On k-Detour Subgraphs of Hypercubes

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Abstract: A spanning subgraph G of a graph H is a *k-detour subgraph* of H if for each pair of vertices $x, y \in V(H)$, the distance, $\mathrm{dist}_G(x, y)$, between x and y in G exceeds that in H by at most k. Such subgraphs sometimes also are called *additive spanners*. In this article, we study k-detour subgraphs of the n-dimensional cube, Q^n, with few edges or with moderate maximum degree. Let $\Delta(k, n)$ denote the minimum possible maximum degree of a k-detour subgraph of Q^n. The main result is that for every $k \geq 2$ and

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$n \geq 21$,

$$e^{-2k} \frac{n}{\ln n} \leq \Delta(k, n) \leq 20 \frac{n \ln \ln n}{\ln n}.$$

On the other hand, for each fixed even $k \geq 4$ and large n, there exists a k-detour subgraph of Q^n with average degree at most $2 + 2^{4-k/2} + o(1)$.

Keywords: *hypercube; additive spanner; k-detour*

1. INTRODUCTION

By $\text{dist}_G(u, v)$, we denote the distance between vertices u and v in a graph G. A spanning subgraph $G = (V, E')$ of a connected graph $H = (V, E)$ is an $f(x)$-*spanner*, if for each pair $\{u, v\} \subset V$, we have $\text{dist}_G(u, v) \leq f(\text{dist}_H(u, v))$. Construction of spanners with few edges and/or low maximum degree has attracted considerable attention in computer science lately. As mentioned in [6], spanners have applications in communication networks [9], broadcasting, routing, and robotics. The reader can look into [6–8,10,11] for more information.

A *k-additive spanner* is a $(k + x)$-spanner. Additive spanners were studied in [1–4,6,7]. In [3,4], 2-additive spanners of the n-dimensional cube, Q^n, were called *detour subgraphs*. The following variations of the notion of a k-additive spanner are closely related to studies in [3,4]. A spanning subgraph G of a graph H is a (k, t)-*detour subgraph* of H if for each pair of vertices $x, y \in V(H)$ with $\text{dist}_H(x, y) \leq t$, we have $\text{dist}_G(x, y) \leq \text{dist}_H(x, y) + k$. A *k-detour subgraph* is a (k, ∞)-detour subgraph, that is, a k-additive spanner.

Erdős et al. [3] studied 2-detour subgraphs and (2,1)-detour subgraphs of Q^n. Recall that the vertices of Q^n are 0-1 vectors of length n and two vectors are adjacent in Q^n if they differ in exactly one coordinate. The *direction* of an edge $xy \in E(Q^n)$ is the coordinate in which x and y differ.

Let $f_{k,t}(n)$ denote the minimum number of edges, and $\Delta_{k,t}(n)$ denote the minimum possible maximum degree of a (k, t)-detour subgraph in Q^n. It was shown in [3] that

$$f_{2,\infty}(n) \leq \frac{3}{4}\sqrt{2n}\, 2^n; \tag{1}$$

$$2(1 - o(1))\, 2^n \leq f_{2,1}(n) \leq \frac{1}{4}\sqrt{6n}\, 2^n; \tag{2}$$

$$\Delta_{2,1}(n) \geq \sqrt{n}. \tag{3}$$

Some of these results were improved in [4]. Namely, it was proved that $f_{2,1}(n) = (3 + o(1))\, 2^n$, that $\lim_{n\to\infty} \frac{f_{2,\infty}(n)}{f_{2,1}(n)} > 1$ and that

$$\sqrt{2n + 0.25} - 0.5 \leq \Delta_{2,1}(n) \leq 1.5\sqrt{2n} - 1. \tag{4}$$

The best lower bound on $f_{2,\infty}(n)$ we know is $(3.000013 - o(1)) \cdot 2^n$ which is far from the upper bound (1). Bass and Sudborough [1] and Liestman and Shermer [7] proved independently that $\Delta_{2,\infty}(n) \le n/2$. The main result of the present article is:

Theorem 1. *For every integer $k \ge 2$ and $n \ge 21$,*

$$\frac{n}{\ln n}\, e^{-2k} \le \Delta_{k,\infty}(n) \le 20\frac{n}{\ln n} \ln\ln n. \tag{5}$$

Theorem 1 significantly improves the upper bounds of [1] and [7], and its lower bound is closely related to the results of [3] and [4] on $\Delta_{2,1}(n)$ and $\Delta_{2,\infty}(n)$, respectively. The gap between the lower and upper bounds in the theorem is relatively tight.

