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# An Ore-type theorem on equitable coloring

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**Abstract**

A proper vertex coloring of a graph is *equitable* if the sizes of its color classes differ by at most one. In this paper, we prove that if $G$ is a graph such that for each edge $xy \in E(G)$, the sum $d(x) + d(y)$ of the degrees of its ends is at most $2r + 1$, then $G$ has an equitable coloring with $r + 1$ colors. This extends the Hajnal–Szemerédi Theorem on graphs with maximum degree $r$ and a recent conjecture by Kostochka and Yu. We also pose an Ore-type version of the Chen–Lih–Wu Conjecture and prove a very partial case of it.


*Keywords:* Equitable coloring; Ore-type; Packing

## 1. Introduction

An *equitable $k$-coloring* of a graph $G$ is a proper $k$-coloring, for which any two color classes differ in size by at most one. It can be viewed as a packing of $G$ with the $|V(G)|$-vertex graph, whose components are cliques with either $\lfloor |V(G)|/k \rfloor$ or $\lceil |V(G)|/k \rceil$ vertices. Recall that two $n$-vertex graphs *pack* if there exists an edge disjoint placement of these graphs into $K_n$. In other words, $G_1$ and $G_2$ *pack* if $G_1$ is isomorphic to a subgraph of the complement of $G_2$ (and vice versa). A number of important graph theoretic problems can be naturally expressed in the language of packing. For example, the classical Dirac's Theorem [5] on the existence of hamiltonian cycles in each $n$-vertex graph with minimum degree at least $n/2$ can be stated in terms of pack-

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ing as follows: *Let* $n \geqslant 3$. *If* $G$ *is an* $n$*-vertex graph and its maximum degree,* $\Delta(G)$*, is at most* $\frac{1}{2}n - 1$*, then* $G$ *packs with the cycle* $C_n$ *of length* $n$.

Similarly, Ore's theorem [14] on hamiltonian cycles is as follows: *If* $n \geqslant 3$ *and* $G$ *is an* $n$*-vertex graph with* $d(x) + d(y) \leqslant n - 2$ *for each edge* $xy \in E(G)$*, then* $G$ *packs with the cycle* $C_n$.

One of the main known results on equitable coloring is the Hajnal–Szemerédi Theorem [6] stating that every graph with maximum degree $\Delta(G) \leqslant r$ has an equitable $(r+1)$-coloring. It has many applications. Alon and Füredi [1], Alon and Yuster [2,3], Janson and Ruciński [7], Pemmaraju [15], and Rödl and Ruciński [16] used this theorem to derive bounds for sums of dependent random variables with limited dependence or to prove the existence of some special vertex partitions of graphs and hypergraphs. We call the Hajnal–Szemerédi Theorem a *Dirac-type* result, since it provides a packing of a graph $G$ with a special graph given a restriction on the maximum degree of $G$. Recently, Kostochka and Yu [11,12] conjectured that the following Ore-type result holds true: *Every graph in which* $d(x) + d(y) \leqslant 2r$ *for every edge* $xy$ *has an equitable* $(r+1)$*-coloring*. Clearly, this conjecture implies the Hajnal–Szemerédi Theorem. In this paper, we prove the following somewhat stronger result.

**Theorem 1.** *Every graph satisfying* $d(x) + d(y) \leqslant 2r + 1$ *for every edge* $xy$*, has an equitable* $(r+1)$*-coloring.*

The proof elaborates the ideas of the original proof of the Hajnal–Szemerédi Theorem [6] and of the recent short proof of it in [8]. Notice that if the bound on maximum degree is weakened from $2r+1$ to $2r+2$, then it is satisfied by $K_{r+2}$ which does not have *any* $(r+1)$-coloring. More subtly, $K_{r+1,r+1}$ also satisfies the weakened bound, but if $r+1$ is odd, then it does not have an equitable $r$-coloring; so the Hajnal–Szemerédi Theorem is tight. Of course, these graphs also show that Theorem 1 is tight. Replacing $r+1$ in the previous discussion by $r$, Chen, Lih and Wu [4] proposed the following analogue of Brooks' Theorem for equitable coloring:

**Conjecture 2.** *(See [4].) Let* $G$ *be a connected graph with* $\Delta(G) \leqslant r$*. If* $G$ *has no equitable* $r$*-coloring, then either* $G$ *is an odd cycle, or* $G = K_{r+1}$*, or* $r$ *is odd and* $G = K_{r,r}$.

