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ON EQUITABLE COLORING OF d-DEGENERATE GRAPHS*

A. V. KOSTOCHKA†, K. NAKPRASIT‡, AND S. V. PEMMARAJU§

Abstract. An *equitable coloring* of a graph is a proper vertex coloring such that the sizes of any two color classes differ by at most 1. A *d-degenerate graph* is a graph G in which every subgraph has a vertex with degree at most d. A star S_m with m rays is an example of a 1-degenerate graph with maximum degree m that needs at least $1 + m/2$ colors for an equitable coloring. Our main result is that every n-vertex d-degenerate graph G with maximum degree at most $n/15$ can be equitably k-colored for each $k \geq 16d$. The proof of this bound is constructive. We extend the algorithm implied in the proof to an $O(d)$-factor approximation algorithm for equitable coloring of an *arbitrary* d-degenerate graph. Among the implications of this result is an $O(1)$-factor approximation algorithm for equitable coloring of planar graphs with fewest colors. A variation of equitable coloring (equitable partitions) is also discussed.

Key words. graph coloring, equitable coloring, d-degenerate graphs

AMS subject classifications. 05C15, 05C35, 05C85

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1. Introduction.

An *equitable coloring* of a graph is a proper vertex coloring such that the sizes of every two color classes differ by at most 1. Equitable colorings naturally arise in some scheduling, partitioning, and load balancing problems [1, 2, 18, 23, 8, 24]. Pemmaraju [21] and Janson and Ruciński [11] used equitable colorings to derive deviation bounds for sums of dependent random variables that exhibit limited dependence. Subsequently, Janson [9] explored equitable colorings with applications to U-statistics, random strings, and random graphs. In these applications, the fewer colors we use, the better.

In contrast with ordinary coloring, a graph may have an equitable k-coloring (i.e., an equitable coloring with k colors) but no equitable $(k + 1)$-coloring. It is easy to check that the complete bipartite graph $K_{7,7}$ has an equitable k-coloring for $k = 2, 4, 6$ and $k \geq 8$ but has no equitable k-coloring for $k = 3, 5, 7$. For a graph G, let $\mathrm{eq}(G)$ denote the smallest k_0 such that G is equitably k-colorable for every $k \geq k_0$.

Finding $\mathrm{eq}(G)$ even for planar graphs G is an NP-complete problem. In particular, determining if a given planar graph with maximum vertex degree 4 has an equitable coloring using at most 3 colors is NP-complete. This can be seen as follows. It is known [6] that determining if a planar graph with maximum vertex degree 4 is 3-colorable is NP-complete. For a given n-vertex planar graph G with maximum vertex degree 4, let G' be obtained from G by adding $2n$ isolated vertices. Then G is 3-colorable if and only if G' is equitably 3-colorable.

This NP-completeness result motivates a series of extremal problems on equitable

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†Department of Mathematics, The University of Illinois, Urbana, IL 61801, and Institute of Mathematics, Novosibirsk, Russia (kostochk@math.uiuc.edu). This author's work was partially supported by NSF grant DMS-0099608 and by grants 02-01-00039 and 00-01-00916 of the Russian Foundation for Basic Research.

‡Department of Mathematics, The University of Illinois, Urbana, IL 61801 (nakprasi@math.uiuc.edu). This author's work was partially supported by NSF grant DMS-0099608.

§Department of Computer Science, The University of Iowa, Iowa City, IA 52242 (sriram@cs.uiowa.edu). This author's work was partially supported by NSF grant DMS-0213305.

colorings. A typical problem would ask us to show that if a graph G is "sparse," then $\mathrm{eq}(G)$ is "small." Here "sparse" might mean that G has a small maximum degree, or small average degree, or is d-degenerate for a small d. Recall that a graph G is *d-degenerate* if every subgraph G' of G has a vertex with degree (in G') at most d. It is well known that forests are exactly 1-degenerate graphs, outerplanar graphs are 2-degenerate, and planar graphs are 5-degenerate. By definition, the vertices of every d-degenerate graph can be ordered $v_1, \ldots, v_n$ in such a way that for every $i \geq 2$, vertex v_i has at most d neighbors v_j with $j < i$.

Hajnal and Szemerédi [7] considered the first version of "sparseness" of a graph. They settled a conjecture of Erdős by proving that every graph G with maximum degree at most Δ has an equitable k-coloring for every $k \geq 1 + \Delta$. In other words, they proved that $\mathrm{eq}(G) \leq \Delta(G) + 1$ for every graph G. In its "complementary" form, this result concerns the decomposition of a sufficiently dense graph into cliques of equal size, which has been used in a number of applications of Szemerédi's regularity lemma [13]. The bound of the Hajnal–Szemerédi theorem is sharp, but it can be improved for some important classes of graphs. In fact, Chen, Lih, and Wu [5] conjectured that every connected graph G with maximum degree $\Delta \geq 2$ has an equitable coloring with Δ colors, except when G is a complete graph or an odd cycle or Δ is odd and $G = K_{\Delta,\Delta}$. They proved the conjecture for graphs with maximum degree at most 3. Lih and Wu [19] proved the conjecture for bipartite graphs and Yap and Zhang [25, 26] proved that the conjecture holds for outerplanar graphs and planar graphs with maximum degree at least 13. In an unpublished paper, Nakprasit extended the result of Yap and Zhang [26] to planar graphs with maximum degree at least 9.

If a graph G has moderate maximum degree Δ and, in addition, is d-degenerate for a small d, then one can get a somewhat better than Δ bound on $\mathrm{eq}(G)$. Meyer [20] proved that every forest (i.e., 1-degenerate graph) with maximum degree Δ has an equitable coloring with $1 + \lceil \Delta/2 \rceil$ colors. This bound is attained at the star S_m with m rays: in every proper coloring of S_m, the center vertex forms a color class, and hence the remaining vertices need at least $m/2$ colors. Kostochka and Nakprasit [15] obtained the upper bound $\mathrm{eq}(G) \leq (d + \Delta + 1)/2$ for d-degenerate graphs with maximum degree Δ in the case $\Delta \geq 27d$. This bound is also sharp.