We also find the order of magnitude of $f_{k,\infty}(n)$ for $k \ge 4$.

Theorem 2. *For every integer $k \ge 4$, $f_{k,\infty}(n) \le (3 + o(1)) \cdot 2^n$.*

Here and throughout the article, $o(1)$ denotes a quantity that tends to zero as n tends to infinity. It is a bit surprising that while each 4-detour graph in Q^n has a few vertices of degree at least $e^{-4}\frac{n}{\ln n}$, it may have average degree as low as $6 + o(1)$ (even strictly less than 6 for n of the form $n = 2^r - 2$). Moreover, for every $\epsilon > 0$, there exists a positive integer k and a k-detour graph G in Q^n such that the average degree of G is at most $2 + \epsilon + o(1)$:

Theorem 3. *For every even integer $k \ge 4$, $f_{k,\infty}(n) \le (1 + 2^{3-k/2} + o(1)) \cdot 2^n$. For every even integer $k \ge 2$, $f_{k,1}(n) \le (1 + 2^{2-k/2} + o(1)) \cdot 2^n$.*

Recall that each spanner is connected and thus each spanner in Q^n has at least $2^n - 1$ edges. We do not know whether the bound in (1) can be improved to something like the bound in Theorem 2.

In the next section, we obtain the left inequality in (5), then in Section 3 we prove the main part of Theorem 1, the upper bound on $\Delta_{k,\infty}(n)$. Finally, in Section 4 we show that the construction of a (2,1)-detour subgraph of [4] with $(3 + o(1))\, 2^n$ edges is also a 4-detour subgraph in Q^n and prove Theorem 3.

2. MAXIMUM DEGREE OF DETOUR SUBGRAPHS IN Q^n—LOWER BOUND

In this section, we prove the lower bound in Theorem 1.

Proof. Since Q^n is bipartite, it is enough to consider k-detours in Q^n for even k.

Let $m = \lfloor \ln n \rfloor$. Let G be a k-detour subgraph of Q^n, and $\Delta(G) = \Delta$ be the maximum degree in G. Since G is connected, $\Delta \ge 2$. Let u be a vertex of G. For each vertex v at distance m from u in Q^n, we have $m \le \mathrm{dist}_G(u, v) \le m + k$. Since the number of walks in G of length j starting at u is at most Δ^j, we have

$$\binom{n}{m} \leq \sum_{i=0}^{k/2} \Delta^{m+2i} < 2\Delta^{m+k}. \tag{6}$$

Recall that $m(m-1) \leq n$ for $n \geq 3$ and therefore

$$\binom{n}{m} = \frac{n^m}{m!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{m-1}{n}\right) \geq \frac{n^m}{m!}\left(1-\frac{m(m-1)}{2n}\right) \geq \frac{n^m}{2m!}.$$

Since $n \geq 21$, we have $m \geq 3$. Hence if (6) holds, then $n^m < 4m!\Delta^{m+k} \leq m^m \Delta^{m+k}$, that is,

$$n < m\Delta^{1+k/m} \leq m\Delta\, n^{k/m}. \tag{7}$$

If $\Delta \leq \frac{n}{\ln n} e^{-2k}$, then (7) yields (since $m = \lfloor \ln n \rfloor$)

$$n < m\frac{n}{\ln n} e^{-2k} n^{k/m} \leq n\, e^{-2k} n^{2k/\ln n} = n,$$

a contradiction. ■

3. MAXIMUM DEGREE OF DETOUR SUBGRAPHS IN Q^n—UPPER BOUND

Now we turn to proving the upper bound of Theorem 1.

Obviously, $\Delta_{k,\infty}(n) \leq n$ for every positive k. Hence, for $n \leq e^{20}$ the upper bound of (5) holds. Now, assume that $n > e^{20} > 10^8$. We will need the following simple fact.