There are more graphs for which Theorem 1 is tight, than those for which the Hajnal–Szemerédi Theorem is tight. For example, for each odd $m < r+1$, the graph $K_{m,2r+2-m}$ satisfies the condition $d(x) + d(y) \leqslant 2r + 2$ for every edge $xy$ and has no equitable $(r+1)$-coloring. We conjecture that the following Ore-type analogue of the Chen–Lih–Wu Conjecture holds (again we replace $r+1$ by $r$).

**Conjecture 3.** *Let* $r \geqslant 3$*. If* $G$ *is a graph satisfying* $d(x) + d(y) \leqslant 2r$ *for every edge* $xy$*, and* $G$ *has no equitable* $r$*-coloring, then* $G$ *contains either* $K_{r+1}$ *or* $K_{m,2r-m}$ *for some odd* $m$.

We also prove that Conjecture 3 holds for $r = 3$.

The structure of the text is as follows. In the next section we prove Theorem 1. The key ingredients of the proof are a recoloring lemma and a discharging proof of the nonexistence of a bad example. In the last section we discuss the Chen–Lih–Wu Conjecture and its extension, Conjecture 3.

Most of our notation is standard; possible exceptions include the following. For a graph $G = (V, E)$, we let $|G| := |V|$, $\|G\| := |E|$ and $\bar{\sigma}(G) := \max\{d(x) + d(y)\colon xy \in E\}$. For a vertex $v$ and set of vertices $X$, $N_X(v) := \{x \in X\colon vx \in E\}$ and $d_X(v) := |N_X(v)|$. As usual, $N(v) =$

$N_V(v)$. If $\mu$ is a function on edges, then $\mu(A,B) := \sum_{xy \in E(A,B)} \mu(x,y)$, where $E(X,Y) := \{xy\colon x \in X \text{ and } y \in Y\}$. If $G$ is a directed graph then $E^-(X) := E(V \setminus X, X\}$. For a set $S$ and an element $x$, we write $S+x$ for $S \cup \{x\}$ and $S-x$ for $S \setminus \{x\}$. For a function $f: V \to Z$, the restriction of $f$ to $W \subseteq V$ is denoted by $f|W$. Functions are viewed formally as sets of ordered pairs.

## 2. Main proof

In this section we prove Theorem 1. For $r=0$ the statement is obvious. Suppose that $r \geqslant 1$ and that the theorem holds for all $0 \leqslant r' < r$. Let $G$ be a graph satisfying $\bar{\sigma}(G) \leqslant 2r+1$. We may assume that $|G|$ is divisible by $r+1$. To see this, suppose that $|G| = s(r+1) - p$, where $p \in [r]$. Let $G'$ be the disjoint union of $G$ and $K_p$. Then $|G'|$ is divisible by $r+1$ and $\bar{\sigma}(G') \leqslant r$. Moreover, the restriction of any equitable $(r+1)$-coloring of $G'$ to $G$ is an equitable $(r+1)$-coloring of $G$. So let $|G| = (r+1)s$.

Suppose for a contradiction, that $G$ is an edge-minimal counterexample to the theorem. Consider any edge $e = xy$ with $d(x) \leqslant d(y)$. By minimality, there exists an equitable $(r+1)$-coloring of $G-e$. Since $G$ is a counterexample, some color class $V$ contains both $x$ and $y$. Since $\bar{\sigma}(G) \leqslant 2r+1$, $d(x) \leqslant r$. Thus there exists a class $W$ such that $x$ has no neighbors in $W$. Moving $x$ to $W$ yields an $(r+1)$-coloring $f$ of $G$ with all classes of size $s$, except for one *small* class $V^-(f) = V - x$ of size $s-1$ and one *large* class $V^+(f) = W + x$ of size $s+1$. We say that such a coloring is *nearly equitable*.

Given a coloring $f$ with a unique small class $V^-$ (but possibly no large class), define an auxiliary digraph $\mathcal{H} = \mathcal{H}(f)$ as follows. The vertices of $\mathcal{H}$ are the color classes of $f$. A directed edge $UW$ belongs to $E(\mathcal{H})$ iff some vertex $y \in U$ has no neighbors in $W$. In this case we say that $y$ is *movable* to $W$. Call $W \in V(\mathcal{H})$ *accessible*, if $V^-$ is reachable from $W$ in $\mathcal{H}$. So $V^-$ is trivially accessible.

