Ore-Type Degree Conditions for a Graph to be *H*-Linked

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Abstract: Given a fixed multigraph H with $V(H) = \{h_1, \ldots, h_m\}$, we say that a graph G is H-linked if for every choice of m vertices $v_1, \ldots, v_m$ in G, there exists a subdivision of H in G such that for every i, v_i is the branch vertex representing h_i. This generalizes the notion of k-linked graphs (as well as some other notions). For a family $\mathcal{H}$ of graphs, a graph G is $\mathcal{H}$-linked if G is H-linked for every $H \in \mathcal{H}$. In this article, we estimate the minimum integer $r = r(n, k, d)$ such that each n-vertex graph with $\sigma_2(G) \geq r$ is $\mathcal{H}$-linked, where $\mathcal{H}$ is the

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family of simple graphs with k edges and minimum degree at least $d \geq 2$. J Graph Theory 58: 14–26, 2008

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1. INTRODUCTION

Let H be a graph. An *H-subdivision* in a graph G is a pair of mappings $f : V(H) \to V(G)$ and g: $E(H)$ into the set of paths in G such that:

(a) $f(u) \neq f(v)$ for all distinct $u, v \in V(H)$;
(b) for every $uv \in E(H)$, $g(uv)$ is an $f(u), f(v)$-path in G, and distinct edges map into internally disjoint paths in G.

A graph G is *H-linked* if every injective mapping $f : V(H) \to V(G)$ can be extended to an H-subdivision in G.

Let S_k be a star with $k+1$ vertices. Then a graph is k-connected if and only if it is S_k-linked, by Fan Lemma due to Dirac [2].

Recall that a graph is *k-linked* if for every list $v_1, \ldots, v_k, w_1, \ldots, w_k$ of $2k$ vertices, there are internally disjoint paths $P_1, \ldots, P_k$ such that P_i connects v_i and w_i for each i. It is known that a graph with at least $2k$ vertices is k-linked if and only if it is M_k-linked, where M_k is a matching of size k.

A graph is *k-ordered*, if for every ordered sequence of k vertices, there is a cycle that encounters the vertices of the sequence in the given order. Let C_k denote the cycle of length k. Clearly, a simple graph G is k-ordered if and only if G is C_k-linked.

Recall that Dirac [1] found sufficient conditions for a simple graph G to be Hamiltonian in terms of the minimum degree, $\delta(G)$, and Ore [15] found similar conditions in terms of $\sigma_2(G)$, the minimum value of the sum $\deg(u) + \deg(v)$ over all pairs $\{u, v\}$ of non-adjacent vertices in G. For a family $\mathcal{H}$ of graphs, let $D_{\mathcal{H}}(n)$ be the minimum integer d such that for each n-vertex graph G with $\delta(G) \geq d$, G is H-linked for every graph $H \in \mathcal{H}$. Similarly, let $R_{\mathcal{H}}(n)$ be the minimum integer r such that each n-vertex graph G with $\sigma_2(G) \geq r$ is H-linked for every graph $H \in \mathcal{H}$. When $\mathcal{H} = \{H\}$, we use the notation $D_H(n) = D_{\mathcal{H}}(n)$ and $R_H(n) = R_{\mathcal{H}}(n)$.

Several authors studied $D_{C_k}(n)$ and $R_{C_k}(n)$, see [3–5,7,10,12,14], and it was shown in [12] that $D_{C_k}(n) = \lfloor n/2 \rfloor + \lceil k/2 \rceil - 1$ for every $n \geq 5k+6$ and in [5] that $R_{C_k}(n) = n + \lceil (3k-9)/2 \rceil$ for every $3 \leq k \leq n/2$. The values $D_{M_k}(n)$ and $R_{M_k}(n)$ were determined in [9] for all n and k. After the concept of H-linkage was introduced independently in [11] and [6], the value $D_{\mathcal{H}}(n)$ was also studied for more general $\mathcal{H}$.

Let $\mathcal{H}(k, d)$ be the collection of simple graphs with k edges and minimum degree at least d. It was shown in [13] that $D_{\mathcal{H}(k,2)}(n) = \lceil (n+k)/2 \rceil - 1$ for all $n \geq 5k+6$. In particular, it occurs that $D_{\mathcal{H}(k,d)}(n) = D_{\mathcal{H}(k,2)}(n)$ for every $d \geq 2$.

The value $D_H(n)$ for an arbitrary multigraph H was discussed in [6,8,12].

In this article, we study the function $R_{\mathcal{H}(k,d)}(n)$. We determine $R_{\mathcal{H}(k,2)}(n)$ for all $n \geq k$.