Claim 1. *If m and n are positive integers such that $n \geq e^{20}$ and $m \leq 0.3\frac{\ln n}{\ln \ln n}$, then $\binom{2m^2}{2m} \leq n^{0.6}$.*

Proof.

$$\binom{2m^2}{2m} \leq \left(\frac{2m^2 e}{2m}\right)^{2m} = \exp\{2m(\ln em)\} \leq \exp\left\{0.6\frac{\ln n}{\ln\ln n}\ln\frac{\ln n}{\ln \ln n}\right\}.$$

Since $\ln\ln n \geq 1$ for $n \geq e^{20}$, the last expression is at most $\exp\{0.6 \ln n\} = n^{0.6}$. ■

Below, we use the standard notation $[n]$ to denote the set $\{1, 2, \ldots, n\}$, for every positive integer n. For a 0-1 vector x and a subset B of the set of coordinates of x, *the projection, $x(B)$, of x on B* is the vector obtained from x be deleting all coordinates not in B.

For a given n, let $m = \lfloor 0.3\frac{\ln n}{\ln\ln n} \rfloor$. Let r be the largest integer such that $2^r - 1 \leq \frac{n}{m}$ and let $q = 2^r - 1$. Denote $s = \lceil \frac{n-q}{2m^2} \rceil$ and partition the set $[n]$ into $2m^2+1$ pairwise disjoint subsets $B_0, B_1, \ldots, B_{2m^2}$, where $B_0 = [q]$ and $|B_i| \in \{s-1, s\}$

for $i = 1, \dots, 2m^2$. For every $2m$-element subset M of $[2m^2]$, let $B_M = \cup_{i \in M} B_i$. Notice that we have defined exactly $\binom{2m^2}{2m}$ sets B_M. Now we build a k-detour graph G in three steps: at Step i we define a graph H_i, and then let $G = H_1 \cup H_2 \cup H_3$.

Step 1. Let H_1 be the subgraph of Q^n spanned by the edges along the coordinates in B_0. Clearly, H_1 is the disjoint union of 2^{n-q} copies of Q^q.

Step 2. Since $q = 2^r - 1$, we can partition the set $V(Q^q)$ into $q + 1$ Hamming codes $D'_1, \dots, D'_{q+1}$. Note that each Hamming code D'_i is a dominating set in Q^q. For $i = 1, \dots, q + 1$, let D_i be the union of D'_i over all 2^{n-q} components of H_1. Thus, $D_1, \dots, D_{q+1}$ form $q + 1$ disjoint dominating sets in H_1. Since $n^{0.4} \geq \ln n$ and $\ln \ln n > 1$ for $n > e^{20}$, we have

$$q \geq \frac{n}{3m} \geq \frac{n}{0.9 \frac{\ln n}{\ln \ln n}} > n^{0.6}. \tag{8}$$

Let $h = \binom{2m^2}{2m}$. By (8) and Claim 1, $h \leq q$. Therefore, we can fix a one-to-one correspondence φ from the family $\{D_1, \dots, D_h\}$ to the family of $2m$-element subsets of $[2m^2]$. Now, for every $x \in V(Q^n)$, we define the neighbors of x in H_2 as follows. If $x \notin \bigcup_{i=1}^h D_i$, then no edges incident with x belong to H_2. If $x \in D_i$ $(1 \leq i \leq h)$ and $\varphi(D_i) = M$, then every edge incident with x whose direction is in B_M belongs to $E(H_2)$. Note that if $x \in D_i$ and y differs from x only in a coordinate $j \notin B_0$, then, by the definition of D_i, the vertex y also belongs to D_i. This shows that H_2 is defined correctly. For every $x \in V(Q^n)$, let $H_2(x)$ denote the component of H_2 containing x. By the definition, if $x \notin \bigcup_{i=1}^h D_i$, then $V(H_2(x)) = \{x\}$ and if $x \in D_i$ for some $1 \leq i \leq h$ and $\varphi(D_i) = M$, then $H_2(x)$ is a subcube of Q^n of dimension $|B_M|$.

An example of Step 2 is shown in Figure 1.

Step 3. For $j = 1, \dots, m$, let $A_j = \bigcup_{i=(j-1)2m+1}^{2mj} B_i$. Consider $F = H_2(x)$, where $x \in D_i$ for some $1 \leq i \leq h$. Suppose that $\varphi(D_i) = M$. As it was mentioned above, F is a subcube of Q^n. Let $z = z(F)$ be the vertex in F with the smallest sum of coordinates and $L_j = L_j(z)$ be the set of vertices in F at distance j from z. If $x \in L_j$ and $j = mp + j'$ where $0 \leq j' \leq m - 1$, then the set, $C(x)$, of edges of H_3 incident with x consists of those with directions in $A_{j'} - B_M$. In order to see that the definition of H_3 is correct, suppose that x_1 differs from x only in coordinate $l \in A_{j'} - B_M$. Since $l \notin B_M$, $x_1 \notin V(F)$. Since $l \notin B_0$, the projections $x(B_0)$ and $x_1(B_0)$ of x and x_1 on B_0 coincide, and therefore $x_1 \in D_i$. Hence, the graph $F_1 = H_2(x_1)$ is the translation of $H_2(x)$ along the coordinate l. Furthermore, $z(F_1)$ is the translation of $z(F)$ along the coordinate l, and hence the distance in H_2 between x_1 and $z(F_1)$ is the same as between x and $z(F)$, namely j. Since B_M and j' are the same for x and x_1, we have $C(x) = C(x_1)$ and hence the edge in direction l incident with x_1 belongs to $E(H_3)$.