Bollobás and Guy [4] initiated a new and important direction of research for equitable colorings. They showed that while $1 + \lceil \Delta/2 \rceil$ is a tight upper bound on the equitable chromatic number of trees, "most" trees can be equitably 3-colored. Their result implies that each n-vertex forest F with $\Delta(F) \leq n/3$ can be equitably 3-colored. This result seems to uncover a fundamental phenomenon in equitable colorings: apart from some "star-like" graphs, most graphs admit equitable colorings with few colors. Another example of this phenomenon was given by Pemmaraju [22]. He showed that every n-vertex outerplanar graph G with $\Delta(G) \leq n/6$ can be equitably 6-colored. In this paper we show that this phenomenon is widely pervasive.

Our main result is the following.

THEOREM 1. *For $d, n \geq 1$, every d-degenerate, n-vertex graph G with $\Delta \leq n/15$ is equitably k-colorable for each $k \geq 16d$.*

The proof of Theorem 1 is constructive and provides an $O(d)$-factor approximation algorithm for equitable coloring with fewest colors of each d-degenerate n-vertex graph G with $\Delta \leq n/15$. Furthermore, many d-degenerate graphs need at least $\Omega(d)$ colors for ordinary coloring, and for such graphs our algorithm gives a constant factor (independent of d) approximation. Then we extend the algorithmic side of Theorem 1 to *all* d-degenerate graphs and show the following.

THEOREM 2. *There exists a polynomial time algorithm that for every equitably s-colorable d-degenerate graph G produces an equitable k-coloring of G for any $k \geq 31ds$.*

The result of Theorem 2 was already used by Bodlaender and Fomin [3] for constructing a polynomial time algorithm for equitable coloring of graphs with a bounded tree width. Theorem 2 gives an $O(d)$-factor approximation algorithm for the problem of the equitable coloring of a d-degenerate graph with fewest colors. For some classes of graphs such as planar graphs, this translates into an $O(1)$-factor approximation algorithm.

The technique used for the proof of Theorem 1 allows us to treat the following variation of equitable coloring. An *equitable k-partition* of a graph G is a collection of subgraphs $\{G[V_1], G[V_2], \dots, G[V_k]\}$ of G induced by the vertex partition $\{V_1, V_2, \dots, V_k\}$ of $V(G)$ where, for every pair V_i and V_j, the sizes of V_i and V_j differ by at most 1. Certainly, every equitable coloring is an equitable partition. Pemmaraju [22] proved that every outerplanar graph has an equitable partition into two forests.

THEOREM 3. *Let $k \geq 3$ and $d \geq 2$. Then every d-degenerate graph has an equitable k-partition into $(d-1)$-degenerate graphs.*

This is an extension of the Bollobás–Guy result [4], which essentially asserts the same for $d = 1$ and $k = 3$. Note that there is no restriction on the maximum degree of a graph in Theorem 3, while such a restriction is important in the Bollobás–Guy theorem.

2. Coloring d-degenerate graphs with $O(d)$ colors. An enumeration $v_1, v_2, \dots, v_n$ of the vertices of a graph G is a *greedy enumeration* (or a *greedy order*) if for every i, $1 \leq i \leq n$, the vertex v_i is a vertex of maximum degree in $G - v_1 - \cdots - v_{i-1}$. Similarly, the enumeration or order is *degenerate* if for every i, $1 \leq i \leq n$, the vertex v_i has minimal degree in $G(\{v_1, \dots, v_i\})$. Note that if $v_1, v_2, \dots, v_n$ is a greedy order on G, then $v_i, v_{i+1}, \dots, v_n$ is a greedy order on $G - v_1 - \cdots - v_{i-1}$, and that if $v_1, v_2, \dots, v_n$ is a degenerate order on G, then $v_1, v_2, \dots, v_i$ is a degenerate order on $G - v_{i+1} - \cdots - v_n$.

If G is d-degenerate, then, by the very definition, in every degenerate order $v_1, v_2, \dots, v_n$ of G, every v_i has at most d neighbors v_j with $j < i$.

The main result of section 2 is Theorem 1 whose statement we repeat below for the reader's convenience.

THEOREM 4 (restatement of Theorem 1). *Every d-degenerate graph with maximum degree at most Δ is equitably k-colorable when $k \geq 16d$ and $n \geq 15\Delta$.*

Proof. Let G be a d-degenerate graph with vertex set V of size n and edge set $E(G)$. Let t be an integer such that $k(t-1) < n \leq kt$ and $k \geq 16d$.

Case 1. $t \leq 15$. We will color the vertices one by one in a degenerate order $v_1, \dots, v_n$ (with some recolorings). Suppose we cannot color vertex v_i. Let Z be the set of color classes containing neighbors of v_i. Since G is d-degenerate, $|Z| \leq d$. If a color class $M \notin Z$ has fewer than t vertices, then we can color v_i with M. Since $n \leq kt$, there is a color class $M_0 \in Z$ with at most $t-1$ vertices. If a vertex w in a color class $M \notin Z$ has no neighbors in M_0, then we can recolor w with M_0 and color v_i with M. Thus, each of the $(k - |Z|)t$ colored vertices outside of Z has a neighbor in M_0. Therefore,

$$(t-1)\Delta \geq (k-d)t\frac{15}{16}kt \geq \frac{15}{16}n.$$

Since $n \geq 15\Delta$, we have

$$(t-1)\frac{n}{15} \geq \frac{15}{16}n,$$

and hence $t - 1 \geq 15^2/16 > 14$, which contradicts the choice of t.

