Minimax degrees of quasiplanar graphs with no short cycles other than triangles

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Abstract

For an edge xy, let $M(xy)$ be the maximum of the degrees of x and y. The *minimax degree* (or M-degree) of a graph G is $M^*(G) = \min\{M(xy)|xy \in E(G)\}$. In order to get upper bounds on the game chromatic number of planar graphs, He, Hou, Lih, Shao, Wang, and Zhu showed that every planar graph G without leaves and 4-cycles has minimax degree at most 8, which was improved by Borodin, Kostochka, Sheikh, and Yu to the sharp bound 7. We show that every planar graph G without leaves and 4- and 5-cycles has M-degree at most 5, which bound is sharp. We also show that every planar graph G without leaves and cycles of length from 4 to 7 has M-degree at most 4, which bound is attained even on planar graphs with no cycles of length from 4 to arbitrarily large number. Besides, we give sufficient conditions for a planar graph to have M-degrees 3 and 2. Similar results are obtained for graphs embeddable into the projective plane, the torus and the Klein bottle.

Dedicated to Professor Ko-Wei Lih on the occasion of his 60th birthday

1 Introduction

By *quasiplanar* graphs we mean those embeddable in the surfaces with nonnegative Euler characteristics; i.e., the plane, the projective plane, the torus and the Klein bottle. For brevity, we will call a graph embeddable into the projective plane *a projective-planar graph*, embeddable into the torus *a toroidal graph*, and embeddable into the Klein bottle *a Kleinian*

graph. Recall that every planar graph is also projective-planar. For an edge xy in a graph G, the *maximum degree* $M(xy)$ is the maximum of the degrees of x and y. The *minimax degree* (for short, M-degree) of a graph G is $M^*(G) = \min\{M(xy)|xy \in E(G)\}$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of a graph G, respectively.

Wernicke [17] proved that $M^*(G) \leq 6$ for every planar graph G with $\delta(G) = 5$. Kotzig [15] proved that $M^*(G) \leq 7$ for every planar graph G with $\delta(G) \geq 4$. Borodin [4] showed that $M^*(G) \leq 10$ for every planar graph G with $\delta(G) \geq 3$, extending Kotzig's similar result [14] on 3-polytopes. (This last upper bound was conjectured by P. Erdős and announced by D. Barnette (see Grünbaum [12], p. 454) to be true but seems to have never been published.) All these bounds are tight.

The M-degree of planar graphs with $\delta(G) \geq 2$ is not bounded from above. For example, $M^*(K_{2,n}) = n$. Note that every cycle in $K_{2,n}$ has length four. He, Hou, Lih, Shao, Wang and Zhu [13] found upper bounds on M-degrees of planar graphs with $\delta(G) \geq 2$ and restrictions on girth. They proved the following results.

Theorem 1 (Corollary 2.1 in [13]) *Let G be a connected planar graph with $\delta(G) \geq 2$. Then*

$$M^*(G) \leq \begin{cases} 5, & \text{if } g(G) \geq 5, \\ 3, & \text{if } g(G) \geq 7, \\ 2, & \text{if } g(G) \geq 11. \end{cases}$$

Theorem 2 (Theorem 2.2 in [13]) *Let G be a connected planar graph with $\delta(G) \geq 2$. If G does not contain 4-cycles, then $M^*(G) \leq 8$.*

He et al. used these bounds to estimate from above the game chromatic number (introduced by Bodlaender [1]) and the game coloring number of the corresponding graphs. Note that the game chromatic number is never greater than the game coloring number. They used the result in [11] that the game coloring number of a tree is at most 4 and the fact that if the edges of a graph G are partitioned into a tree and a graph H, then the game coloring number of G is at most $4 + \Delta(H)$. In particular, Theorem 2 yields that every quadrangle-free planar graph can be decomposed into a forest and a graph with maximum degree at most 7, and therefore the game chromatic number and the game coloring number of every such graph G are at most 11. The bounds in Theorem 1 are tight. It is also mentioned in [13] that it is not known whether 8 in Theorem 2 is the exact bound.

Borodin, Kostochka, Sheikh, and Yu [9] proved that the M-degree of any quadrangle-free graph G with $\delta(G) \geq 2$ is at most 7 if G is projective-planar, and is at most 8 if G is toroidal and Kleinian. Both bounds are tight.

An induced cycle $C = v_1 v_2 \ldots v_{2k}$ in a graph G is called 2-*alternating* if $d(v_1) = d(v_3) = \ldots = d(v_{2k-1}) = 2$. This notion, introduced in [3], as well as its subsequent variations, turned out to be useful in some coloring and edge-decomposition problems on quasiplanar graphs (see, for example, [4, 10, 6, 7, 13, 16, 18]).