This finishes the construction of the graph $G = H_1 \cup H_2 \cup H_3$.

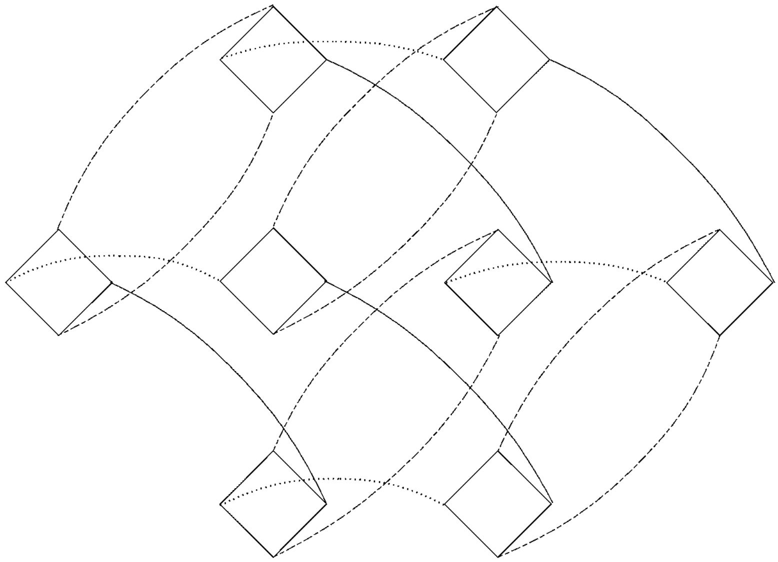

FIGURE 1. An example of Step 2: Edges of H_2 (a matching) added to the graph H_1 (squares).

Claim 2. $\Delta(G) \leq 20\frac{n}{\ln n}\ln\ln n$.

Proof. By construction, $\Delta(H_1) \leq \frac{n}{m}$, and $\Delta(H_2) + \Delta(H_3) \leq 2ms + 2ms = 4ms$. Recall that $s = \lceil\frac{n-q}{2m^2}\rceil$ and $q \geq \frac{n}{2m} - 1$. We prove first that

$$\frac{n}{2m} \geq 2m^2 + 1. \tag{9}$$

Since $n \geq e^{20}$ and $\ln\ln n > 2$, (9) follows from

$$n \geq 4(0.15\ln n)^3 + 2(0.15\ln n),$$

which holds for every $n \geq 20$.

By (9), $s \leq \frac{n}{2m^2}$ and hence $\Delta(G) \leq \frac{n}{m} + 4m\frac{n}{2m^2} = \frac{3n}{m}$. Since $n \geq 500$, we have $0.3\frac{\ln n}{\ln\ln n} > 1$ and therefore,

$$m = \left\lfloor 0.3\frac{\ln n}{\ln\ln n}\right\rfloor \geq \frac{1}{2}\left(0.3\frac{\ln n}{\ln\ln n}\right).$$

Thus, $\Delta(G) \leq 3n\frac{20\ln\ln n}{3\ln n} = \frac{20n\ln\ln n}{\ln n}$. ■

Let $B \subset [n]$. A subgraph H of Q^n is a (k, B)-*detour graph*, if the inequality $\mathrm{dist}_H(x, y) \leq \mathrm{dist}_{Q^n}(x, y) + k$ holds for each x and y such that $x(B) = y(B)$.

Claim 3. *If G is a $(2, B_0)$-detour graph, then G is a 2-detour graph.*

Proof. Suppose that G is a $(2, B_0)$-detour graph and x and y are arbitrary vertices of G. Let x' be the vertex such that $x'(B_0) = y(B_0)$ and $x'([n] - B_0) = x([n] - B_0)$.