**Lemma 4.** *If $G$ has a nearly equitable coloring, whose large class $V^+$ is accessible, then it has an equitable coloring with the same number of colors.*

**Proof.** Let $\mathcal{P} = V_1, \dots, V_k$ be a path in $\mathcal{H}$ from $V_1 := V^+$ to $V_k := V^-$. This means that for each $j = 1, \dots, k-1$, $V_j$ contains a vertex $y_j$ that has no neighbors in $V_{j+1}$. So, if we move $y_j$ to $V_{j+1}$ for $j = 1, \dots, k-1$, then we obtain an equitable coloring with the same number of color classes. $\square$

Let $\mathcal{A} = \mathcal{A}(f)$ denote the family of accessible classes and $\mathcal{B}$ denote the family of non-accessible classes. Then $V^- \in \mathcal{A}$ and, by Lemma 4, $V^+ \in \mathcal{B}$. Set $A := \bigcup \mathcal{A}$, $B := \bigcup \mathcal{B}$, $m := |\mathcal{A}| - 1$ and $q := |\mathcal{B}| = r - m$. Then $|A| = (m+1)s - 1$ and $|B| = qs + 1$.

*Each vertex $y \in B$ has a neighbor in each class of $\mathcal{A}$ and so satisfies $d_A(y) \geqslant m+1$.* (1)

It follows that

$$\bar{\sigma}\big(G[B]\big) \leqslant \bar{\sigma}(G) - 2(m+1) \leqslant 2q - 1.$$

Thus by the minimality of $r$,

*Every subgraph of $G[B]$ has an equitable $q$-coloring.* (2)

For an accessible class $U \in \mathcal{A}(f)$, define $\mathcal{S}_U := \mathcal{S}_U(f)$ to be the set of classes $X \in \mathcal{A}$ such that there is an $X, V^-$-path in $\mathcal{H}(f) - U$ and $\mathcal{T}_U := \mathcal{T}_U(f) := \mathcal{A} \setminus (\mathcal{S}_U + U)$. Call $U$ *terminal*, if

$\mathcal{S}_U = \mathcal{A} - U$; otherwise $U$ is non-terminal. Note that if $U$ is non-terminal then $\mathcal{T}_U \neq \emptyset$. Trivially, $V^-$ is non-terminal unless $m = 0$, in which case it is terminal.

Define a non-empty family $\mathcal{A}' := \mathcal{A}'(f) \subseteq \mathcal{A}(f)$ as follows. If $m = 0$ then set $\mathcal{A}' := \mathcal{A} = \{V^-\}$. Otherwise, $V^-$ is a non-terminal class, and so such classes exist. Choose a non-terminal $U$ so that $|\mathcal{T}_U|$ is minimum and set $\mathcal{A}' := \mathcal{T}_U$. Let $A' := A'(f) := \bigcup \mathcal{A}'$ and $t := t(f) := |\mathcal{A}'|$.

**Lemma 5.** *The family $\mathcal{A}'$ satisfies the following*:

(P1) *Every class in $\mathcal{A}'$ is terminal.*
(P2) $d_A(x) \geqslant m - t$ *for all* $x \in A'$.

**Proof.** If $m = 0$ then the only accessible class $V^-$ is terminal. So $\mathcal{A}' = \mathcal{A}$ satisfies (P1) and (P2) trivially. Otherwise $m > 0$ and $\mathcal{A}' = \mathcal{T}_U$ for some non-terminal $U \in \mathcal{A}$. Consider $X \in \mathcal{T}_U$. Then $\mathcal{T}_X \subset \mathcal{T}_U$. By the minimality of $\mathcal{T}_U$, $X$ is terminal. So (P1) holds true.

No class in $\mathcal{A}' = \mathcal{T}_U$ has an out-neighbor in $\mathcal{S}_U$. It follows that every vertex in $A'$ has a neighbor in each of the $m - t$ classes of $\mathcal{S}_U$. So (P2) holds true. □

An edge $zy$ is *solo* if $z \in W \in \mathcal{A}'$, $y \in B$ and $N_W(y) = \{z\}$. The ends of solo edges are called *solo vertices* and vertices linked by solo edges are called *special neighbors* of each other.

Our interest in terminal classes and solo edges stems from the following lemma.