Suppose G is a planar graph with $\delta(G) \geq 2$, and instead of forbidding all 4-cycles we forbid only 2-alternating 4-cycles. Then $M^*(G) \leq 15$ (see [2]). Furthermore, if G has no 2-alternating cycles at all, then $M^*(G) \leq 13$ (see [3]), where both bounds are tight.

Douglas West asked for similar upper bounds on M-degrees of quasi-plane graphs without leaves and 4-faces. We proved in [8] that $M^*(G) \leq 15$ if G is embedded into the plane or the projective plane and $M^*(G) \leq 18$ if G is embedded into the torus or the Klein bottle. Both bounds are sharp.

Wang [16] proved that $M^*(G) \leq 6$ for every toroidal or Kleinian graph G that has no leaves, 4-cycles, and 5-cycles. This bound is sharp.

The purpose of this paper is to prove Theorems 3 to 7 below:

Theorem 3 *Let G have no leaves and be either a projective-planar graph without 4- and 5-cycles, or be a toroidal or Kleinian graph without cycles of length from 4 to 6. Then $M^*(G) \leq 5$. Both bounds on the length of forbidden cycles are sharp.*

As explained in [13], this implies

Corollary 1 *Let G be either a projective-planar graph without 4- and 5-cycles, or a toroidal or Kleinian graph without cycles of length from 4 to 6. Then*
(i) G has an edge-partition into a forest and a subgraph H with $\Delta(H) \leq 4$;
(ii) the game chromatic number and the game coloring number of G is at most 8.

Theorem 4 *Let G have no leaves and be either a projective-planar graph without cycles of length from 4 to 7, or be a toroidal or Kleinian graph without cycles of length from 4 to 8. Then $M^*(G) \leq 4$. This bound is attained even on planar graphs having no cycles of length from 4 to arbitrarily large number. Both bounds on the length of forbidden cycles are sharp.*

This theorem and the next two theorems have corollaries similar to Corollary 1 on edge-partitions and game chromatic number of the corresponding graphs. We will not state them explicitly.
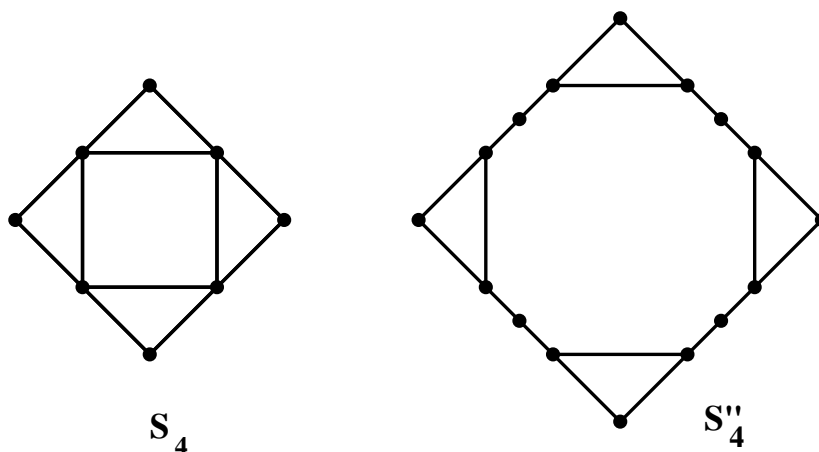

Figure 1: Sunflower S_4 on the left and split sunflower S_4'' on the right.

For $k \geq 4$, by the *k-sunflower*, S_k, we mean the graph obtained from k-cycle C_k as follows: for every edge xy add a new vertex $w = w(xy)$ and edges wx and wy. For example, S_4 is depicted in Fig. 1 (left). This graph has k vertices of degree 2 and k vertices of degree 4. Observe that S_k has no cycles of length from 4 to $k - 1$, while $M^*(S_k) = 4$.

Therefore, to be able to find a graph with $M^*(G) \leq 3$, we consider graphs without 3-cycles sharing a vertex (without a *butterfly*).

Theorem 5 *Suppose G is a graph without butterflies and leaves. If G is projective-planar and has no cycles of length from 4 to 9, or if G is a toroidal or Kleinian graph that has no cycles of length from 4 to 10, then $M^*(G) \leq 3$. Both bounds are sharp.*

Let $d_T(G)$ be the minimum distance between triangles in G. Saying that $d_T(G) \geq 1$ is another way to say that G has no butterflies. Our next result is a sufficient condition for a graph to have $M^*(G) = 2$. The following construction of *split sunflowers* says that condition $d_T(G) \geq 2$ is not sufficient: we take a sunflower S_k and split each 4-vertex into two vertices (of degree 3) joined by a path of length two as in Fig. 1 (right). The obtained graph S_k'' has $M^*(S_k'') = 3$ and has no cycles of length $4 \leq l \leq 3k - 1$.