Case 2. $t \geq 16$. Let $t = \beta_1 4^m + \beta_2 4^{m-1} + \cdots + \beta_{m+1}$, where β_j is an integer, $0 \leq \beta_j \leq 3$. For $i = 1, 2, \ldots, m+1$, define $l_i = \beta_1 4^{i-1} + \beta_2 4^{i-2} + \cdots + \beta_i$. For notational convenience, let $l_0 = 0$. We have that $l_i = 4l_{i-1} + \beta_i$ for each $i = 1, 2, \ldots, m+1$ and also that $t = l_{m+1}$.

We now partition V into sets $C_1, C_2, \ldots, C_{m+1}$ and color the vertices in C_i at the ith phase of the algorithm. We use the values of $l_1, l_2, \ldots, l_m$ to control the sizes of these sets. For convenience, set $A_0 = B_0 = C_0 = \emptyset$. For each $i = 1, 2, \ldots, m$, we construct sets A_i and B_i and let $C_i = A_i \cup B_i$. We use C'_i to denote the vertices in the sets constructed thus far. In other words, for each $i = 0, 1, \ldots, m+1$, we let C'_i denote $\cup_{j=0}^{i} C_j$. For each $i = 1, 2, \ldots, m$, A_i is constructed by selecting vertices in $G - C'_{i-1}$ as follows. Arrange the vertices of $G - C'_{i-1}$ in a greedy ordering and let A_i be the first $(l_i - l_{i-1})k$ vertices in this ordering. B_i is selected from vertices in $G - C'_{i-1} - A_i$ as follows. Initially set $B_i = \emptyset$ and, while there is a vertex $w \in G - C'_{i-1} - A_i - B_i$ that has at least $13d$ neighbors in $A_i \cup B_i \cup C'_{i-1}$, add w to B_i. Repeat this process until every vertex $w \in G - C'_{i-1} - A_i - B_i$ has fewer than $13d$ neighbors in $C'_{i-1} \cup A_i \cup B_i$. This completes the construction of A_i and B_i and we simply set $C_i = A_i \cup B_i$. After constructing $C_1, C_2, \ldots, C_m$, we set $C_{m+1} = V(G) - C'_m$.

Now let $b_i = |B_i|$ for each $i = 0, 1, 2, \ldots, m$ and let $e(H)$ denote the number of edges in a graph H. It follows from our construction that for each $i = 0, 1, \ldots, m$,

$$e(G[C'_i]) \geq 13d \sum_{j=0}^{i} b_j.$$

On the other hand, $G[C'_i]$ is a d-degenerate graph and has $l_i k + \sum_{j=0}^{i} b_j$ vertices, and so $e(G[C'_i]) < (l_i k + \sum_{j=0}^{i} b_j)d$. It follows that $\sum_{j=0}^{i} b_j < (l_i k/12)$, or in other words, for each $i = 1, \ldots, m$,

$$|C'_i| < \frac{13}{12} l_i k. \tag{1}$$

Since $C'_{m+1} = V(G)$, we also know that $|C'_{m+1}| \leq tk = l_{m+1}k$.

We will color C_1 with k colors in such a way that each color class has at most $\lceil \frac{7}{6} l_1 \rceil$ vertices. We color vertices in C_1 one by one in a degenerate order. Hence when we color vertex $u \in C_1$, there are at least $k - d$ color classes that do not contain neighbors of u. Since

$$|C_1| < \frac{13 l_1 k}{12} \leq \frac{13 l_1 k}{12} \frac{16(k-d)}{15k} < \frac{7}{6} l_1 (k-d),$$

there exists a color class M of size less than $\frac{7}{6} l_1$ that does not contain neighbors of u. We color u with color M.

We now show how to color the rest of the sets $C_2, C_3, \ldots, C_{m+1}$. For $2 \leq i \leq m+1$, at the ith phase we start with G such that all vertices in C'_{i-1} have been colored. At this phase we will color the vertices in C_i in a degenerate order in such a way that (i) every color class is of size at most L_i, where $L_i = \lceil \frac{7}{6} l_i \rceil$ for $2 \leq i \leq m$, and $L_{m+1} = t$; (ii) the vertices in C'_{i-1} will *not* be recolored.

CLAIM 2.1. *For every* $i \geq 2$, $L_{i-1}/L_i \leq 2/5$.

Proof. Recall that $l_i \geq 4l_{i-1}$ for every $i \geq 2$. If $i = m+1$, then $L_i = l_i = t \geq 16$. Therefore,

$$\frac{L_m}{L_{m+1}} = \frac{\lceil 7l_m/6 \rceil}{t} \leq \frac{7l_m/6 + 5/6}{t} \leq \frac{7}{6 \cdot 4} + \frac{5/6}{16} = \frac{11}{32} < \frac{2}{5}.$$

If $2 \leq i \leq m$, then $L_i = \lceil \frac{7l_i}{6} \rceil$. If $l_{i-1} \geq 2$, then $l_i \geq 8$ and

$$\frac{L_{i-1}}{L_i} \leq \frac{7l_{i-1}/6 + 5/6}{7l_i/6} \leq \frac{1}{4} + \frac{5/6}{7 \cdot 8/6} = \frac{19}{56} < \frac{2}{5}.$$

Finally, if $l_{i-1} = 1$, then $L_{i-1} = 2$ and $L_i \geq 5$. This proves the claim. ∎