**Lemma 6.** *Suppose that $W \in \mathcal{A}'$. If $z \in W$ is solo then $z$ has a neighbor in every class of $\mathcal{A} - W$. In particular $d_A(z) \geqslant m$.*

**Proof.** Suppose for a contradiction that $z$ has a special neighbor $y \in B$ and no neighbor in $X \in \mathcal{A} - W$. Since $W$ is terminal, there exists a path $\mathcal{P}$ from $X$ to $V^-$ in $\mathcal{H} - W$. Move $z$ to $X$ and $y$ to $W$. By hypothesis $X^* := X + z$ is independent and, since $xy$ is solo, $W^* := W + y - z$ is independent. This yields a nearly equitable coloring $f^*$ of $G[A + y]$ with $V^+(f^*) = X + z$. Moreover $\mathcal{P}^* := \mathcal{P} + V^+(f^*) - X$ is a path from $V^+(f^*)$ to $V^-(f^*)$ in $\mathcal{H}(f^*)$. By Lemma 4, $G[A + y]$ has an equitable $(m + 1)$-coloring $h_1$. By (2), $G[B] - y$ has an equitable $q$-coloring $h_2$. Thus $h_1 \cup h_2$ is an equitable $(r + 1)$-coloring of $G$, a contradiction. □

We now come to a delicate point in the argument. Define an *obstruction* to be a nearly equitable $(r + 1)$-coloring $f$ such that

(C1) $m(f) = |\mathcal{A}(f)| - 1$ is maximum; and
(C2) subject to (C1), $t(f) = |\mathcal{A}'(f)|$ is minimum.

Given an obstruction $f$, the next lemma allows us to switch the colors of the ends of a solo edge to obtain a new obstruction. When this operation is applied in the proof of Claim 10, we consider two cases depending on whether $t \geqslant q$. It is important that after switching, we are still in the same case. This is guaranteed by conditions (C1) and (C2).

**Lemma 7.** *Suppose that $f$ is an obstruction, $W \in \mathcal{A}'$ and $z \in W$ is a solo vertex with a special neighbor $y \in B$. Set $A^- := A - z$. Then $G$ has an obstruction $g$ such that*

$$g|A^- = f|A^- \quad \textit{and} \quad g(y) = f(z). \tag{3}$$

**Proof.** Set $W^* := W + y - z$. Since $zy$ is a solo edge, $W^*$ is independent. Thus switching $y$ and $z$ we obtain an equitable $(m+1)$-coloring $h_1$ of $G[A^*]$ from $f$, where $A^* := A + y - z$. Our plan is to extend $h_1$ to an obstruction. Any such extension will satisfy (3). For this we will need the following analysis of $\mathcal{H}(h_1)$.

Set $\mathcal{H}_0 := \mathcal{H}(f)[\mathcal{A}(f)]$. For $X \in \mathcal{A}$, let $X^* := X$ if $X \neq W$; otherwise let $X^* := W^*$. Then $\mathcal{H}_0 - W = \mathcal{H}(h_1) - W^*$. Moreover, by (1) and Lemma 6, neither $y$ nor $z$ is movable to any class in $\mathcal{H}_0 - W$. It follows that the out-neighborhood of $W$ in $\mathcal{H}_0$ is the same as the out-neighborhood of $W^*$ in $\mathcal{H}(h_1)$. In other words,

$$*: \mathcal{H}_0 - E^-(W) \to \mathcal{H}(h_1) - E^-\bigl(W^*\bigr)$$

is an isomorphism. Let $\mathcal{P} := X_1 \dots X_t$ and $\mathcal{P}^* := X_1^* \dots X_t^*$ be the image of $\mathcal{P}$. Then $\mathcal{P}$ is a path in $\mathcal{H}_0$ with $W \notin V(\mathcal{P}) - X_1$ iff $\mathcal{P}^*$ is a path in $\mathcal{H}_1$ with $W^* \notin V(\mathcal{P}^*) - X_1^*$. Since $W$ is terminal by (P1), it follows that every class of $h_1$ is accessible in $\mathcal{H}(h_1)$, i.e. $\mathcal{A}^*(f) = \mathcal{A}(h_1)$, where $\mathcal{A}^*(f)$ is the image of $\mathcal{A}(f)$.

