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Note

Packing of graphs with small product of sizes

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Abstract

We show that for every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that for every $n > n_0$, two n-vertex graphs G_1 and G_2 with $e(G_1)e(G_2) \leqslant (1-\epsilon)n^2$ pack, unless they belong to a well-defined family of exceptions. This extends a well-known result by Sauer and Spencer.

Keywords: Graph packing

1. Introduction

How many edges should an n-vertex graph have to contain every graph with at most n vertices and at most m edges? Erdős and Stone [7] proved that for every positive integer d and positive c and sufficiently large n, every graph G of order n with at least $(n^2/2)(1-1/d)+cn^2$ edges contains a complete $(d+1)$-partite graph with t vertices in each part, where t tends to infinity with n. It follows that this G contains every d-colorable graph on t vertices, and, in particular, that G contains every graph with less than $\binom{d+1}{2}$ edges. Later Bollobás, Erdős, and Simonovits [2] showed that $t \geqslant a\log n/(d\log(1/c))$ for some positive constant a and conjectured that this can be improved as follows: $t \geqslant b\log n/\log(1/c)$. Chvátal and Szemerédi [6] verified this conjecture by proving the following theorem.

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Theorem 1. *(See Chvátal and Szemerédi [6].) For each positive integer d and each $c > 0$, there is an $n_0 = n_0(d, c)$ such that for each $n \geqslant n_0$, every graph of order n with at least $(n^2/2)(1 - 1/d) + cn^2$ edges contains a complete $(d + 1)$-partite graph with at least $(\log n)/(500 \log(1/c))$ vertices in each part.*

Bollobás and Eldridge [1] considered also Turán-type conditions guaranteeing that an n-vertex graph contains every subgraph with αn edges for $\alpha < 1/2$. They proved a bound in this direction in the language of packing and posed a conjecture which was proved by Brandt [5]. Recall that two graphs *pack* if one of the graphs is contained in the complement of the other. The next theorem is a somewhat simplified version of Brandt's result.

Theorem 2. *(See Brandt [5].) For every $0 < \alpha < 1/2$, there exists $n_0 = n_0(\alpha)$ such that if $n > n_0$, $e(G_1) \leqslant \alpha n$, and $e(G_2) \leqslant \frac{1}{3\sqrt{\alpha}} n^{3/2}$, then G_1 and G_2 pack.*

Bollobás, Kostochka, and Nakprasit [3] extended Theorem 2 to the case $\alpha \geqslant \frac{1}{2}$. A simplified version of it is as follows.

Theorem 3. *(See Bollobás, Kostochka and Nakprasit [3].) Let $1/2 \leqslant \alpha < 1$. Let G_1 and G_2 be graphs of order $n > (\frac{40}{1-\alpha})^6$ such that $e(G_1) \leqslant \alpha n$, $e(G_2) \leqslant \frac{1}{3} n^{3/2}$, and $\Delta(G_2) < n - 1 - \frac{\sqrt{n}}{\sqrt{2\alpha}(1-\alpha)}$. Then G_1 and G_2 pack.*

Sauer and Spencer [8] proved the following bound in terms of the product of the sizes of graphs.

Theorem 4. *(See Sauer and Spencer [8].) Two n-vertex graphs G_1 and G_2 pack, if*

$$e(G_1)e(G_2) < \binom{n}{2}.$$

The following examples of graphs that do not pack show that the condition $e(G_1)e(G_2) < \binom{n}{2}$ cannot be weakened without introducing other restrictions.

Example 1. $G_1 = K_n$ and $G_2 = K_2 \cup \overline{K}_{n-2}$.

Example 2. $G_1 = K_{1,n-1}$ and G_2 has no isolated vertices.

Note that in Example 2, if n is even and G_2 is a perfect matching, then $e(G_1)e(G_2) = \binom{n}{2}$. Also note that $e(G_1) + e(G_2)$ can be around $3n/2$. Bollobás and Eldridge [1] proved that this may happen only if one of the graphs has an all-adjacent vertex or n is small. In a bit simplified form, their result is as follows.

Theorem 5. *(See Bollobás and Eldridge [1].) Let G_1 and G_2 be graphs of order $n > 10$ such that $\Delta(G_1), \Delta(G_2) \leqslant n - 2$ and $e(G_1) + e(G_2) \leqslant 2n - 3$. Then G_1 and G_2 pack.*

This bound is also sharp as the following examples show.