Then $\mathrm{dist}_{Q^n}(x, y) = \mathrm{dist}_{Q^n}(x, x') + \mathrm{dist}_{Q^n}(x', y)$. On the other hand, x and x' are in the same component of H_1 and hence $\mathrm{dist}_G(x, x') = \mathrm{dist}_{Q^n}(x, x')$. Since G is a $(2, B_0)$-detour graph, $\mathrm{dist}_G(x', y) \leq \mathrm{dist}_{Q^n}(x', y) + 2$. Therefore,

$$\begin{aligned}\mathrm{dist}_G(x, y) &\leq \mathrm{dist}_G(x, x') + \mathrm{dist}_G(x', y) \leq \mathrm{dist}_{Q^n}(x, x') + \mathrm{dist}_{Q^n}(x', y) + 2\\ &= \mathrm{dist}_{Q^n}(x, y) + 2.\end{aligned}$$

This proves the claim. ■

To finish the proof of the upper bound, we will show that G is a $(2, B_0)$-detour graph. Let x and y be arbitrary vertices in G such that $x(B_0) = y(B_0)$. Suppose that the set of coordinates in which x and y differ is $J = \{j_1, \ldots, j_w\}$. Recall that $B_0 \cap J = \emptyset$. We consider two cases.

Case 1. $w \leq 2m$.

Let M be any $2m$-element subset of $[2m^2]$ such that $B_M \supset J$ and let i be the index such that $\varphi(D_i) = M$. Let x' be the vertex in D_i at distance at most one from x in H_1 (it maybe a neighbor of x or x itself). Let y' be the vertex in D_i at distance at most one from y. Since x' differs from x and y' differs from y in the same coordinate (or $x = x'$ and $y = y'$, simultaneously), the set of coordinates in which y' differs from x' is exactly J. In particular, $\mathrm{dist}_{Q^n}(x', y') = \mathrm{dist}_{Q^n}(x, y)$. Furthermore, by the definition, x' and y' are in the same component of H_2 and hence

$$\mathrm{dist}_G(x', y') = \mathrm{dist}_{Q^n}(x', y'). \tag{10}$$

Thus,

$$\mathrm{dist}_G(x, y) \leq 2 + \mathrm{dist}_G(x', y') = 2 + \mathrm{dist}_{Q^n}(x', y') = 2 + \mathrm{dist}_{Q^n}(x, y).$$

Case 2. $w > 2m$.

Let M be any $2m$-element subset of $[2m^2]$ such that $B_M \supset \{j_1, \ldots, j_{2m}\}$ and let i be the index such that $\varphi(D_i) = M$. Let x' be the vertex in D_i at distance at most one from x in H_1 and y' be the corresponding vertex in D_i for y. As in Case 1, $\mathrm{dist}_{Q^n}(x', y') = \mathrm{dist}_{Q^n}(x, y)$. Hence, if (10) holds, then we are done as in Case 1. Thus, our goal is to prove (10).

Let $F' = H_2(x')$ be the component of H_2 containing x' and $z = z(F')$ be the vector in F' with the smallest sum of its coordinates. Let $Q = Q^{n-q}$ be the set of vectors v with $v(B_0) = x'(B_0)$. Since all vectors in Q have the same projection on B_0, the subgraph of H_2 induced by Q consists of $2^{n-q-|B_M|}$ disjoint copies of F'. We can partition $V(Q)$ into levels as follows: level 0 consists of vertices of the kind $z(F)$ for every component F of H_2 in Q; for every $i \geq 1$, level i consists of vertices at distance i in H_2 from $z(F)$ in the corresponding component F of H_2. Then every edge of H_2 connects vertices of neighboring levels, and every edge in $E(Q) - E(H_2)$ connects vertices of the same level.

Let x'' be the vector in F' such that $x''(B_M) = y'(B_M)$ and $x''([n] - B_M) = x'([n] - B_M)$. By the choice of M, x' and x'' differ in at least $2m$ coordinates. Let P be a shortest x', x''-path in $H_2(x')$ such that first it goes farther and farther from z and then comes closer to z with every step. We can split P into two paths: the ascending part $P_1 = (x' = x_0, x_1, \ldots, x_f)$ and the descending part P_2. Let j_i be the direction in which x_i differs from x_{i-1}. Since the length of P is at least $2m$, we may assume w.l.o.g. that $|V(P_1)| \geq m+1$. Then P_1 visits some $m+1$ consecutive levels of the cube F' with z as zero vector. Recall that the set $C(v)$ of directions of edges in H_3 incident with a vertex $v \in V(Q)$ depends only on the level of v in Q, and that every direction $j \in [n] - B_0 - B_M$ appears in $\cup_{i=0}^{m-1} C(x_i)$.