Set $B^- := B - y$. By (2), $G[B^-]$ has an equitable $q$-coloring $h_2$. Using (1), Lemma 6, and the fact that $W$ is terminal, we have

$$2r + 1 \geqslant d(z) + d(y) = d_A(z) + d_A(y) + d_B(z) + d_B(y) \geqslant 2m + 1 + d_B(z) + d_B(y).$$

In other words,

$$2q \geqslant d_B(z) + d_B(y).$$

Since $z$ is adjacent to $y$, $d_{A^*}(z) \geqslant d_A(z) + 1 \geqslant m + 1$ and $d_{B^-}(z) \leqslant 2q - 1$. If there exists a class $X \subseteq B^-$ of $h_2$ such that $z$ has no neighbors in $X$, then move $z$ to $X$ to obtain a $q$-coloring $h_3$ of $G[B^*]$, where $B^* := B^- + z$. Then $g := h_1 \cup h_3$ is a nearly equitable $(r+1)$-coloring of $G$. Otherwise $d_{B^-}(z) \geqslant q$ and $d(z) \geqslant q + m + 1 = r + 1$. Since $d_{B^-}(z) \leqslant 2q - 1$, some class $X$ of $h_2$ contains exactly one neighbor $w$ of $z$. Switch $z$ and $w$ to obtain a $q$-coloring $h_4$ of $G[B^*] - w$. Then $f' = h_1 \cup h_4$ is an equitable coloring of $G - w$ with one small class $V^-$ and no large class. Since $d(z) \geqslant r + 1$ and $z$ is adjacent to $w$, $d(w) \leqslant r$. It follows that $w$ can be added to some class of $f'$, yielding a large class.

First suppose that $w$ can be added to $X^* \subseteq A^*$. This yields a nearly equitable coloring $h'$ of $A^* + w$ with large class $X^* + w$. Since $X^* \in \mathcal{A}(h_1)$ we have $X^* + w \in \mathcal{A}(h')$. By Lemma 4, there exist an equitable $(m+1)$-coloring $h''$ of $G[A^* + w]$. Then $h'' \cup h_4$ is an equitable $(r+1)$-coloring of $G$, a contradiction. So $w$ can be moved to some $X \subseteq B^*$. Let $g$ be the nearly equitable $(r+1)$-coloring obtained from $h_1 \cup h_4$ by moving $w$ to $X$. Regardless of the case, we have constructed a nearly equitable $(r+1)$-coloring $g$ that satisfies (3). We still must show that $g$ satisfies (C1) and (C2).

First, we show that $g$ satisfies (C1). Since $f$ satisfies (C1) it suffices to show that $m(f) \leqslant m(g)$, which follows from $\mathcal{A}^*(f) = \mathcal{A}(h_1) \subseteq \mathcal{A}(g)$. So $g$ satisfies (C1) and $\mathcal{A}(h_1) = \mathcal{A}(g)$. Now we show that $g$ satisfies (C2). Suppose that $\mathcal{A}'(f) = \mathcal{T}_U$, where $U$ is non-terminal in $\mathcal{H}(f)$. Since $f$ satisfies (C2), it suffices to show that $t(g) \leqslant t(f)$. We will do this by showing that $W^* \in \mathcal{T}_U(g)$ and $\mathcal{S}_U(f) \subseteq \mathcal{S}_U(g)$. Then $U$ is non-terminal in $\mathcal{H}(g)$ and $t(g) \leqslant |\mathcal{T}_U(g)| \leqslant |\mathcal{T}_U(f)| = t(f)$. Suppose that $\mathcal{P}^*$ is a $W^*, V^-$-path in $\mathcal{H}_g$. Then $V(\mathcal{P}^*) \subseteq \mathcal{A}(g) = \mathcal{A}(h_1)$. So its inverse $\mathcal{P}$ under $^*$ is a $W, V^-$-path in $\mathcal{H}(f)$. Since $W \in \mathcal{T}_U$, $U$ must be a vertex of $\mathcal{P}$ and thus $\mathcal{P}^*$. So $W^* \in \mathcal{T}_U(g)$. Now suppose that $X \in \mathcal{S}_U(f)$. Then there exists an $X, V^-$-path $\mathcal{P}$ in $\mathcal{H}(f) - U$. It follows that $\mathcal{P}^*$ is a path in $\mathcal{H}(h_1) - U \subseteq \mathcal{H}(g) - U$ and so $X \in \mathcal{S}_U(g)$. So (C2) holds and $g$ is an obstruction. $\square$