Example 3. $G_1 = G_2 = K_3 \cup K_{1,n-4}$.

Example 4. $G_1 = K_{1,n-2} \cup K_1$ and G_2 is 2-regular.

Example 5. $G_1 = K_{1,n-3} \cup K_2$, n is divisible by 3, and $G_2 = K_3 \cup \cdots \cup K_3$.

Teo and Yap [9] showed that for $n \geqslant 13$, Examples 3–5 are the only pairs (G_1, G_2) of n-vertex graphs with $e(G_1) + e(G_2) = 2n - 2$ that do not pack.

In this paper we strengthen Theorem 4 by describing (for large n) the pairs (G_1, G_2) of n-vertex graphs with $e(G_1)e(G_2) \leqslant (1-\epsilon)n^2$ that do not pack.

Theorem 6. *For every $\epsilon > 0$, there exists N, such that for all $n > N$, if two n-vertex graphs G_1 and G_2 with*

$$e(G_1)e(G_2) \leqslant (1-\epsilon)n^2 \tag{1}$$

do not pack, then one of the following holds:

(i) *one of the graphs is K_n and the other has exactly one edge*; *or*
(ii) *one of the graphs has maximum degree $n-1$ and the other has minimum degree at least one*; *or*
(iii) *one of the graphs is a triangle, and the other has independence number two.*

Observe that there are exponentially many pairs (G_1, G_2) of n-vertex graphs satisfying (ii) or (iii) with $e(G_1)e(G_2) \leqslant 0.9n^2$. Although n-vertex graphs with independence number two and fewer than $(1-\epsilon)n^2/3$ edges may have a complicated structure, we can in polynomial time check any graph whether it possesses this property. We believe that it will be sufficiently harder to describe the pairs (G_1, G_2) of n-vertex graphs with $e(G_1)e(G_2) \leqslant (1+\epsilon)n^2$ that do not pack even for small positive ϵ. Note that Examples 3–5 fall into this category. Yet another example is as follows.

Example 6. $G_1 = K_4$, n is divisible by 3, and $G_2 = K_{n/3} \cup K_{n/3} \cup K_{n/3}$.

In the proof of Theorem 6 we will make use of the following fact.

Theorem 7. *(See Bollobás, Kostochka and Nakprasit [4].) Let $d \geqslant 2$. Let G_1 be a d-degenerate graph of order n and maximum degree Δ_1 and G_2 a graph of order n and maximum degree at most Δ_2. If $40\Delta_1 \ln \Delta_2 < n$ and $40d\Delta_2 < n$, then G_1 and G_2 pack.*

Recall that a graph is *d-degenerate* if every subgraph of it contains a vertex of degree at most d.

2. Proof of Theorem 6

Fix an $0 < \epsilon < 0.1$. Let n be large. Suppose that Theorem 6 does not hold for ϵ and n, i.e. that there are n-vertex graphs G_1 and G_2 satisfying (1) that do not pack and do not belong to the families described by (i)–(iii). We may assume that $e(G_1) \leqslant e(G_2)$. So, by (1), $e(G_1) < \sqrt{1-\epsilon}n < (1-\epsilon/2)n$. Let $\alpha = e(G_1)/n$. By above,

$$0 < \alpha < 1 - \epsilon/2. \tag{2}$$

Let Δ_i, $i = 1, 2$, denote the maximum degree of G_i. By Theorems 2 and 3, if $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$ and $\Delta_2 < n - 1 - \frac{\sqrt{n}}{1-\alpha}$, then G_1 and G_2 pack. So we consider the following two cases: (1) $\Delta_2 \geqslant n - \frac{\sqrt{n}}{1-\alpha}$ and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$, and (2) $e(G_2) \geqslant \frac{1}{3}n^{\frac{3}{2}}$.

Case 1. $\Delta_2 \geqslant n - \frac{\sqrt{n}}{1-\alpha}$ and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$. Let $w \in V(G_2)$ be a vertex of maximum degree Δ_2 in G_2.

If $e(G_1) < (1 - \epsilon/2)n/2$, then since n is large and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$, G_1 and G_2 pack by Theorem 2. So we assume that $e(G_1) \geqslant (1 - \epsilon/2)n/2$. Note that then by (1), $e(G_2) \leqslant (2 - \epsilon)n$.