Suppose we want to color a vertex v. Let $M_1, \ldots, M_k$ be the current color classes. Let Y_0 denote the set of color classes of cardinality less than L_i. If some $M_j \in Y_0$ contains no neighbors of v, then we color v with M_j and work with the next vertex. Otherwise, let Y_0-*candidate* be a vertex $w \in V - C'_{i-1}$ such that there exists a color class $M(w) \in Y_0$, with $w \notin M(w)$ and $N_G(w) \cap M(w) = \emptyset$. Let Y_1 be the set of color classes containing a Y_0-candidate. If a member M_j of Y_1 does not contain a neighbor of v, then we color v with M_j and recolor some Y_0-candidate $w \in M_j$ with $M(w)$. For $h \geq 1$, let a Y_h-*candidate* be a vertex $w \in C_i - \cup_{M \in Y_0 \cup \cdots \cup Y_h} M$ such that there exists $M(w) \in Y_h$ with $N_G(w) \cap M(w) = \emptyset$. Let Y_{h+1} be the set of color classes containing a Y_h-candidate. If a member M_j of Y_{h+1} does not contain a neighbor of v, then we color v with M_j and similarly to the above recolor a sequence of candidates. Finally, let $Y = \cup_{j=0}^{\infty} Y_j$ and $y = |Y|$. Then by the above, Y possesses the following properties:

(a) Every color class in Y contains a neighbor of v.

(b) Every vertex $u \in C_i - \cup_{M \in Y} M$ has a neighbor in every $M \in Y$ (otherwise the color class of u would be in Y).

We will prove that there is at least one color class M in Y that does not contain neighbors of v. Suppose this is not the case.

Observe that each vertex $u \in C_i$ has less than $13d$ neighbors in C'_{i-1} (by the construction of B_{i-1}) and at the moment of coloring has at most d neighbors among vertices of C_i colored earlier (since vertices are considered in a degenerate order). So when we color a vertex $u \in C_i$, there are less than $(13+1)d$ color classes that have neighbors of u. By property (a) of Y, $y < 14d$.

CLAIM 2.2. $y < 8d/7$.

Proof. Let $S = \cup_{M \in Y} M$ and $T = C_i - S$. By property (b) of Y, at least $y|T|$ edges connect T with S. Since G is d-degenerate, we conclude that $y|T| < d(|S| + |T|)$, i.e., that $(y-d)|T| < d|S|$. Clearly, $|S| \leq yL_i$. By the definition of Y_0, every color class outside of Y_0 has size exactly L_i, and each $k - y$ color class outside of Y contains at most L_{i-1} vertices in C'_{i-1}. Hence

$$|T| \geq (k-y)(L_i - L_{i-1}).$$

By Claim 2.1, $\frac{L_i - L_{i-1}}{L_i} \geq 1 - \frac{2}{5} = \frac{3}{5}$ for every $i \geq 2$. Therefore,

$$(y-d)(k-y)\frac{3}{5} < dy.$$

Since $k \geq 16d$, the last inequality yields that $(y-d)(16d-y)\frac{3}{5} < dy$. This implies the following inequality for $\gamma = y/d$:

$$\gamma^2 - \frac{46}{3}\gamma + 16 > 0.$$

Therefore, either $\gamma > (23+\sqrt{385})/3 \sim 14.207\ldots$ or $\gamma < (23-\sqrt{385})/3 \sim 1.1261\ldots < 8/7$. The former is impossible since $y \leq 14d$, and thus the latter holds. This proves the claim. ∎

Subcase 2.1. $2 \leq i \leq m$. The total number of colored vertices is at least $L_i(k-y)$, which by Claim 2.2 is greater than

$$\left\lceil \frac{7l_i}{6} \right\rceil \left(k - \frac{8d}{7}\right) \geq \frac{7l_i}{6}\frac{13k}{14} = \frac{13l_ik}{12}.$$

This contradicts (1) for $j = i-1$.

Subcase 2.2. $i = m+1$. Let D_i be the highest degree in $G[V - C_i']$.

Claim 2.3. $l_1\Delta + (l_2 - l_1)D_1 + (l_3 - l_2)D_2 + \cdots + (l_{m+1} - l_m)D_m \leq 3\Delta + 4.25dt$.

Proof. Observe that

$$|E(G)| \geq \sum_{\substack{1\leq i\leq m\\ 1\leq j\leq l_ik}} \deg_{V-C'_{i-1}-\{v_1^i,\ldots,v_{j-1}^i\}}(v_j^i)\ldots.$$

By the definition of A_i, for $v_j^i \in A_i$,

$$\deg_{G[V-C'_{i-1}-\{v_1^i,\ldots,v_{j-1}^i\}]}(v_j^i) \geq D_i \quad\text{and}\quad |A_i| = (l_i - l_{i-1})k.$$

Thus,

$$|E(G)| \geq k(l_1D_1 + (l_2 - l_1)D_2 + (l_3 - l_2)D_3 + \cdots + (l_m - l_{m-1})D_m).$$

Since $|E(G)| < dn \leq dtk$, we have

$$l_1D_1 + (l_2 - l_1)D_2 + (l_3 - l_2)D_3 + \cdots + (l_m - l_{m-1})D_m < dt. \tag{2}$$

Note that

$$\frac{l_{i+1} - l_i}{l_i - l_{i-1}} = \frac{4l_i + \beta_{i+1} - l_i}{4l_{i-1} + \beta_i - l_{i-1}} \leq \frac{3(4l_{i-1} + \beta_i) + 3}{3l_{i-1} + \beta_i} = 4 + \frac{3 - \beta_i}{3l_{i-1} + \beta_i} \leq 4 + \frac{1}{l_{i-1}}.$$

For $i \geq 3$, we obtain $l_{i+1} - l_i \leq (4 + \frac{1}{4})(l_i - l_{i-1})$. Also $(l_2 - l_1) - 4.25l_1 = \beta_2 - 1.25l_1$. Therefore,

$$\begin{aligned}&4.25\,(l_1D_1 + (l_2 - l_1)D_2 + (l_3 - l_2)D_3 + \ldots + (l_m - l_{m-1})D_m)\\ &\quad\geq (l_2 - l_1)D_1 + (l_3 - l_2)D_2 + \cdots + (l_{m+1} - l_m)D_m + (1.25l_1 - \beta_2)D_1.\end{aligned}$$

