number of bad vertices is at most

$$F(x_3) = \frac{4ma}{3} + 5d\Delta_2 + (12.3 \ln \Delta_2 - 2x_3)\Delta_1 + x_3\left(\frac{1.02d25m}{7}\right).$$

We want to prove that $F(x_3) < n$ for every $0 \leqslant x_3 \leqslant 4.1 \ln \Delta_2$. Since $F(x_3)$ is linear, it suffices to check this inequality for $x_3 = 4.1 \ln \Delta_2$ and $x_3 = 0$. By (1) and (2),

$$\begin{aligned} F(4.1 \ln \Delta_2) &\leqslant \frac{4}{3}\frac{7n}{25} + 5\frac{n}{40} + 4.1(\ln \Delta_2)\Delta_1 + 4.1 \ln \Delta_2 \frac{1.02 \cdot 1.05 \cdot 25d\Delta_2}{7 \ln \Delta_2} \\ &\leqslant \frac{28n}{75} + \frac{5n}{40} + \frac{4.1n}{40} + 4.1\frac{3.825n}{40} \\ &< n(0.3734 + 0.125 + 0.1025 + 0.3921) < n. \end{aligned}$$

Similarly,

$$F(0) \leqslant \frac{28n}{75} + \frac{5n}{40} + 12.3(\ln \Delta_2)\Delta_1 \leqslant n(0.374 + 0.125 + 0.3075) < n.$$

It follows that either there is a vertex $w' \in V(G_1)$ placed on a vertex $v' \in V(G_2)$ or a non-occupied vertex $v' \in V(G_2)$ such that (a) $w' \notin W'_m$, (b) w' is not a blue–red 'neighbor' of v, and (c) v' is not a red–blue 'neighbor' of w.

By (b), if we move w' from v' onto v, no parallel red and blue edges occur. By (c), if we place w onto the freed vertex v', then again no parallel edges occur. By (a), we did not move vertices of W'_m. This proves the theorem.

4. Packing many graphs

The idea of packing many d-degenerate graphs with moderate maximal degree is to pack them consecutively, one by one, and to control the maximal degrees of intermediate graphs. To do this, we need the following version of Theorem 2.

Theorem 8. *Let n, d, Δ_1 and Δ_2 be positive integers such that $d \geqslant 2$ and*

$$1000d\Delta_1 < \frac{n}{\ln n}. \tag{7}$$

Let $z = \frac{n}{100d}$ and $\Delta_2 \leqslant z$. Let G_1 be a d-degenerate graph of order n and maximal degree at most Δ_1 and G_2 a graph of order n with at most $\frac{n^2}{1500d}$ edges and maximal degree at most Δ_2. Then there is a packing of G_1 and G_2 such that the maximal degree of the resulting graph $H = G_1 \cup G_2$ is at most $\max\{0.0028\frac{n}{d}, \Delta(G_2) + 10.5d\}$.

Proof. If $\Delta_1 = 1$ then the statement follows from the fact that the complement of G_2 is hamiltonian. Let $\Delta_1 \geqslant 2$. Then by (7), $n \geqslant 2000d \ln n$, which yields $\ln n \geqslant 10$ and therefore

$$z = \frac{n}{100d} \geqslant 200. \tag{8}$$

The proof below will follow the lines of that of Theorem 2 with small changes. In particular, we think of the edges of G_1 as red, and of the edges of G_2 as blue.

Since $e(G_2) \leqslant \frac{n^2}{1500d}$, there exists a subset V_0 of $V(G_2)$ with $|V_0| = \lceil 0.5n \rceil$ such that $\deg_{G_2}(v) \leqslant \frac{n}{375d}$ for every $v \in V_0$.

Let $m = \lceil \frac{z}{\ln z} \rceil$. By (8), Lemma 7 applies to $G = G_2$ with $V' = V_0$ and $\Delta = z$. Let $(V_1, \dots, V_m)$ be a partition of V_0 whose existence is guaranteed by Lemma 7. We may assume that $|V_i| \geqslant |V_{i+1}|$ for all i. Define $V_i' = V_1 \cup V_2 \cup \dots \cup V_i$. Then $|V_i'| \geqslant in/2m$ for each $i \leqslant m$.

We now choose disjoint subsets of $V(G_1)$ to be sets $W_1, W_2, \dots, W_m$. For notational convenience, set $A_0 = B_0 = W_0 = \emptyset$. For each $i = 1, 2, \dots, m$, we construct sets A_i and B_i and set $W_i = A_i \cup B_i$.