Suppose that $f$ is an obstruction and $z \in A'$ is a solo vertex with a special neighbor $y \in B$. Let $S^y$ be the set of special neighbors of $y$ in $A'$. By (1), $y$ has a neighbor in every class of $\mathcal{A}$; moreover if $W \in \mathcal{A}'$ and $y$ does not have a solo neighbor in $W$, then $y$ has at least two neighbors in $W$. Thus

$$d_{A'}(y) \geqslant 2t - |S^y| \quad \text{and} \quad d_A(y) \geqslant m + 1 + t - |S^y|. \tag{4}$$

Set $c_y := \max\{d_B(z)\colon z \in S^y\}$ if $S^y \neq \emptyset$; otherwise $c_y := 1$. Similarly, set $c'_y := \max\{d_B(z)\colon z \in N_{A'}(y) \setminus S^y\}$ if $N_{A'}(y) \neq S^y$; otherwise $c'_y := 1$. Define a weight function $\mu$ on $E(A', B)$ by

$$\mu(xy) := \frac{q}{d_B(x)}.$$

We shall finish our proof by proving the following three contradictory claims.

**Claim 8.** *For all obstructions $f$, there exists a vertex $y \in B$ such that $\mu(A', y) < t$.*

**Claim 9.** *For all obstructions $f$ and all vertices $y \in B$, if $\mu(A', y) < t$ then $y$ is solo. Moreover, in this case, either $c_y \geqslant q + 1$ or $c'_y \geqslant 2q + 1$.*

**Claim 10.** *There exists an obstruction $f$ such that $\mu(A', y) \geqslant t$ for all solo vertices $y \in B$.*

**Proof of Claim 8.** For any $x \in A$, if $N_B(x) \neq \emptyset$ then

$$\mu(x, B) = \sum_{y \in N_B(x)} \frac{q}{d_B(x)} = q;$$

otherwise $\mu(x, B) = 0$. Regardless,

$$\mu(x, B) \leqslant q.$$

Thus

$$\begin{aligned} qst = q|A'| &\geqslant \sum_{x \in A'} \mu(x, B) = \mu(A', B) = \sum_{y \in B} \mu(A', y) \\ &\geqslant |B| \min_{y \in B} \mu(A', y) > qs \min_{y \in B} \mu(A', y) \end{aligned}$$

and so $\mu(A', y) < t$ for some $y \in B$. $\square$

**Proof of Claim 9.** Let $\mu(A', y) < t$. Let $\mathcal{S} := \{W \in \mathcal{A}'\colon S^y \cap W \neq \emptyset\}$ and $\mathcal{D} := \mathcal{A}' \setminus \mathcal{S}$. First suppose that $c'_y \leqslant 2q$. Then

$$\begin{aligned} t > \mu(A', y) &= \sum_{W \in \mathcal{S}} \sum_{x \in N_W(y)} \frac{q}{d_B(x)} + \sum_{W \in \mathcal{D}} \sum_{x \in N_W(y)} \frac{q}{d_B(x)} \\ &\geqslant |\mathcal{S}| \frac{q}{c_y} + 2|\mathcal{D}| \frac{q}{c'_y} \geqslant |\mathcal{S}| \frac{q}{c_y} + |\mathcal{D}|. \end{aligned}$$

Thus $|\mathcal{D}| < t$ and so $|S^y| = |\mathcal{S}| > 0$. Thus $y$ is solo. Moreover, $\frac{q}{c_y} < 1$ and so $c_y \geqslant q + 1$.

Now suppose that $d_B(x) \geqslant 2q+1$ for some $x \in N_{A'}(y)$. Using (P2) and (4),

$$\begin{aligned}2r+1 &\geqslant d(x)+d(y) = d_A(x)+d_B(x)+d_A(y)+d_B(y)\\ &\geqslant (m-t)+(2q+1)+\big(m+1+t-|S^y|\big)\\ &= 2(m+q+1)-\left|S^y\right|\\ &= 2r+2-\left|S^y\right|\end{aligned}$$