If G_1 has an isolated vertex, say w', then let $G_1' = G_1 - w'$, and $G_2' = G_2 - w$. We have $e(G_1') = e(G_1)$ and

$$e(G_2') = e(G_2) - \Delta(G_2) \leqslant (1 - \epsilon)n + \frac{\sqrt{n}}{1-\alpha} < (1 - \epsilon/2)n \tag{3}$$

for $n > (\frac{2}{\epsilon(1-\alpha)})^2$. By (2) and (3), for such n and $i \in \{1, 2\}$, we have

$$\Delta(G_i') \leqslant e(G_i') < (1 - \epsilon/2)n \leqslant (n - 1) - 2.$$

By Theorem 5, G_1' and G_2' pack. Thus G_1 and G_2 pack as well (by placing w' at w).

Assume now that G_1 has no isolated vertices. Since (ii) does not hold, G_2 has no vertex of degree $n - 1$. Since every connected graph H containing a cycle has $|E(H)| \geqslant |V(H)|$ and $e(G_1) < (1 - \epsilon/2)n$, G_1 has at least $\frac{\epsilon n}{2}$ tree components. So there is a tree component T of G_1 with at most $\frac{n}{\epsilon n/2} = \frac{2}{\epsilon}$ vertices. We will first place on the vertices of G_2 the vertices of T, and then find a placement of the remaining vertices.

Let $t = |V(T)|$ and let the vertices of T be ordered $u_1, u_2, \ldots, u_t$ in such a way that u_1 is a leaf and for every $i = 2, \ldots, t$, vertex u_i has exactly one neighbor in $\{u_1, u_2, \ldots, u_{i-1}\}$. Place u_1 at w. Since $d_{G_2}(w) < n - 1$, we may place u_2 at a non-neighbor w_2 of w in G_2. Let $G_1' = G_1 - u_1$ and $G_2' = G_2 - w$. Suppose now that $2 \leqslant i \leqslant t - 1$ and we have already placed $u_2, \ldots, u_i$ on vertices $w_2, \ldots, w_i$ of G_2'. By the ordering of $V(T)$, u_{i+1} has exactly one neighbor $u_j \in \{u_1, u_2, \ldots, u_i\}$. Observe that

$$e(G_2') \leqslant e(G_2) - \Delta_2 < (1 - \epsilon/2)n. \tag{4}$$

Therefore, w_i has at least $(1 - \epsilon/2)n - 2$ non-neighbors in G_2'. At most $i \leqslant t - 1 \leqslant \frac{2}{\epsilon} - 1$ of these vertices are already occupied by $u_2, \ldots, u_i$. Thus for large n, there is a non-neighbor w_{i+1} of w_i not yet occupied. Place there u_{i+1}. This way, we place all vertices in $V(T)$ on vertices of G_2 without conflicts.

Let $G_1'' = G_1 - V(T)$ and $G_2'' = G_2 - \{w_1, w_2, \ldots, w_t\}$. If we find a packing of G_1'' with G_2'', then we obtain a packing of G_1 with G_2, a contradiction. By (4), for large n,

$$e(G_1'') + e(G_2'') \leqslant \big((1 - \epsilon/2)n - (t - 1)\big) + (1 - \epsilon/2)n \leqslant 2(n - t) - 3.$$

By (2) and (4), for $i = 1, 2$, $\Delta(G_i'') \leqslant e(G_i'') < (1 - \epsilon/2)n \leqslant n(G_i'') - 2$, and hence neither G_1'' nor G_2'' has an all-adjacent vertex. Thus by Theorem 5, G_1'' and G_2'' pack.