We let $A_i' = \bigcup_{j=0}^{i} A_j$, $B_i' = \bigcup_{j=0}^{i} B_j$ and $W_i' = \bigcup_{j=0}^{i} W_j$.

Let $a = \lfloor \frac{n}{6m} \rfloor$. Arrange the vertices of $G_1 - W_{i-1}'$ in a greedy order and let A_i be the set of the first a vertices in this order. Select B_i from the vertices in $G_1 - W_{i-1}' - A_i$ as follows. Initially set $B_i = \emptyset$ and while there is a vertex $w \in G_1 - W_{i-1}' - A_i - B_i$ that has at least $4d$ neighbors in $A_i \cup B_i \cup W_{i-1}'$, add w to B_i. Repeat this process until every vertex $w \in G_1 - W_{i-1}' - A_i - B_i$ has fewer than $4d$ neighbors in $W_{i-1}' \cup A_i \cup B_i$. Then we simply set $W_i = A_i \cup B_i$. As in the proof of Theorem 2, we find that $|W_i'| < \frac{4ia}{3}$.

Step i, $1 \leqslant i \leqslant m$: Having packed W_{i-1}' into V_{i-1}', we continue packing W_i into V_i'. We put the vertices of W_i into a reverse degenerate order and place them one by one in this order.

At the moment of its embedding, each vertex w in W_i has at most d embedded red neighbors in W_i and fewer than $4d$ red neighbors in W_{i-1}'. Each of these red neighbors, w_j, is placed on a vertex v_j of G_2 that has at most $5i \ln z$ blue neighbors in V_i' by Lemma 7(a). Hence w has fewer than $25di \ln z$ red–blue 'neighbors' in V_i' onto which we cannot place w because of arising parallel edges and at most $|W_i'| - 1 < \frac{4ia}{3} - 1$ vertices in V_i' already occupied by vertices of W_i'. Thus if

$$X = 25di \ln z + \frac{4ia}{3} \leqslant |V_i'|, \tag{9}$$

then there are free vertices in V_i' to accommodate w without creating parallel red and blue edges. Since $z \geqslant 200$, we have $m \geqslant 37$ and therefore

$$\frac{1.03z}{\ln z} \geqslant m. \tag{10}$$

Thus, recalling that $z = n/100d$,

$$X < i\left(\frac{25d1.03z}{m} + \frac{4n}{18m}\right) < \frac{i}{m}\left(0.2575n + \frac{2n}{9}\right) = \frac{ni}{2m}\left(0.515 + \frac{4}{9}\right) \leqslant \frac{ni}{2m} \leqslant |V_i'|.$$

This proves (9).

Final step: Consider a reverse degenerate order of the vertices of $G' = G_1 - W_m'$, and pack them in this order into G_2 without rearranging the vertices in W_m'. Suppose that it is the turn of a vertex $w \in V(G_1)$ to be packed. Let $v \in V(G_2)$ be not occupied by a vertex of G_1. As above, there are fewer than $4d$ red neighbors of w in W_m' (by construction), and at most d red neighbors in G' that are already packed. So w has fewer than $5d$ red neighbors that had been packed previously. Each red neighbor of w has at most Δ_2 blue neighbors. Thus w has at most $5dz$ red–blue 'neighbors' that are bad for placing w on them.

Let D_i denote the maximum degree of $G_1 - W_i'$. Suppose that v has exactly x_i blue neighbors in V_i, $i = 1, \dots, m$. Then the number, $\mathrm{br}(v)$, of blue–red 'neighbors' of v with the intermediate vertices in $V_i' \cap N_{G_2}(v)$ is at most $\sum_{j=1}^{i} x_j D_j$. By Lemma 7, this is at most $5 \ln z \sum_{j=1}^{i} D_j$.