Theorem 1. *Let $k \geq 3$ and $\mathcal{H}(k,2)$ be the family of simple graphs with k edges and minimum degree at least 2. Then*

$$R_{\mathcal{H}(k,2)}(n) = \begin{cases} \lceil n + (3k-9)/2 \rceil, & \text{if } n > 2.5k - 5.5, \\ \lceil n + (3k-8)/2 \rceil, & \text{if } 2k \leq n \leq 2.5k - 5.5, \\ 2n - 3, & \text{if } k \leq n \leq 2k - 1. \end{cases}$$

Observe that the value of $R_{\mathcal{H}(k,2)}(n)$ is not always the same as $R_{C_k}(n)$, and also that $R_{\mathcal{H}(k,2)}(n) > 2D_{\mathcal{H}(k,2)}(n)$ when k and n are large.

We also give upper bounds on $R_{\mathcal{H}(k,d)}(n)$ for $d \geq 3$ when $n > 2k$ and $k \geq d(d+1)$. These bounds are exact if d divides k and differ by 1 from the lower bounds in other cases. Note that simple graphs with k edges and minimum degree at least d exist only if $k \geq d(d+1)/2$.

Theorem 2. *Let $d \geq 3$ and $k \geq d^2$. Let $\mathcal{H}(k,d)$ be the family of simple graphs with k edges and minimum degree at least d. Then for $n \geq k - 1 + d + 3k/d$,*

$$R_{\mathcal{H}(k,d)}(n) \leq n + k - d - 3 + \left\lceil \frac{k+1}{d} \right\rceil. \tag{1}$$

Furthermore, if $k \geq d(d+1)$, then $R_{\mathcal{H}(k,d)}(n) \geq n + k - d - 4 + \lceil \frac{k+1}{d} \rceil$. If, in addition, d divides k, then we have equality in (1).

In both, Theorems 1 and 2, we actually prove a stronger statement: in the subdivisions of H that we find, each edge of H is replaced by a path of length at most 3.

Unlike in the situation with $D_{\mathcal{H}(k,d)}(n)$, we have $R_{\mathcal{H}(k,d)}(n) < R_{\mathcal{H}(k,2)}(n)$ when $d > 2$ and $k \geq d(d+1)$.

In the next section, we prove the upper bounds for Theorem 2. Then in Section 3, we show how to modify this proof in order to get the upper bounds for Theorem 1. In Section 4, we prove the lower bounds by giving examples.

2. UPPER BOUNDS FOR THEOREM 2

Let $d \geq 2$ and $k \geq d^2$. Let G be a graph with $n \geq k - 1 + d + 3k/d$ vertices and $\sigma_2(G) > n + k(1 + 1/d) - 3 - d$. Let H be any simple graph with k edges and minimum degree at least d.

Let $f : V(H) \to V(G)$ be an injective mapping and $S = f(V(H))$. Let $E(H) = \{e_i^0 = u_i^0 v_i^0 : 1 \leq i \leq k\}$. For $1 \leq i \leq k$, let $u_i = f(u_i^0)$, $v_i = f(v_i^0)$, $e_i = (u_i, v_i)$, $\beta_i = \frac{1}{\deg_H(u_i^0)}$, and $\gamma_i = \frac{1}{\deg_H(v_i^0)}$. Since $\delta(H) \geq d$, we have $s = |S| = |V(H)| \leq 2k/d$.

Assume that $u_iv_i \in E(G)$ for $k - w + 1 \le i \le k$, where $w \ge 0$.

Construct the auxiliary bipartite graph B with partite sets W_1 and W_2 as follows. Let $W_1 = \{e_1, \ldots, e_{k-w}\}$, $W_2 = V(G) - S$, and let a pair (e_i, v) be an edge in B if $v \in N_G(u_i) \cap N_G(v_i)$. If B has a matching saturating W_1, then this matching gives the required linkage. Otherwise, let m be the size of a maximum matching in B.

By Ore's theorem on maximum matchings in bipartite graphs, there is a $Q \subseteq W_1$ with $k - w - m = |Q| - |N_B(Q)|$. Denote $R = N_B(Q)$ and $L = V(G) - S - R$.

We may assume that $Q = \{e_i : i = 1, \ldots, q\}$. Let $Q' = \bigcup_{i=1}^{q}\{u_i, v_i\}$ (the elements of Q are ordered pairs, and the elements of Q' are all the elements of these pairs). Note that $|Q| = q$, $|Q'| \le 2q$, $|R| = q - k + w + m$, and $|L| = n - s + k - q - m - w$.

Let P be a maximum matching in B. By the definition of Q and König-Ore Theorem on matchings in bipartite graphs, every maximum matching in B covers all vertices in $W_1 - Q$. Hence, we may assume that only vertices in $D = \{e_1, \ldots, e_{k-w-m}\}$ are not covered by this matching. Let $D' = \{u_i, v_i : 1 \le i \le k - w - m\}$. Consider the linkage $\mathcal{P}$ corresponding to P. Let Z be the set of vertices of $V(G) - S$ not participating in the linkage. Clearly, $|Z| = n - s - m$.