It follows that $|S^y| \geqslant 1$ and so $y$ is again solo. $\square$

**Proof of Claim 10.**

CASE 1: $t \geqslant q$. Choose an obstruction $f$ such that $|E(A', B)|$ is minimum. Let $y \in B$ be solo and choose $z \in S^y$ so that $d_B(z) = c_y$. Let $g$ be an obstruction satisfying the conclusion of Lemma 7. Set $A^- := A' - z$ and $B^- := B - y$. By the choice of $f$, $|E(A'(f), B(f))| \leqslant |E(A'(g), B(g))|$ and so $d_{A^-}(y) + d_{B^-}(z) \leqslant d_{A^-}(z) + d_{B^-}(y)$. Recalling that $y$ is adjacent to $z$,

$$d_{A'}(y) + d_B(z) \leqslant \left\lfloor \frac{(d_{A^-}(y) + d_{B^-}(y) + d_{A^-}(z) + d_{B^-}(z))}{2} \right\rfloor + 2.$$

Note that $d_{B^-}(y) = d_B(y)$ and (1) implies

$$d_{A^-}(y) = d_{A'} - 1 \leqslant d_A(y) - |\mathcal{A} \setminus \mathcal{A}'| - 1 = d_A(y) - (m+1-t) - 1.$$

Arguing analogously (but using Lemma 6 this time) for $z$, we obtain $d_{B^-}(z) = d_B(z) - 1$ and

$$d_{A^-}(z) = d_{A'}(z) \leqslant d_A(z) - |\mathcal{A} \setminus \mathcal{A}'| = d_A(y) - (m+1-t).$$

Combining terms and using $d(y) + d(z) \leqslant 2r+1$, we have

$$\begin{aligned}d_{A'}(y) + d_B(z) &\leqslant \left\lfloor \frac{d(y) - (m+2-t) + d(z) - 1 - (m+1-t)}{2} \right\rfloor + 2\\ &\leqslant \left\lfloor \frac{2r+1-2m+2t}{2} \right\rfloor\\ &= t+q.\end{aligned} \tag{5}$$

By (1), $d_{A'}(y) \geqslant t$ and hence $d_B(z) \leqslant q$. By the choice of $z$, $c_y = d_B(z) \leqslant q$. By (4) and (5),

$$\left|S^y\right| \geqslant 2t - d_{A'}(y) \geqslant 2t - \big(t+q-d_B(z)\big) = t-q+c_y.$$

So

$$\mu(A', y) \geqslant \sum_{z \in S^y} \frac{q}{d_B(z)} \geqslant \left|S^y\right| \frac{q}{c_y} \geqslant (t-q+c_y)\frac{q}{c_y} = (t-q)\frac{q}{c_y} + q \geqslant t.$$

CASE 2: $q \geqslant t$. Choose an obstruction $f$ such that $\|G[B]\|$ is as large as possible. Then $d_B(z) \leqslant d_B(y) + 1$ for all solo edges $zy$ with $z \in A'$. Thus, using Lemma 6 and (1),

$$\begin{aligned}2r+1 &\geqslant d_A(z) + d_B(z) + d_A(y) + d_B(y) \geqslant 2m+1+d_B(z)+d_B(y),\\ 2q &\geqslant d_B(z) + d_B(y) \geqslant 2d_B(z) - 1,\\ q &\geqslant d_B(z).\end{aligned}$$

Since $z$ was arbitrary, $c_y \leqslant q$. If $\mu(A', y) < t$, then, by Claim 9, $y$ has a neighbor $x \in A'$ such that $d_B(x) \geqslant 2q+1$. Moreover $d_B(y) \geqslant c_y - 1$ by the maximality of $\|G[B]\|$. So, using (P2), (1) and (4),

$$2r+1 \geqslant d(x)+d(y) \geqslant (m-t+2q+1)+\bigl(m+1+t-\bigl|S^y\bigr|+c_y-1\bigr)$$
$$=2r+1-\bigl|S^y\bigr|+c_y.$$

Thus $|S^y| \geqslant c_y$. So $\mu(A', y) \geqslant |S^y|\frac{q}{c_y} \geqslant q \geqslant t$. □

Since Claims 8–10 are contradictory, this completes the proof of the theorem.