As in the proof of Theorem 2, we have $D_2 + \cdots + D_m < dn/a$. Therefore,

$$\mathrm{br}(v) < 5 \ln z\left(\Delta_1 + \frac{dn}{a}\right). \tag{11}$$

Now, let the set of *bad* vertices be the union of the set of vertices in G_2 onto which the vertices of W'_m are placed, the set of red–blue 'neighbors' of w, and the set of blue–red 'neighbors' of v. Here by blue–red 'neighbors' we mean vertices of G_1 already placed on vertices of G_2. We have $|W'_m| \leqslant \frac{4ma}{3}$. Also, by (8) and (10), $\frac{n}{6m} \geqslant \frac{n \ln z}{6\cdot 1.03z} > \frac{100d \ln z}{6.18} > 150$ and hence $1.01a > \frac{n}{6m}$. Therefore, the total number of bad vertices is at most

$$\begin{aligned}
&\frac{4ma}{3} + 5dz + 5\ln z\left(\Delta_1 + \frac{dn}{a}\right)\\
&\quad\leqslant \frac{4}{3}\cdot\frac{n}{6} + \frac{5dn}{100d} + 5\Delta_1 \ln z + 5\ln z\frac{dn6m1.01}{n}\\
&\quad\leqslant n\left(\frac{2}{9} + \frac{1}{20} + \frac{1}{200} + 5\ln z\frac{6.06d1.03z}{n\ln z}\right)\\
&\quad\leqslant n\left(0.28 + 30.30\frac{1.03}{100}\right) < 0.7n.
\end{aligned}$$

Similarly to the proof of Theorem 2, this means that there exists either a vertex $w' \in V(G_1)$ placed on a vertex $v' \in V(G_2)$ or a non-occupied vertex $v' \in V(G_2)$ such that (a) $w' \notin W'_m$, (b) w' is not a blue–red 'neighbor' of v, and (c) v' is not a red–blue 'neighbor' of w. And again, we can safely move w' from v' onto v and place w onto the freed vertex v' without creating parallel red and blue edges. Thus the procedure will result in a graph $H = G_1 \cup G_2$. Let us now estimate the degrees of the vertices in H. Suppose that a vertex $u \in V(H)$ is the result of identifying a vertex $w \in V(G_1)$ with a vertex $v \in V(G_2)$.

Case 1. $w \in W'_m$. Then $v \in V'$ and therefore $\deg_{G_2}(v) \leqslant \frac{n}{375d}$. By (7), $\deg_{G_1}(w) \leqslant \frac{n}{1000d \ln n}$ and by (8), $\ln n \geqslant \ln 20000d \geqslant \ln 40000 > 10$. Thus, $\deg_H(u) \leqslant \frac{n}{d}(\frac{1}{375} + 0.0001) < 0.0028\frac{n}{d}$.

Case 2. $w \notin W'_m$. Then w has fewer than $4d$ neighbors in W'_m, since otherwise it would be included into B'_m. If w has more than $6.5d$ neighbors in $G' = G_1 - W'_m$, then every $w' \in A'_m$ should have more than $6.5d$ neighbors in G'. But in this case,

$$\left|E(G_1)\right| > 6.5d\left|A'_m\right| = 6.5dma \geqslant \frac{6.5}{1.01}dm\frac{n}{6m} > dn,$$

which contradicts the fact that G_1 is d-degenerate. Thus, $\deg_{G_1}(w) \leqslant 4d + 6.5d = 10.5d$ and therefore, $\deg_H(u) \leqslant 10.5d + \Delta_2$. This proves the theorem. □

Now we are ready to prove our second main result, Theorem 6.

Proof of Theorem 6. We shall prove by induction that for every i, $1 \leqslant i \leqslant q$, we can pack $F_1, \ldots, F_i$ so that the resulting graph, H_i, satisfies

$$\Delta(H_i) \leqslant 0.0028\frac{n}{d} + 10.5(i-1)d. \tag{12}$$

Since $H_1 = F_1$, for $i = 1$ inequality (12) follows from (3). Suppose that $i > 1$ and (12) holds for $H_{i-1} = F_1 \cup \cdots \cup F_{i-1}$. Let us check that $G_1 = F_i$ and $G_2 = H_{i-1}$ satisfy the conditions of Theorem 8. Indeed, (7) follows from (3),

$$\Delta_2 \leqslant 0.0028\frac{n}{d} + 10.5(q-2)d \leqslant \frac{n}{d}\left(0.0028 + \frac{10.5}{1500}\right) < \frac{n}{100d},$$

and $|E(G_2)| < (q-1)dn < \frac{n^2}{1500d}$. Thus, by Theorem 8, we can pack G_1 and G_2 so that the maximal degree of the resulting graph, H_i, is at most

$$\max\left\{0.0028\frac{n}{d}, \Delta(G_2) + 10.5d\right\} \leqslant 0.0028\frac{n}{d} + 10.5(i-2)d + 10.5d.$$

This proves the induction step and so completes the proof of the theorem. □

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