Claim 1. *Let $\phi_1(k, d, s, w) = k - 1 + d + \lceil \frac{k+1}{d} \rceil - s - w$ and $\phi_2(k, d, s) = k + 1 - d + \lceil \frac{k+1}{d} \rceil - s$. Then for every $1 \le i \le k - w - m$,*

$$|(N(u_i) \cap N(v_i)) - S| \ge \max\{\phi_1(k, d, s, w), \phi_2(k, d, s)\}. \tag{2}$$

In particular, $m \ge \max\{\phi_1(k, d, s, w), \phi_2(k, d, s)\}$.

Proof. Since u_iv_i is not an edge in G, vertices u_i and v_i together have at most $2s - 4$ neighbors in S (counted with multiplicities). On the other hand, since $\delta(H) \ge d$, the number of such neighbors is at most $2s - 2(d + 1) + w$. But $\deg_G(u_1) + \deg_G(v_1) \ge n + k + \lceil \frac{k+1}{d} \rceil - 3 - d$. It follows that u_i and v_i have at least $(n + k + \lceil \frac{k+1}{d} \rceil - 3 - d) - \max\{2s - 4, 2s - 2(d + 1) + w\} - (n - s)$ common neighbors outside of S, which yields (2).

The second statement of the claim follows from (2) and Hall's Theorem. ■

Remark. *Recall that Hall's Theorem yields that if the degree of each vertex $u \in W_1$ in a bipartite graph $B = (W_1, W_2; E_B)$ is at least ϕ, then there is a maximum matching covering any given ϕ vertices in W_1 (provided that $\phi \le |W_1|$).*

Claim 2. *One can choose a maximum matching P (of size m, by definition) in B in such a way that for every $1 \le i \le k - w - m$, either of u_i and v_i has at least $d - 0.5w$ non-neighbors in S.*

Proof. By the remark above, we can cover by edges in P any $\lceil \phi_2(k, d, s) \rceil$ vertices in Q. We will choose them as follows. For every $v \in S$, let $w(v)$ denote the number of edges in $W = \{e_{k-w+1}, \ldots, e_k\}$ incident to v. Order $y_1, \ldots, y_s$, the vertices of S in such a way that $\deg_H(f^{-1}(y_i)) - w(y_i) \le \deg_H(f^{-1}(y_{i+1})) - w(y_{i+1})$ for every $i \le s - 1$. Note that the value $\deg_H(f^{-1}(y_i)) - w(y_i)$ is a lower bound on the number of non-neighbors (in the graph G) of y_i in S (not counting y_i itself).

We first include in P the edges covering all vertices in Q that correspond to the pairs $\{u_i, v_i\}$ containing y_1, then those containing y_2, and so on, until we include $\lceil \phi_2(k, d, s) \rceil$ vertices. After this, by the remark above, we can complete P to a maximum matching. Observe that for $d \geq 2$, $k \geq d^2$, and $s \leq 2k/d$,

$$\phi_2(k, d, s) \geq k + 1 - d + \frac{k+1}{d} - \frac{2k}{d} = \frac{1}{d} + (d-1)\left(\frac{k}{d} - 1\right) > (d-1)^2.$$

Also, for every d, $1 + (d-1)^2 \geq 2(d-1)$. We will show that such choice of P provides that for every $i \leq k - w - m$ each of u_i and v_i has at least $d - 0.5w$ non-neighbors in S.

Let $F = (S, E_F)$ be the graph with the vertex set S and edge set $E_F = \{u_i v_i : k - w + 1 \leq i \leq k\}$. By definition, if a vertex $y \in S$ has exactly $d - l$ non-neighbors in S (in G), then $\deg_F(y) \geq l$. Also, we care only about $l \geq 1$.

If $\deg_H(f^{-1}(y_1)) - w(y_1) \geq d$, then, by the choice of y_1, there is nothing to prove. If $\deg_H(f^{-1}(y_1)) - w(y_1) \leq d - 1$ and $\deg_H(f^{-1}(y_2)) - w(y_2) \geq d$, then our matching P covers all vertices corresponding to pairs $\{u_i, v_i\}$ containing y_1, and every other vertex in S has at least d non-neighbors in S. If $\deg_H(f^{-1}(y_2)) - w(y_2) \leq d - 1$, then P covers all vertices corresponding to pairs $\{u_i, v_i\}$ containing y_1 and y_2. The only case of a simple graph F with w edges in which some three vertices have degree greater than $w/2$ is the graph K_3, in which case the trouble occurs if $\deg_H(f^{-1}(y_1)) - w(y_1) = \deg_H(f^{-1}(y_2)) - w(y_2) = \deg_H(f^{-1}(y_3)) - w(y_3) = d - 2$. But for every $d \geq 2$, we have $1 + (d-1)^2 \geq 3(d-2)$ and all pairs $\{u_i, v_i\}$ containing y_3 with $i \leq k - w$ will be covered by P. This proves the claim. ■

Lemma 1. $n - s - m \geq 2(k - m - w)$.