Comparing with (2), we get

$$(l_2 - l_1)D_1 + (l_3 - l_2)D_2 + \cdots + (l_{m+1} - l_m)D_m < 4.25dt + \beta_2D_1 - 1.25l_1D_1.$$

Hence

$$l_1\Delta + (l_2 - l_1)D_1 + (l_3 - l_2)D_2 + \cdots + (l_{m+1} - l_m)D_m \leq l_1\Delta + 4.25dt + \beta_2D_1 - \frac{5}{4}l_1D_1.$$

In order to prove the claim it is now enough to show that

$$l_1\Delta + \beta_2D_1 - \frac{5}{4}l_1D_1 \leq 3\Delta. \tag{3}$$

Recall that $l_1 \leq 3$ and $\beta_2 \leq 3$. If $\beta_2 \leq \frac{5}{4}l_1$, then (3) is evident. If $\beta_2 > \frac{5}{4}l_1$, then

$$l_1\Delta + \beta_2 D_1 - \frac{5}{4}l_1 D_1 \leq l_1\Delta + \left(\beta_2 - \frac{5}{4}l_1\right)\Delta \leq \beta_2\Delta \leq 3\Delta.$$

This proves (3) and thus the claim. ◻

Let $M_1 \in Y_0$. By construction, every M_j contains at most L_i vertices in C'_i. So the number of neighbors of M_1 is at most

$$\begin{aligned}
&L_1\Delta + (L_2 - L_1)D_1 + \cdots + (L_{m+1} - L_m)D_m \\
&= \left\lceil \frac{7l_1}{6} \right\rceil \Delta + \left(\left\lceil \frac{7l_2}{6} \right\rceil - \left\lceil \frac{7l_1}{6} \right\rceil\right) D_1 + \cdots + \left(t - \left\lceil \frac{7l_m}{6} \right\rceil\right) D_m \\
&= \left\lceil \frac{7l_1}{6} \right\rceil (\Delta - D_1) + \left\lceil \frac{7l_2}{6} \right\rceil (D_1 - D_2) + \cdots + \left\lceil \frac{7l_m}{6} \right\rceil (D_{m-1} - D_m) + tD_m \\
&\leq \frac{7l_1}{6}(\Delta - D_1) + \frac{5}{6}(\Delta - D_1) + \frac{7l_2}{6}(D_1 - D_2) + \frac{5}{6}(D_1 - D_2) \\
&\quad + \cdots + \frac{7l_m}{6}(D_{m-1} - D_m) + \frac{5}{6}(D_{m-1} - D_m) + tD_m \\
&\leq \left(\frac{7l_1}{6} + \frac{5}{6}\right)\Delta + \frac{7}{6}\left((l_2 - l_1)D_1 + (l_3 - l_2)D_2 + \cdots + (l_{m+1} - l_m)D_m\right).
\end{aligned}$$

On the other hand, as in the proof of Claim 2.2, every color class outside of Y_0 has size exactly $L_{m+1} = t$, and each of the $k-y$ color classes outside of Y contains at most L_m vertices in C'_m. Hence, the number of neighbors of M_1 is at least $(k-y)(t-L_m)$. Note that

$$t - L_m = t - \left\lceil \frac{7l_m}{6} \right\rceil \geq t\left(1 - \frac{\frac{7l_m}{6} + \frac{5}{6}}{t}\right) = t\left(1 - \frac{7}{4\cdot 6} - \frac{5}{6\cdot 16}\right) = \frac{21}{32}t.$$

Hence by Claim 2.3 we have

$$(k-y)(t-L_m) \geq \left(k - \frac{8d}{7}\right)\frac{21}{32}t.$$

Comparing this with the upper bound above and applying Claim 2.3 we get

$$\left(k - \frac{8d}{7}\right)\frac{21}{32}t \leq \frac{5}{6}\Delta + \frac{7}{6}\left(3\Delta + 4.25dt\right).$$

Since $\Delta \leq n/15 \leq kt/15$, this reduces to

$$\left(k - \frac{8d}{7}\right)\frac{21}{32} \leq \frac{5}{6\cdot 15}k + \frac{7}{6}\left(\frac{3}{15}k + 4.25d\right),$$

which gives

$$\left(\frac{21}{32} - \frac{1}{18} - \frac{7}{6}\frac{1}{5}\right)k \leq \left(\frac{21}{32}\frac{8}{7} + \frac{7\cdot 4.25}{6}\right)d.$$

It follows that

$$\frac{k}{d} \leq \frac{68.5}{12}\frac{1440}{529} = \frac{8220}{529} < 15.6,$$

which contradicts $k \geq 16d$. This proves the theorem. ◻

ALGORITHM. The above proof implies a simple algorithm for equitable k-coloring of any n-vertex d-degenerate graph with $\Delta(G) \le n/15$. We first partition $V(G)$ into sets C_i, $1 \le i \le m+1$, as described in the first part of the proof. Then for each $i = 1, 2, \ldots, m+1$, we attempt to color vertices of C_i in degenerate order. It is possible that in the process some vertices may have to be recolored, but these recolorings are restricted to the set currently being colored, namely, C_i. The algorithm clearly runs in polynomial time and it can be implemented in $O(n^3)$ time; we do not give details here.

3. Constant-factor approximation algorithm. The algorithm above can be thought of as providing an $O(d)$-factor approximation algorithm for equitable coloring with fewest colors of an n-vertex d-degenerate graph with maximum degree at most $n/15$. In this section, we extend this to an $O(d)$-factor algorithm for equitable coloring of an *arbitrary* d-degenerate graph. This implies an $O(1)$-factor algorithm for planar graphs. The main result in this section is the following.