## 3. On two conjectures

The Chen–Lih–Wu conjecture has been proved for some classes of graphs such as bipartite graphs [13], outerplanar graphs [17], planar graphs with maximum degree at least 13 [18], graphs with average degree five times less than their maximum degree [10] and others. In particular, Chen, Lih and Wu [4] proved that their conjecture holds for $r = 3$ and we [9] have extended their result to $r = 4$. We will use their theorem:

**Theorem 11.** *If $G$ is a connected graph with $\Delta(G) \leqslant 3$ distinct from $K_4$ and $K_{3,3}$, then $G$ has an equitable 3-coloring.*

If we consider the Ore-type setting, then for every odd $m \leqslant r$, the graph $G_{r,m} = K_{m,2r-m}$ has $\bar{\sigma}(G_{r,m}) = 2r$ and has no equitable $r$-coloring. However, we believe that Conjecture 3 stated in the introduction holds true. To support the conjecture, we prove that it is true for $r = 3$. Note that the word 'connected' is not present in the statement, but this is an equivalent form: If $G$ satisfies $\bar{\sigma}(G_{r,m}) \leqslant 2r$ and contains $H \in \{K_4, K_{3,3}, K_{5,1}\}$, then $H$ is a component of $G$.

**Theorem 12.** *If $G$ is a graph with $d(x)+d(y) \leqslant 6$ for each $xy \in E(G)$ and if $G$ does not contain any of the graphs $K_4$, $K_{3,3}$ and $K_{5,1}$, then $G$ has an equitable 3-coloring.*

**Proof.** Let $G$ be an edge-minimal counterexample to the theorem. Let $v$ be a vertex of the maximum degree in $G$. If $d(v) = 5$, then $G$ contains $K_{5,1}$, a contradiction to our assumption. By Theorem 11, $d(v) > 3$. Hence $d(v) = 4$. Let $w_1, w_2, w_3, w_4$ be the neighbors of $v$. Under the constraints on the graph, $d(w_i) \leqslant 2$ for each $i = 1, 2, 3, 4$. For $i = 1, 2, 3, 4$, let $u_i$ be the neighbor of $w_i$ distinct from $v$, if it exists.

CASE 1: $u_1$ does not exist or $u_1 = w_2$. Consider $G' = G - v - w_1 - w_2$. Since $G'$ is a proper subgraph of $G$, it satisfies the conditions of the theorem. By the minimality of $G$, there exists an equitable 3-coloring $f$ of $G'$. We extend $f$ to an equitable 3-coloring of $G$ as follows: Choose a color $\alpha \in \{1, 2, 3\} - f(w_3) - f(w_4)$ as $f(v)$, then choose a color $\beta \in \{1, 2, 3\} - f(u_2) - \alpha$ as $f(w_2)$, and finally choose the color $\gamma \in \{1, 2, 3\} - \alpha - \beta$ as $f(w_1)$.

So, below all $u_i$ exist and are distinct from all $w_j$.

CASE 2: $u_3 = u_4$. Consider $G'' = G - \{v, w_1, w_2, w_3, w_4, u_3\}$. By the minimality of $G$, there exists an equitable 3-coloring $f$ of $G''$. We extend $f$ to the whole $G$ as follows. First assign to $u_3$ and $v$ a color $\alpha$ distinct from the colors of neighbors of $u_3$ in $G''$ (there are at most two such neighbors). Then for $i = 1, 2$, let $f(w_i) \in \{1, 2, 3\} - f(u_i) - \alpha$. Finally, for $i = 3, 4$, let $f(w_i) \in \{1, 2, 3\} - f(w_{i-2}) - \alpha$. Since each color appears exactly twice on $\{v, w_1, w_2, w_3, w_4, u_3\}$, we have an equitable 3-coloring of $G$.

Thus below all $u_i$ are distinct and the only remaining case is as follows.

CASE 3: All $u_i$ exist and are distinct; furthermore the set $\{w_1, w_2, w_3, w_4\}$ is independent. Let $G'''$ be the graph obtained from $G - v$ by merging $w_1$ with $w_3$ into a new vertex $w_1^*$ and merging $w_2$ with $w_4$ into a new vertex $w_2^*$. Since the two new vertices have degree exactly 2 each, $G'''$ does not contain any of $K_4$, $K_{3,3}$ and $K_{5,1}$. Hence there exists an equitable 3-coloring $f$ of $G'''$. We may assume that $f(w_1^*) = 1$. If $f(w_2^*) \neq 1$, then we may assume that $f(w_2^*) = 2$ and let $f(w_1) = f(w_3) = 1$, $f(w_2) = f(w_4) = 2$, and $f(v) = 3$.

Suppose that $f(w_1^*) = f(w_2^*) = 1$. We may assume that $f(u_4) = 2$. Then we let $f(w_1) = f(w_2) = f(w_3) = 1$, $f(w_4) = 3$, and $f(v) = 2$.

Thus in all cases we find an equitable 3-coloring of $G$, a contradiction. □

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