Proof. Assume that the lemma is false, that is, that $n - s - m < 2(k - m - w)$. It follows that $m < 2k - 2w + s - n$. Taking into account (2), we get

$$k - 1 + d + \left\lceil \frac{k+1}{d} \right\rceil - s - w < 2k - 2w + s - n$$

and hence $n < k - w + 2s + 1 - d - \lceil \frac{k+1}{d} \rceil$. Since $s \leq \lfloor \frac{2k}{d} \rfloor$, we have $n < k + \lfloor \frac{3k-1}{d} \rfloor + 1 - d$, a contradiction to the condition $n \geq k + 1 - d + 3k/d$. ■

By Lemma 1, for every $i = 1, \ldots, k - m - w$, we can assign a vertex $z_i \in Z$ to u_i and a vertex $z'_i \in Z$ to v_i so that the assigned vertices in Z are all distinct. Also, for every $k - w - m + 1 \leq i \leq k - w$, let y_i be the common neighbor of u_i and v_i corresponding to the matching P above.

Lemma 2. *There exists an assignment $\mathcal{A}$ such that every z_i is adjacent to u_i and every z'_i is adjacent to v_i.*

Proof. For $i = 1, \ldots, k - m - w$, let $X_i = \{u_i, v_i, z_i, z'_i\}$, for $i = k - w - m + 1, \ldots, k - w$, let $X_i = \{u_i, v_i, y_i\}$, and for $i = k - w + 1, \ldots, k$, let $X_i = \{u_i, v_i\}$. Let $X = \bigcup_{i=1}^{k} X_i$ and $Y = \{y_{k-w-m+1}, \ldots, y_{k-w}\}$.

For $i \leq k - m - w$, u_i (or v_i) is *senior* if it has at least $k - w - m$ neighbors outside of $S \cup Y$, otherwise it is *junior*.

Claim 3. *The vertices $z_1, z'_1, \ldots, z_{k-m-w}, z'_{k-m-w}$ can be chosen so that for each senior vertex u_i (respectively, v_i), z_i is adjacent to u_i (respectively, z'_i is adjacent to v_i).*

Proof. Consider the auxiliary bipartite graph $F = (F_1, F_2; E_F)$, where F_1 is the set of senior vertices (taken with multiplicities) and $F_2 = V(G) - S - Y$. We join a vertex in F_1 with a vertex in F_2 if the corresponding vertices are adjacent in G. Then our claim is equivalent to the existence of a matching saturating F_1 in F.

Suppose that F has no matching saturating F_1. Then by Hall's Theorem there exists $T \subset F_1$ such that $|N_F(T)| \leq |T| - 1$. By the definition of senior vertices, $|T| > |N_F(T)| \geq k - w - m$. Then since $|F_1| \leq 2(k - w - m)$, there is some i such that $u_i, v_i \in T$. Note that $(N(u_i) \cap N(v_i)) - S - Y = \emptyset$ by the maximality of m. Therefore, $|N_F(T)| \geq (k - w - m) + (k - w - m) \geq |F_1|$, a contradiction to the choice of T. ■

Among choices of z_i and z'_i satisfying Claim 3, choose one with the maximum number of edges of the kind $z_i u_i$ and $z'_i v_i$. Suppose that $u_1 z_1 \notin E(G)$. Then u_1 is junior and hence has at most $k - w - m - 1$ neighbors outside of $S \cup Y$. By the choice, u_1 has no neighbors outside of X.

Claim 4.

$$|N(u_1) \cap (X - S)| + |N(z_1) \cap S| \leq k(1 + 1/d) - 1 - w\frac{d-1}{d}.$$

Proof. For $i = 1, \ldots, k$, let p_i denote the sum of the number of neighbors of u_1 in $\{z_i, z'_i\}$ (if $k - w - m + 1 \leq i \leq k - w$, then $z_i = z'_i = y_i$; if $i \geq k - w + 1$, then $\{z_i, z'_i\} = \emptyset$) plus β_i if $u_i \in N_G(z_1)$ and plus γ_i if $v_i \in N_G(z_1)$. By the definition,

$$\sum_{i=1}^{k} p_i = |N(u_1) \cap (X - S)| + |N(z_1) \cap S|.$$

Thus, we need to estimate $\sum_{i=1}^{k} p_i$. For $j = 0, 1, 2$, let

$$I_j = \{i : u_1 \text{ has exactly } j \text{ neighbors in } \{z_i, z'_i\}\ \}.$$

If $i \in I_0$, then $p_i \leq \beta_i + \gamma_i \leq 2/d$. Moreover, if $i \leq k - w - m$, then z_1 cannot be adjacent to both u_i and v_i by the maximality of m. Hence $p_i \leq 1/d$ if $i \leq k - m - w$.