THEOREM 5. *Every n-vertex d-degenerate graph G with maximum degree at most Δ is equitably k-colorable for any k, $k \ge \max\{62d,\, 31d\frac{n}{n-\Delta+1}\}$.*

Proof. Let G be an n-vertex d-degenerate graph. Let $G_0 = G$, $h = 30d-1$ and, for $j = 1, \ldots, h$, let w_j be a vertex of the maximum degree in G_{j-1} and $G_j = G_{j-1} - w_j$.

CLAIM 3.1. *For every $v \in V(G_h)$, $\deg_{G_h}(v) < n/30$.*

Proof. If $\deg_{G_h}(v) \ge n/30$ for some $v \in V(G_h)$, then also $\deg_{G_{j-1}}(w_j) \ge n/30$ for every $j = 1, \ldots, 30d-1$, and hence $|E(G)| \ge 30d(n/30) = dn$. This is a contradiction, since any n-vertex d-degenerate graph has fewer than dn edges. ∎

CLAIM 3.2. *There are pairwise disjoint independent sets $M_1, M_2, \ldots, M_h$ such that for every j, $1 \le j \le h$,*

(i) $w_j \in \bigcup_{s=1}^{j} M_s$,

(ii) $\lfloor n/k \rfloor \le |M_j| \le \lceil n/k \rceil$, *and*

(iii) $nj/k \le \sum_{s=1}^{j} |M_s| < 1 + nj/k$.

Proof. Let $X_1 = V(G) - w_1 - N_G(w_1)$. Clearly, $|X_1| \ge n - \Delta - 1$. Since G is d-degenerate, X_1 contains an independent set M_1' of size at least $\frac{|X_1|}{d+1} \ge \frac{n-\Delta-1}{d+1}$. Since

$$\frac{n}{k} \le \frac{n-\Delta+1}{31d} < \frac{n-\Delta}{d+1},$$

$|M_1'| > \frac{n}{k} - \frac{1}{d+1}$. Hence, we can choose a subset M_1'' of M_1' of size $\lceil \frac{n}{k} \rceil - 1$ and let $M_1 = M_1'' + w_1$. By construction, M_1 satisfies properties (i)–(iii) for $j = 1$.

Suppose we have constructed $M_1, M_2, \ldots, M_{j-1}$ satisfying (i)–(iii) for some $j \le h$. Let $x_j = w_j$ if $w_j \notin \bigcup_{s=1}^{j-1} M_s$, and let x_j be any vertex outside $\bigcup_{s=1}^{j-1} M_s$ otherwise. Let $X_j = V(G) - \bigcup_{s=1}^{j-1} M_s - x_j - N_G(x_j)$. Since G is d-degenerate, X_j contains an independent set M_j' of size at least $\frac{|X_j|}{d+1}$. Suppose that $|M_j'| < -1 + n/k$. In view of (iii), this means that

$$\frac{n - 1 - (j-1)\frac{n}{k} - 1 - \Delta}{d+1} < \frac{n}{k} - 1.$$

For $n > k$ and $d \ge 1$, the last inequality yields $n - \Delta + 1 < \frac{(j+d)n}{k} + 1 < \frac{31dn}{k}$. But this contradicts the choice of k. Thus, we can choose a subset of M_j' that together with x_j forms an independent set M_j'' of size $\lceil n/k \rceil$. If

$$|M_j''| + \sum_{s=1}^{j-1} |M_s| < \frac{jn}{k} + 1,$$

then we let $M_j = M''_j$; otherwise we get M_j by deleting a vertex $v \neq x_j$ from M''_j. Note that in the latter case, $\lfloor n/k \rfloor \neq \lceil n/k \rceil$, and thus (i)–(iii) hold in both cases. This proves the claim. □

Let G' be the graph obtained by deleting vertices in $M_1 \cup M_2 \cup \cdots \cup M_h$ from G and let $V' = V(G')$.

CLAIM 3.3. $|V'| \geq 16n/31$.

Proof. By (iii) of Claim 3.2, $|V'| \geq n - (30d-1)n/k - 1 \geq n - 30dn/k$. Since $k \geq 62d$, we get $|V'| \geq 32n/62$. □

By Claims 3.1 and 3.3,

$$\frac{|V'|}{\Delta(G')} \geq \frac{32n}{62} \cdot \frac{30}{n} > 15.$$

Since $k-h \geq 62d - 30d = 32d$, by Theorem 1, G' is equitably $(k-h)$-colorable. Hence G is equitably k-colorable. This proves the theorem. □

COROLLARY 1. *Every d-degenerate graph with n vertices and maximum degree at most $1+n/2$ is equitably k-colorable when $k \geq 62d$.*

Now we are ready to prove Theorem 2, which we state again for convenience.

THEOREM 6 (restatement of Theorem 2). *There exists a polynomial time algorithm that, given a d-degenerate graph G with $\chi_{eq}(G) \leq s$, can equitably color G with k colors for any k, $k \geq 31ds$.*

Proof. Assume that a graph G on n vertices with maximum degree Δ admits an equitable coloring ϕ with s colors. Let $v \in V(G)$ have degree Δ. The color class of v contains at most $n - \Delta$ vertices. Thus no other color class can contain more than $n - \Delta + 1$ vertices. Hence,

$$s > \frac{n}{n - \Delta + 1}. \tag{4}$$

Also, if G has at least one edge, $s \geq 2$. If $\Delta \leq 1 + n/2$, then by Corollary 1 G can be equitably k-colored for any $k \geq 62d$. Since $62d \leq 31ds$, G can be equitably k-colored for any $k \geq 31ds$. If $\Delta > 1 + n/2$, then $31d\frac{n}{n-\Delta+1} > 62d$ and therefore by Theorem 5, G can be equitably k-colored for any $k \geq 31d\frac{n}{n-\Delta+1}$. It follows from inequality (4) that G can be equitably k-colored for any $k \geq 31ds$.