Below, we construct a path P_0 in G from $x_0 = x'$ to y' of length $\text{dist}_{Q^n}(x', y')$ as follows. If $C(x_0) \cap J \neq \emptyset$, then we move along every of the directions in $C(x_0) \cap J$ exactly once. Then we move in the direction j_1. Similarly, we now move along every of the directions in $C(x_1) \cap J$ exactly once and then move in the direction j_2. Repeat this procedure m times, and we come at the vertex y'' such that $y''(B_M) = x_m(B_M)$ and $y''([n] - B_M) = y'([n] - B_M)$. In other words, y'' is in the component F'' of H_2 that contains y', and the position of y'' with respect to y' in F'' is that of x_m with respect to x'' in F'. Now we simply take a shortest path from y'' to y' in F''. Since with each step of the above constructed path, we shortened the distance to y' in Q, we made exactly $\text{dist}_Q(x', y')$ steps. This proves (10).

4. ON k-DETOUR SUBGRAPHS IN Q^n WITH FEW EDGES

We recall a construction from [4]. Let $n_1 = \lceil n/2 \rceil$ and $n_2 = n - n_1$. We view Q^n as the Cartesian product $Q^{n_1} \times Q^{n_2}$ and write every vector $v \in V(Q^n)$ in the form $v = (v_1, v_2)$, where $v_1 \in V(Q^{n_1})$ and $v_2 \in V(Q^{n_2})$. By a well known result due to Kabatyanskii and Panchenko [5], for $i \in \{1, 2\}$, the graph Q^{n_i} has a dominating set D_i with

$$|D_i| = 2^{n_i}\left(\frac{1}{n_i} + o\left(\frac{1}{n_i}\right)\right) = 2^{n_i}\left(\frac{2}{n} + o\left(\frac{1}{n}\right)\right). \tag{11}$$

Let $S_1 = \{(v_1, v_2) \in V(Q^n) : v_1 \in D_1, v_2 \in V(Q^{n_2})\}$, $S_2 = \{(v_1, v_2) \in V(Q^n) : v_1 \in V(Q^{n_1}), v_2 \in D_2\}$, and $S = S_1 \cup S_2$. Let G be the spanning subgraph of Q^n whose edges are all the edges of Q^n incident to at least one vertex in S. By the definition and (11),

$$|S| \leq |S_1| + |S_2| = |D_1|2^{n_2} + |D_2|2^{n_1} = 2^n\left(\frac{4}{n} + o\left(\frac{1}{n}\right)\right). \tag{12}$$

Claim 4. ([4]) $|E(G)| \leq (3 + o(1))2^n$.

Proof. For $i = 1, 2$, each vertex $v \in S_i$ is adjacent to at least n_{3-i} other vertices in S_i. Therefore, taking (12) into account,

$$|E(G)| \le n|S| - \frac{n_2}{2}|S_1| - \frac{n_1}{2}|S_2| \le \left(\frac{3n}{4} + \frac{1}{4}\right)|S| = 2^n\,(3 + o\,(1))\,.$$

■

Claim 5. *For every $u \in S_1$ and $v \in S_2$,*

$$\mathrm{dist}_G(u, v) = \mathrm{dist}_{Q^n}(u, v).$$

Proof. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Let $x = (u_1, v_2)$. Since $u \in S_1$ and $v \in S_2$, we have $u_1 \in D_1$ and $v_2 \in D_2$. It follows that all vectors $w = (w_1, w_2)$ with $w_1 = u_1$ are in S_1. Thus, $\mathrm{dist}_G(u, x) = \mathrm{dist}_{Q^n}(u, x)$. Similarly, $\mathrm{dist}_G(x, v) = \mathrm{dist}_{Q^n}(x, v)$. This proves the claim. ■

The next claim concludes the proof of Theorem 2.

Claim 6. *G is a 4-detour graph in Q^n.*

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be arbitrary vertices in G. Recall that D_i is a dominating set in Q^{n_i} for $i = 1, 2$. Hence, x has a neighbor $u = (u_1, x_2) \in S_1$ and y has a neighbor $v = (y_1, v_2) \in S_2$. Applying Claim 4 finishes the proof. ■

To prove Theorem 3, we need the following simple fact.