If $i \in I_1$, then z_1 cannot be adjacent to both u_i and v_i, since otherwise we switch z_1 with the element of $\{z_i, z_i'\}$ adjacent to u_1. Thus, in this case, $p_i \le 1 + \max\{\beta_i, \gamma_i\} \le 1 + 1/d$.

Let $i \in I_2$. If z_1 is adjacent to, say, u_i, then we switch z_1 with z_i and get a better assignment. Thus, in this case, $p_i = 2$.

Since u_1 is junior, $|I_2| \le |I_0| - 1 - w$ and hence some $i' \le k - m - w$ belongs to I_0. It follows that

$$\sum_{i=1}^{k} p_i \le p_{i'} + \frac{2}{d}(|I_0| - 1) + \frac{d+1}{d}|I_1| + 2|I_2|$$
$$\le p_{i'} + w\frac{2}{d} + \frac{d+1}{d}(k - 1 - w) \le \frac{d+1}{d}k - 1 - w\frac{d-1}{d}.$$

This proves the claim. ■

By Claim 2, $|N(u_1) \cap S| \le s - 1 - d + w/2$. Since $|N(u_1) \cap (V(G) - X)| = \emptyset$, Claims 2 and 4 yield

$$\begin{aligned}\deg(u_1) + \deg(z_1) &= |N(u_1) \cap S| + |N(u_1) \cap (X - S)| \\ &\quad + |N(z_1) \cap (V(G) - S)| + |N(z_1) \cap S| \\ &\le (s - 1 - d + \frac{w}{2}) + |N(u_1) \cap (X - S)| \\ &\quad + (n - 1 - s) + |N(z_1) \cap S| \\ &\le n - 2 - d + \frac{w}{2} + k(1 + 1/d) - 1 - w\frac{d-1}{d},\end{aligned}$$

a contradiction to $\deg(u_1) + \deg(z_1) > n + k(1 + 1/d) - 3 - d$. ■

Lemma 3. *The assignment $\mathcal{A}$ in Lemma 2 can be chosen in such a way that for every $1 \le i \le k - m - w$, z_i is adjacent to z_i'.*

Proof. Choose an assignment $\mathcal{A}$ satisfying Lemma 2 so that as many as possible z_i are adjacent to corresponding z_i'. Define $X_i (i = 1, \ldots, k)$, X, and Y as in the proof of Lemma 2. We may renumber (u_i, v_i) so that, for some $0 \le l \le k - w - m$, we have $z_i z_i' \in E(G)$ if $l + 1 \le i \le k - w - m$ and $z_i z_i' \notin E(G)$ if $1 \le i \le l$. If $l = 0$, then the lemma holds. Suppose that $l \ge 1$.

For $i = 1, \ldots, k$, let q_i denote the number of neighbors of X_1 (with multiplicities) in $\{z_i, z_i'\}$ (if $k - w - m + 1 \le i \le k - w$, then $z_i = z_i' = y_i$) plus β_i times the number of neighbors of u_i in $\{z_1, z_1'\}$ and plus γ_i times the number of neighbors of v_i in $\{z_1, z_1'\}$. By the definition, $\sum_{i=1}^{k} q_i$ is equal to the total number of neighbors of X_1 in X (counted with multiplicities) minus the total number of the neighbors of the set $\{u_1, v_1\}$ in S (also counted with multiplicities).

Since $l \geq 1$, each member of X_1 has exactly one neighbor in X_1, and hence $q_1 = 2 + \beta_1 + \gamma_1$. Clearly, $q_i \leq 2(\beta_i + \gamma_i) \leq 4/d$ for $k - w + 1 \leq i \leq k$.

Claim 5.

$$q_i \leq \begin{cases} 6 + 2/d, & \text{for } 2 \leq i \leq k - w - m, \\ 4 + 2/d, & \text{for } k - w - m + 1 \leq i \leq k - w. \end{cases}$$

Proof.

Case 1. $2 \leq i \leq k - w - m$. By the maximality of m, neither of z_1 and z_1' is a common neighbor of u_i and v_i and neither of z_i and z_i' is a common neighbor of u_1 and v_1. Thus, $q_i \leq 6 + 2\max\{\beta_i, \gamma_i\} \leq 6 + 2/d$.