The fact that such an equitable k-coloring can be constructed in polynomial time is implied by the proof of Theorem 5. The algorithm is sketched here. First identify the high degree vertices $w_1, w_2, \ldots, w_h$ in G and construct the color classes $M_1, M_2, \ldots, M_h$ containing these vertices as in Claim 3.1. Construction of these color classes uses as a subroutine an algorithm that finds an independent set of size at least $m/(d+1)$ in a given m-vertex, d-degenerate graph. The following greedy algorithm suffices for this task: pick a minimum degree vertex, delete the vertex and its neighbors, and repeat until no vertices are left. Since at every step we deleted at most $d+1$ vertices, the number of steps will be at least $m/(d+1)$. Once the color classes $M_1, M_2, \ldots, M_h$ are constructed and the colored vertices are deleted, we are left with a graph whose maximum vertex degree is less than $n/30$. We color the vertices in this graph using the algorithm from the previous section. This phase dominates the running time of the algorithm, and hence we have an $O(n^3)$ algorithm. □

4. Equitable partitions of *d*-degenerate graphs. It is easy to see that any d-degenerate graph G can be partitioned into two $(d-1)$-degenerate graphs: construct a degenerate ordering and color the vertices in this order red or blue using the rule that a vertex v is colored red if it has less than d red neighbors; otherwise, color v blue. While this procedure leads to a partition into $(d-1)$-degenerate graphs, this partition need not be equitable. In fact, the only partition of the star S_m with m rays (which is 1-degenerate) into two independent sets (which are 0-degenerate) is that in which one set contains one vertex and the other contains the rest. Similarly, any partition of S_m into k 0-degenerate sets has one 1-element set and some set with at least m/k elements. In this section we show that if we have $d \geq 2$ and we allow for a third set, then we can provide equitability. This extends the Bollobás–Guy result [4] to arbitrary $d \geq 2$ and also provides a tool for obtaining equitable colorings that use few colors. Specifically, we will prove Theorem 3.

THEOREM 7 (restatement of Theorem 3). *Let $k \geq 3$ and $d \geq 2$. Then every d-degenerate graph can be equitably partitioned into k $(d-1)$-degenerate graphs.*

Proof. We prove the result by contradiction, assuming that the above claim is false. Let G be a smallest (with respect to the number of vertices) counterexample to the theorem. Let $n = |V(G)|$. Then $n > dk$, because otherwise, any equitable vertex partition is good enough. A simple observation that forms the basis of the proof is the following.

CLAIM 4.1. *Let $v_1, v_2, \dots, v_m$ be a d-degenerate vertex ordering of a d-degenerate graph H. If $H - v_m$ has a k-partition $(W_1, \dots, W_k)$, where every W_i induces a $(d-1)$-degenerate subgraph, then among $W_1 + v_m, \dots, W_k + v_m$ at most one is not $(d-1)$-degenerate. Furthermore, if $W_i + v_m$ is not $(d-1)$-degenerate, then v_m has d neighbors and W_i contains all d neighbors of v_m.*

Proof. By the definition of a d-degenerate vertex ordering, the degree of v_m is at most d. If W_i has fewer than d neighbors of v_m, then we can append v_m to a $(d-1)$-degenerate ordering of W_i. ∎

CLAIM 4.2. *The minimum degree of G is d and n is divisible by k.*

Proof. Suppose that $n = k \cdot s + r$, where $1 \leq r \leq k$. We can choose a degenerate ordering of G such that the last vertex in the ordering, v_n, is a vertex of minimum degree. By the minimality of G, there exists an equitable k-partition $(W_1, \dots, W_k)$ of $V(G) - v_n$ into sets inducing $(d-1)$-degenerate graphs. Note that exactly $r-1$ of these sets have size $s+1$ and the remaining $k-r+1$ sets are of size s. Since $k - r + 1 \geq 1$, there is at least one W_i of size s. If $\deg_G(v_n) \leq d-1$, then adding v_n to any set W_i of size s creates the desired equitable k-partition of G. This contradicts the choice of G and so we have that $\deg_G(v_n) \geq d$.

If k does not divide n, then we have $r < k$. This implies that there are $k-r+1 \geq 2$ sets of size s and, by Claim 4.1, we can add v_n to at least one of these sets of size s. Again, this contradicts the choice of G as a minimal counterexample and implies that k divides n. ∎

Given a vertex ordering $R = \{v_1, \dots, v_n\}$ of a graph H and an edge $e = v_i v_j \in E(H)$, we denote $l_R(e) = i$ and $r_R(e) = j$ if $i < j$. From all d-degenerate orderings of $V(G)$ choose a *special* ordering $U = (u_1, \dots, u_n)$, where the maximum index $l_U(e)$ of an edge $e \in E(G)$ is maximized. Let i_0 be the maximum of $l_U(e)$ over all the edges in the special ordering U. For convenience, we use U_i to denote the set $\{u_i, u_{i+1}, \dots, u_n\}$ for each i, $1 \leq i \leq n$.

CLAIM 4.3. *The vertex u_{i_0} is adjacent to u_i for every $i_0 < i \leq n$, and the set U_{i_0+1} is independent.*

Proof. The second part of the claim is directly implied by the definition of i_0. Suppose that for some $j > i_0$, the vertex u_j is not adjacent to u_{i_0}. Then all the neighbors of u_j are in $V(G) - U_{i_0}$. So moving u_j from its current position to just before

u_{i_0} creates another d-degenerate ordering of $V(G)$. In this ordering the maximum index of the left end of an edge is $i_0 + 1$, which contradicts the choice of the special ordering U. □

Now we are ready to prove the theorem.