Lemma 4. *For each positive integers k, t and n, $f_{k+2,t}(n + 1) \le f_{k,t}(n) + 2^n$. This also holds if $t = \infty$.*

Proof. Consider the graph Q^{n+1} as the union of two copies, Q and R, of Q^n joined by a perfect matching M. For each $v \in V(R)$, let $M(v)$ be the neighbor of v in Q. Let G' be a (k, t)-detour graph in Q with $f_{k,t}(n)$ edges. Define $E(G) = E(G') \cup M$.

To check that G is a $(k + 2, t)$-detour graph in Q^{n+1}, consider arbitrary vertices x and y in Q^{n+1} at distance at most t. If both x and y are in Q, then, by the definition of G', $\mathrm{dist}_G(x, y) \le \mathrm{dist}_Q(x, y) + k$. If $x \in V(Q)$ and $y \in V(R)$, then

$$\mathrm{dist}_G(x, y) = 1 + \mathrm{dist}_G(x, M(y)) \le 1 + \mathrm{dist}_Q(x, M(y)) + k = \mathrm{dist}_{Q^{n+1}}(x, y) + k.$$

Finally, if both x and y are in R, then

$$\begin{aligned}\mathrm{dist}_G(x, y) &= 2 + \mathrm{dist}_G(M(x), M(y)) \le 2 + \mathrm{dist}_Q(M(x), M(y)) + k\\ &= \mathrm{dist}_{Q^{n+1}}(x, y) + k + 2.\end{aligned}$$

This proves the lemma. ■

Now we finish the proof of Theorem 3 by induction on k. The base case for the first statement is the case $k = 4$ which holds by Theorem 2. Suppose that for some even $k \geq 4$, we have $f_{k,\infty}(n-1) \leq (1 + 2^{3-k/2} + o(1)) \cdot 2^{n-1}$. Then by the above lemma, we get

$$f_{k+2,\infty}(n) \leq (1 + 2^{3-k/2} + o(1)) \cdot 2^{n-1} + 2^{n-1} = (1 + 2^{3-(k+2)/2} + o(1)) \cdot 2^n.$$

The proof for $f_{k,1}$ is the same; only the base case is $k = 2$ which was proved in [4] (see the construction at the beginning of this section).

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REFERENCES

[1] D. Bass and I. H. Sudborough, Vertex-symmetric spanning subnetworks of hypercubes with small diameter, Proceedings of PDCS'99, The 11th IASTED International Conference on Parallel and Distributed Computing Systems, 1999, pp. 7–12.

[2] V. Chepoi, F. Dragan, and C. Yan, Additive spanners for k-chordal graphs, Algorithms and Complexity, Springer LNCS 2653, 2003, pp. 96–107.

[3] P. Erdős, P. Hamburger, R. E. Pippert, and W. D. Weakley, Hypercube subgraphs with minimal detours, J Graph Theory 23(2) (1996), 119–128.

[4] P. Hamburger, A. Kostochka, and A. Sidorenko, Hypercube subgraphs with local detours, J. Graph Theory 30(2) (1999), 101–111.

[5] G. A. Kabatyanskiĭ and V. I. Panchenko, Packings and coverings of the Hamming space by balls of unit radius, Problems Inform Transmission 24(4) (1988), 261–272, translated from Problemy Peredachi Informatsii 24(4) (1988), 3–16.

[6] D. Kratsch, H. Le, H. Müller, E. Prisner, and D. Wagner, Additive tree spanners, SIAM J Discrete Math 17(2) (2004), 332–340.

[7] A. L. Liestman and T. C. Shermer, Additive graph spanners, Networks 23(4) (1993), 343–363.

[8] D. Peleg and A. A. Schäffer, Graph spanners, J Graph Theory 13(1) (1989), 99–116.

[9] D. Peleg and J. D. Ullman, An optimal synchronizer for the hypercube, Proceedings of the Sixth Annual ACM Symposium on Principles of Distributed Computing, 1987, pp. 77–85.

[10] J. Soares, Graph spanners: A survey, Congressus Numer 89 (1992), 225–238.

[11] G. Venkatesan, U. Rotics, M. S. Madanlal, J. A. Makowsky, and C. Pandu Rangan, Restrictions of minimum spanner problems, Inform Comput 136(2) (1997), 143–164.