Case 2. $k - w - m + 1 \leq i \leq k - w$. If u_1 or v_1 is not adjacent to y_i, then $q_i \leq 3 + 2(\beta_i + \gamma_i) \leq 4 + 2/d$ and we are done. Thus, we may assume that $u_1 y_i, v_1 y_i \in E(G)$. If $q_i > 4 + 2/d$, then z_1 and z_1' together contribute more than $2 + 2/d$ to q_i. In this case, either z_1 or z_1' is adjacent to both u_i and v_i. This contradicts the maximality of m. So, $q_i \leq 4 + 2/d$. ■

Claim 6. *For each $v \notin X$, $|N(v) \cap \{u_1, v_1, z_1, z_1'\}| \leq 2$.*

Proof. Otherwise, we can swap v with either z_1 or z_1' so that the new assignment is better than $\mathcal{A}$. ■

Let $F = \deg(u_1) + \deg(v_1) + \deg(z_1) + \deg(z_1')$. Since $u_1 v_1 \notin E(G)$ and $z_1 z_1' \notin E(G)$, we have $F > 2n + 2k(1 + 1/d) - 6 - 2d$. On the other hand, in view of the claims above, and the fact that for every $k - w + 1 \leq j \leq k$, $|\{u_1, v_1\} \cap \{u_j, v_j\}| \leq 1$, we have

$$\begin{aligned} F &\leq 2(n - |X|) + \sum_{i=1}^{k} q_i + \big(2(s-1) - (|N_H(u_1^0)| + |N_H(v_1^0)|) + w\big) \\ &\leq 2(n - s - 2(k - w - m) - m) + 2 + 2/d + (k - w - m - 1)(6 + 2/d) \\ &\quad + m(4 + 2/d) + 4w/d + (2(s-1) - 2d + w) \\ &\leq 2n + 2k(1 + 1/d) - 6 - 2d - w(1 - 2/d) \\ &\leq 2n + 2k(1 + 1/d) - 6 - 2d. \end{aligned}$$

This contradiction proves Lemma 3 and hence Theorem 2. ■

3. UPPER BOUNDS FOR THEOREM 1

Let $d = 2$. If $k = 3$, then the statement follows from the original result of Ore [15].

Let $k \geq 4$. Analyzing the proof of Theorem 2, we find that in order to prove Theorem 1 we need to modify only the proof of Lemma 1. In this section, we prove this lemma under conditions of Theorem 1.

Proof of Lemma 1. Choose $(u_1, v_1) \in Q$ such that u_1 and v_1 together have the minimum number of neighbors in S (with multiplicities) among all pairs in Q. One of u_1 and v_1 is not adjacent to at least half of vertices of L. We may assume that v_1 is this vertex and v_1 is not adjacent to $L_1 \subseteq L$ with $|L_1| \geq 0.5|L|$. Let x be the number of non-neighbors of v_1 in S. We have $1 \leq x \leq |N_H(v_1^0)|$, since $u_1v_1 \notin E(G)$. Thus

$$\deg(v_1) \leq n - 1 - \left(x + \frac{n - s + k - q - m - w}{2}\right). \tag{3}$$

Let $u \in L_1$ and $T = Q' - N(u)$. Let $|T| = t$. Then at least one end of each pair of Q is not adjacent to u, that is, every pair in Q should have at least one end in T. This means that

$$\sum_{v \in f^{-1}(T)} \deg_{H[f^{-1}(Q')]}(v) \geq q. \tag{4}$$

Since $\delta(H) \geq 2$, we have

$$\sum_{v \in f^{-1}(T)} \deg_{H[f^{-1}(Q')]}(v) + 2(s - t) \leq 2k, \tag{5}$$

that is,

$$t \geq q/2 + s - k. \tag{6}$$

By definition,

$$\deg(u) \leq n - 1 - t. \tag{7}$$

Therefore,

$$\deg(v_1) + \deg(u) \leq 2(n - 1) - \left(x + \frac{n - s + k - q - m - w}{2}\right) - (q/2 + s - k). \tag{8}$$

Since $uv_1 \notin E(G)$,

$$\deg(v_1) + \deg(u) \geq n + (3k - c)/2, \tag{9}$$

where $c = 8$ if $n \leq 2.5k - 5.5$ and $c = 9$ otherwise. Thus, we have

$$2(k - m - w) \leq n - s - m - w + (c - 4) - 2x. \tag{10}$$

If $2(k - m - w) \leq n - s - m$, we are done. Otherwise, $c - 4 - w - 2x \geq 1$.

If $c = 8$, then $w + 2x \leq 3$. Note that $w + x \geq 2$ and $x \geq 1$, $x = w = 1$ and in the above argument, $|L_1| = 0.5|L|$ and we also achieve equalities in (3)–(10). Since

$|L_1| = 0.5|L|$, the roles of u_1 and v_1 are interchangeable. But when we consider u_1, since H is simple, $x \geq 2$. Therefore, $2(k - m - w) \leq n - s - m$.

Now let $c = 9$ and $n > 2.5k - 5.5$. Note that since $w + x \geq 2$, we have the following two cases.

Case 1. $(x, w) = (2, 0)$, or $(x, w) = (1, 2)$. In either situation, $2(k - w - m) = n - s - m + 1$. Moreover, $|L_1| = 0.5|L|$ and we achieve equalities in (3)–(10). More specifically:

(I) Since $|L_1| = 0.5|L|$, the roles of u_1 and v_1 are interchangeable.
(II) By (5), in H, every vertex in $f^{-1}(S - T)$ has degree 2.
(III) By (6), q is even and $t = q/2 + s - k$. By (9), k is odd. Thus $|Q| \leq k - 3$.
(IV) By (7), (8), $\deg(v_1) = n - 1 - x - |L|/2$ and $\deg(u) = n - 1 - t$.
(V) By (10), $2(k - m - w) = n - s - m - w + 1$, that is, $n = 2k + s - m - w - 1$.