Case 1. $i_0 \geq n-k+1$. Let $G' = G - U_{n-k+1}$. By the minimality of G, $V(G')$ has an equitable partition $(W_1, \ldots, W_k)$ into sets inducing $(d-1)$-degenerate graphs. Now we attempt to consecutively add $u_{n-k+1}, u_{n-k+2}, \ldots, u_n$ (in this order) so that (a) we add one vertex to every set, and (b) every new set still induces a $(d-1)$-degenerate graph. For vertices $u_{n-k+1}, u_{n-k+2}, \ldots, u_{n-1}$ we can do this by Claim 4.1. Suppose that after adding vertices $u_{n-k+1}, u_{n-k+2}, \ldots, u_{n-1}$, W_i is the only set to which no vertex has been added. The trick with u_n is that one of its neighbors is u_{i_0}, which has already been added to a set different from W_i. Thus u_n has at most $(d-1)$ neighbors in W_i and therefore the set $W_i \cup \{u_n\}$ still induces a $(d-1)$-degenerate graph.

Case 2. $i_0 \leq n-k$. Let $G'' = G - U_{i_0}$. By the minimality of G, $V(G'')$ has an equitable partition $(W_1, \ldots, W_k)$ into sets inducing $(d-1)$-degenerate graphs. For $i > i_0$, call a set W_ℓ $1 \leq \ell \leq k$ *i-incompatible* if all $d-1$ neighbors of u_i different from u_{i_0} are in W_ℓ. By Claim 4.1, for every $i > i_0$, there could be at most one i-incompatible set. However, a set W_ℓ may be i-incompatible for several i. By Claim 4.1, u_{i_0} can be added to any one of at least $k-1$ sets among the W_i's. Let $S = \{W_i \mid 1 \leq i \leq k \text{ and } u_{i_0} \text{ can be added to } W_i\}$. There exists some set $W_{\ell'} \in S$ such that $W_{\ell'}$ is i-incompatible with at most $(n-i_0)/|S|$ values of $i > i_0$. Since $k \geq 3$, $|S| \geq 2$ and so $(n-i_0)/|S| \leq (n-i_0)/2$. Now add u_{i_0} to $W_{\ell'}$. Any u_i, $i > i_0$, for which $W_{\ell'}$ is i-incompatible, can be added to any set other than $W_{\ell'}$. Distribute such u_i's among sets other than $W_{\ell'}$ so that the sizes of new sets do not exceed $s = n/k$. The remaining u_i's can be added to any set. Thus, we add these in an arbitrary way so that the size of every W_l becomes $s = n/k$. □

ALGORITHM. The algorithm implied by the above proof is sketched here; the correctness of the algorithm follows from the proof. An equitable k-partition of a given n-vertex graph G is constructed recursively. If G contains a vertex of degree less than d or if n is not divisible by k, we construct a d-degenerate ordering of G and, assuming that v is the last vertex in this ordering, construct an equitable k-partition of $G - v$ and then add v to one of the k sets. Otherwise, we construct a special d-degenerate ordering U of G, referred to in the proof, as follows. Let L_0 be the set of vertices in G with degree at most d. If L_0 contains a pair of adjacent vertices, say u and v, then U is obtained by constructing an arbitrary d-degenerate ordering of $G - u - v$ and appending u and v to this. Otherwise, let L_1 be the set of vertices in $G - L_0$ with degree at most d. By definition, every vertex in L_1 has a neighbor in L_0. Find a vertex $v \in L_1$ with fewest neighbors in L_0. Let S denote the set of neighbors of v in L_0. U is obtained by constructing an arbitrary d-degenerate ordering of $G - v - S$ and appending v followed by vertices in S to this. Once U is constructed, we determine whether Case 1 (respectively, Case 2) of the proof applies and accordingly construct an equitable k-partition of $G' = G - U_{n-k+1}$ (respectively, $G'' = G - U_{i_0}$) and add vertices in U_{n-k+1} (respectively, U_{i_0}) to the sets in the partition. It is easy to see that $O(n^2)$ time suffices for the algorithm, though it seems likely that with more care this can be implemented in subquadratic time.

Remark. In [17], a list analogue of equitable coloring was considered. A *list assignment* L for a graph G assigns to each vertex $v \in V(G)$ a set $L(v)$ of allowable colors. An L-coloring of G is a proper vertex coloring such that for every $v \in V(G)$ the color on v belongs to $L(v)$. For example, when colors represent time periods and vertices are jobs, the list model incorporates the restriction that not all time periods are suitable for all jobs. A list assignment L for G is *k-uniform* if $|L(v)| = k$ for all $v \in V(G)$.

Given a k-uniform list assignment L for an n-vertex graph G, we say that G is *equitably L-colorable* if G has an L-coloring of G such that every color has at most $\lceil n/k \rceil$ vertices. A graph G is *equitably list k-colorable* if G is equitably L-colorable whenever L is a k-uniform list assignment for G.

Because some colors in the lists may occur rarely, one cannot ensure using each color, and most of the techniques previously used for ordinary equitable colorings do not work well for equitable list colorings. In particular, it is not absolutely clear how to adapt the proofs of Theorems 1 and 2 for equitable colorings. However, the idea of the proof of Theorem 3 could be adapted to prove its list version as follows.

THEOREM 8. *Let $k \geq 3$ and $d \geq 2$. Suppose that every vertex v of a d-degenerate graph G on n vertices is given a list $L(v)$ of k colors. Then the vertices of G can be colored from their lists in such a way that every color class induces a $(d-1)$-degenerate subgraph of G and contains at most $\lceil n/k \rceil$ vertices.*

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