Case 2. $e(G_2) \geqslant \frac{1}{3}n^{\frac{3}{2}}$. Then

$$e(G_1) \leqslant (1 - \epsilon)n^2/\left(\frac{1}{3}n^{\frac{3}{2}}\right) = 3(1 - \epsilon)\sqrt{n}. \tag{5}$$

Since $e(G_1) < 3\sqrt{n}$, G_1 has at least $n - 6\sqrt{n} > n/2$ isolated vertices. Let $v_1, v_2, \ldots, v_n$ be an ordering of $V(G_2)$ such that v_i has maximum degree in $G_2[v_i, \ldots, v_n]$. Let $G_2' =$

$G_2[v_{\lfloor n/2\rfloor+1},\dots,v_n]$ and Δ'_2 be the maximum degree of G'_2. Then $0 \leqslant e(G'_2) - \Delta'_2 \leqslant e(G_2) - \Delta'_2(n+1)/2$, and thus $\Delta'_2 \leqslant 2e(G_2)/n$. Let G'_1 be the graph obtained after removing $\lfloor n/2\rfloor$ isolated vertices from G_1. Let $n' = \lceil n/2\rceil$.

If G_1 is a forest, then let $d = 2$. Otherwise, let d be the maximum positive integer such that G_1 is d-degenerate. Since $e(G_1) \geqslant d(d+1)/2$,

$$d \leqslant \left\lfloor \sqrt{2e(G_1)} \right\rfloor. \tag{6}$$

By (5), $40\Delta_1 \ln \Delta'_2 < 40e(G_1)\ln n < n'$. Since G'_1 and G'_2 do not pack, Theorem 7 yields $40d\Delta'_2 \geqslant n'$. So

$$n/2 \leqslant 40d\Delta'_2 \leqslant 40\sqrt{2e(G_1)}\,\frac{2e(G_2)}{n} \leqslant 80\sqrt{2e(G_1)}\,\frac{(1-\epsilon)n^2}{ne(G_1)} = \frac{80\sqrt{2}(1-\epsilon)n}{\sqrt{e(G_1)}}.$$

That is, $e(G_1) \leqslant (160\sqrt{2}(1-\epsilon))^2 < 10^5$. Let $c_0 = e(G_1)$. If $c_0 = 1$ and G_1 and G_2 do not pack, then $G_2 = K_n$ and hence (i) holds.

If $c_0 = 2$ and G_1 and G_2 do not pack, then the complement $\overline{G}_2$ of G_2 is contained either in a matching (if G_1 is a 2-path), or in $K_{1,n-1}$ or in K_3 (if G_1 has two isolated edges). In all cases, G_2 has at least $\binom{n}{2} - n$ edges. Therefore, $e(G_1)e(G_2) \geqslant n^2 - 3n$, a contradiction to (1).

The case $G_1 = K_3$ and G_1 and G_2 do not pack is the other way to express (iii).

So, we have $3 \leqslant c_0 \leqslant 10^5$ and $G_1 \neq K_3$. Hence the size of the complement $\overline{G'_2}$ of G'_2 is at least

$$\binom{n}{2} - (1-\epsilon)\frac{n^2}{c_0} = \frac{n^2}{2}\left(1 - \frac{2}{c_0}\right) + \frac{\epsilon}{c_0}n^2 - \frac{n}{2} \geqslant \frac{n^2}{2}\left(1 - \frac{2}{c_0}\right) + \frac{\epsilon}{2c_0}n^2.$$

By Theorem 1, $\overline{G'_2}$ contains complete $(\lfloor 0.5c_0\rfloor + 1)$-partite graph with $t \geqslant \frac{\log n}{500\log(c_0/\epsilon)} > 10^5$ vertices in each part. Thus, if

$$\chi\big(G'_1\big) \leqslant 1 + \lfloor 0.5c_0\rfloor, \tag{7}$$

then $\overline{G'_2}$ contains G'_1, i.e., G'_1 and G'_2 pack and hence G_1 and G_2 pack. This is certainly the case if $c_0 = 3$ and $G_1 \neq K_3$. If $c_0 \in \{4,5\}$, then $\chi(G'_1) \leqslant 3$ and $1 + \lfloor 0.5c_0\rfloor = 3$. Similarly, if $c_0 \in \{6,7\}$, then $\chi(G'_1) \leqslant 4$ and $1 + \lfloor 0.5c_0\rfloor = 4$.

Let $c_0 \geqslant 8$ and $k = \chi(G'_1)$. Since G_1 is d-degenerate, (6) yields $k \leqslant 1 + d \leqslant 1 + \lfloor\sqrt{2c_0}\rfloor$. But for each real $c_0 \geqslant 8$, we have $\sqrt{2c_0} \leqslant 0.5c_0$ and so (7) holds. This proves the theorem.

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