If $(x, w) = (1, 2)$, then u_1 has exactly one non-neighbor in S too, otherwise, instead of considering v_1, we consider u_1 and thus $x = w = 2$, we are done. Hence, either of u_1^0 or v_1^0 has degree 2 in H. If $q \geq 3$, then there is a pair (u_1', v_1') in Q with more non-neighbors in S than (u_1, v_1), a contradiction. Thus $q = 2$. But then $t \leq 1$. Thus, $\deg(u) \geq n - 2$ and u is a common neighbor of another pair in Q, a contradiction to the choice of Q.

If $(x, w) = (2, 0)$, then in H, u_1^0, v_1^0 both have degree 2. In fact, we may choose any pair of Q and the same argument works. Thus, every end of an edge in Q has degree 2 in H. Together with (II), this yields that H is 2-regular. Hence $s = k$. Therefore, $t = q/2$ and $\deg(u) = n - 1 - q/2$, that is, u is adjacent to every vertex of $S - Q$. Note that for every $v \in Q$, $\deg(v) = n - 1 - |L|/2$, that is, v is adjacent to every vertex of R. Observe that $|R| \geq n + (3k - 9)/2 - (n - 2) - (k - 2) = (k + 3)/2$. Thus by V), $n = 3k - 1 - m = 3k - 1 - (k - |Q| + |R|) = 2k - 1 + |Q| - |R| \leq 2k - 1 + (k - 3) - (k + 3)/2 = 2.5k - 5.5$, a contradiction.

Case 2. $w = x = 1$. Then $v_1 u_j \in E(G)$ for some j. We observe that since $w = 1$, u_1 and v_1 have at most $s - 3 + s - 2 = 2s - 5$ neighbors in S (counting with multiplicities). Then u_1 and v_1 together have at least $n + (3k - 9)/2 - 2s + 5$ edges to $V(G) - S$, and hence at least $(3k + 1)/2 - s \geq (k + 1)/2$ common neighbors in $V(G) - S$. It follows that $q \geq 1 + (k + 1)/2 \geq 3$. Thus, we are able to choose a pair in Q such that either each end of the pair has at least 2 non-neighbors in S, or one end of the pair is u_j, and the other end has fewer neighbors in S, a contradiction to the choice of u_1, v_1. ■

4. EXAMPLES

In this section, we give three examples to prove the lower bounds of Theorems 2 and 1.

Example 1. Let $d \geq 2$. Let $V(G) = Q_1 \cup Q_2 \cup L \cup T$, where $|Q_1| = |Q_2| = \lfloor k/d \rfloor$, $|L| = k - 1$, and $|T| = n - 2\lfloor k/d \rfloor - k + 1$. Since $k \geq d^2$, there exists a bipartite simple graph $G_1 = (Q_1, Q_2; E_1)$ such that the degrees of all vertices are at least d and at most $d + 1$. Moreover, if k/d is an integer, then there exists a d-regular bipartite graph $G_1 = (Q_1, Q_2; E_1)$. Let the complement of our graph G be the union of the complete bipartite graph $G(Q_1, T)$ with the partite sets Q_1 and T and the graph G_1. If d divides k, then each vertex in Q_1 has degree $2k/d - 1 - d + k - 1$, each vertex in T has degree $n - 1 - k/d$, and the degree of each vertex in Q_2 is $n - 1 - d$. Since $k \geq d^2$, when d divides k, we have

$$\begin{aligned}\sigma_2(G) &= (2k/d - 1 - d + k - 1) + min\{n - 1 - k/d, n - 1 - d\} \\ &= n + k(1 + 1/d) - 3 - d.\end{aligned}$$

If d does not divide k, then each vertex in Q_1 has degree at least $2\lfloor k/d \rfloor - 1 - d - 1 + k - 1$, each vertex in T has degree $n - 1 - \lfloor k/d \rfloor$, and each vertex in Q_2 has degree at least $n - 1 - d - 1$. Since $k \geq d(d + 1)$, we have $n - 1 - \lfloor k/d \rfloor \leq n - d - 2$ and therefore

$$\sigma_2(G) = n + k + \left\lfloor \frac{k}{d} \right\rfloor - 4 - d = n + k + \left\lceil \frac{k+1}{d} \right\rceil - 5 - d.$$

Take H to be the bipartite graph $G_1 = (Q_1, Q_2; E_1)$. We claim that G has no H-subdivision in which the branch vertices are the original vertices of H. If G had such a subdivision, then every path of this subdivision corresponding to an edge in H would contain a vertex in L, but $|L| < k$.

This example shows that $(n + k(1 + 1/d) - 3 - d) + 1$ is a lower bound for $R_{\mathcal{H}(k,d)}(n)$ for each $n \geq 2k$ if $k \geq d^2$ and $d \geq 2$ divides k. When $d \geq 3$ does not divide k and $k \geq d(d + 1)$, then the example yields the bound $R_{\mathcal{H}(k,d)}(n) \geq n + k - d - 4 + \lceil \frac{k+1}{d} \rceil$.

Example 2. Let $d = 2$, $k \geq 3$ be odd, $0 \leq r \leq k - 3$, and $2k \leq n \leq 2.5k - 5.5$. Let $V(G) = T \cup Q_1 \cup Q_2 \cup L_1 \cup L_2 \cup R$, where $|T| = 3$, $|Q_1| = |Q_2| = (k - 3)/2$, $|L_1| = |L_2| = k - 2 - r$, and $|R| = r$. Define the complement, $\overline{G}$, of G as follows:

$$\begin{aligned}E(\overline{G}) = \{uv : u, v \in T\} &\cup \{uv : u \in Q_1, v \in L_1\} \\ &\cup \{uv : u \in Q_2, v \in L_2\} \cup E(C_{k-3}),\end{aligned}$$

where C_{k-3} is a spanning cycle in $Q_1 \cup Q_2$ and vertices of Q_1 and Q_2 alternate on C_{k-3}.

For $u \in Q_1 \cup Q_2$, $\deg(u) = (k - 3) + r + (k - 2 - r) = 2k - 5$; for $v \in L_1 \cup L_2$, $\deg(v) = n - 1 - (k - 3)/2$; for each other vertex u, $\deg(u) \geq n - 3$. Thus, $\sigma_2(G) = \min\{2k - 5 + n - 1 - (k - 3)/2, 2(2k - 5)\}$. For $n \leq 2.5k - 5.5$, we have $\sigma_2(G) = n + (3k - 9)/2$.

Let $H = C_3 \cup C_{k-3}$ and take T and $Q_1 \cup Q_2$ as the sets of branching vertices for C_3 and C_{k-3}, respectively. We claim that G has no H-subdivision with these branch vertices. Indeed, each path corresponding to an edge in C_{k-3} contains either a vertex in R or a vertex in L_1 plus a vertex in L_2. Each path corresponding to an edge in C_3 contains a vertex in $R \cup L_1 \cup L_2$. If we spend all r vertices in R for paths corresponding to edges in C_{k-3}, we still need $3 + 2(k-3-r) = 2k-3-2r$ vertices from $L_1 \cup L_2$, but have there only $2k-4-2r$ of them.

Note that $n(G) = k + r + 2(k-2-r) = 3k-4-r$. Thus, this example shows that $R_{\mathcal{H}(k,d2)}(n) > n + (3k-9)/2$ for each $n \in [2k-1, 2.5k-5.5]$ and odd $k \geq 3$.

Example 3. Let $d = 2$, $k \geq 3$ be odd, and $n > 2.5k - 5.5$. An example for $k = 3$ and $H = C_3$ is an n-vertex graph G that is the union of two complete graphs sharing exactly one vertex. This graph G has no cycle through two vertices separated by the cut vertex and $\sigma_2(G) = n(G) - 1$.

Let $k \geq 5$ and $H = C_k$. Let $V(G) = T \cup Q_1 \cup Q_2 \cup L \cup T$, where $|Q_1| = (k-1)/2$, $|Q_2| = (k+1)/2$, $|L| = k-2$, and $|T| = n - 2k + 2$. Define the complement, $\overline{G}$, of G as follows:

$$E(\overline{G}) = \{uv : u \in Q_1, v \in T\} \cup E(C_k),$$

where C_k is a spanning cycle in $Q_1 \cup Q_2$ and vertices of Q_1 and Q_2 alternate on C_k apart from one edge of C_k connecting two vertices in Q_2.

Each vertex in Q_1 has degree $(k-3) + (k-2)$, each vertex in T has degree $n - 1 - (k-1)/2$, each vertex in Q_2 has degree $n-3$, and vertices in L are all-adjacent. Since $k \geq 5$, we have

$$\sigma_2(G) = (2k-5) + \min\{n - 1 - (k-1)/2, n-3\} = n + (3k-11)/2.$$

We claim that G has no H-subdivision with $Q_1 \cup Q_2$ as the set branch vertices arranged so that no edge of G connects the images of adjacent vertices of H. Indeed, each path in G corresponding to an edge in H, apart from one, should contain a vertex in L, but $|L| = k-2$, a contradiction.

This shows that $R_{\mathcal{H}(k,2)}(n) \geq n + (3k-9)/2$ for each $n > 2.5k - 5.5$ and odd $k \geq 3$.

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