Theorem 6 *Suppose G is a graph with $\delta(G) \geq 2$ and $d_T(G) \geq 3$. If G is projective-planar and has no cycles of length from 4 to 25, or if G is a toroidal or Kleinian graph that has no cycles of length from 4 to 30, then $M^*(G) = 2$. Both bounds are sharp.*

Let $N(S)$ denote the Euler characteristics of a surface S. Recall that $N(S) = 2$ if S is the plane, $N(S) = 1$ if S is the projective plane, and $N(S) = 0$ if S is the torus or the Klein bottle; for the other surfaces $N(S) < 0$.

In [9], we proved that any graph G without 4-cycles and leaves embedded into a surface S with $N(S) < 0$ and having more than $-72N(S)$ edges has $M^*(G) \leq 8$.

From the proofs of Theorems 3–6, we will deduce the following.

Theorem 7 *If a graph G without leaves is embeddable into a surface S with $N(S) < 0$ and has more than $-2049N(S)$ edges, then the following statements hold:*
  (i) if G has no 4- and 5-cycles, then $M^*(G) \leq 6$;
  (ii) if G has no cycles of length from 4 to 6, then $M^*(G) \leq 5$;
  (iii) if G has no cycles of length from 4 to 8, then $M^*(G) \leq 4$;
  (iv) if G has no butterflies and cycles of length from 4 to 10,
      then $M^*(G) \leq 3$;
  (v) if $d_T(G) \geq 3$ and no cycles of length from 4 to 30, then $M^*(G) = 2$.

Thus, a large graph on a fixed surface S with $N(S) < 0$ behaves in terms of M^* as a graph embeddable into the torus or the Klein bottle.

In the next section, we show that all restrictions on cycle lengths in Theorems 3-6 are sharp. In Sections 3 through 7, we prove Theorem 3 through Theorem 7 respectively.

2 Extremal examples

Examples for Theorem 3. In [9], an infinite family of quadrangle-free planar graphs G with $M^*(G) = 7$ is described. A construction of a toroidal or Kleinian graph without C_4 and C_5 with $M^*(G) = 6$ is as follows: take a cubic graph G' of girth six (there are many such graphs embeddable into the torus or the Klein bottle) and for every edge xy add a new vertex $w = w(xy)$ and edges wx and wy. The degree of each "old" vertex in the new graph G is six, and the "new" vertices form an independent set. Thus $M^*(G) = 6$ and G has neither 4-cycles nor 5-cycles.

Examples for Theorem 4. First observe that the k-sunflower has no cycles of length from 4 to $k - 1$, while $M^*(S_k) = 4$.
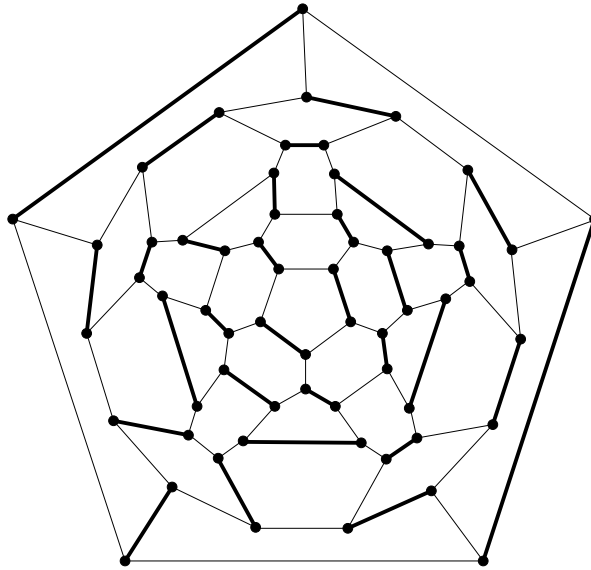

Figure 2: A preparatory graph G' to get a planar graph G with $M^*(G) = 5$ and with no cycles of length from 4 to 6.

Now we show that the bounds on the length of forbidden cycles are sharp. Consider the truncated icosahedron in Fig. 2. It has the property that each cycle that is not the boundary of a face has length at least 9. Subdivide every edge of the bold matching of the graph G' in this figure with a vertex, and for every thin edge xy, add a vertex $w = w(xy)$ adjacent only with x and y. Since bold edges form a matching in G', every vertex of G' has in the new graph G degree exactly five. Since every 5-face in G' contains two bold edges, and every 6-face contains at least one edge, the new graph has no 4-, 5-, or 6-cycles.

A construction of a toroidal or Kleinian graph without C_4–C_7 with $M^*(G) = 5$ is as follows. There are many cubic graphs G' that can be embedded into the torus or into the Klein bottle in such a way that (a) every face is a 6-face, (b) every cycle that is not the boundary of a face has length at least 10, and (c) there is a proper edge 3-coloring of G'

such that in each 6-face the opposite edges have the same color. Fix a color class C in such an edge coloring. Subdivide each edge of C with a vertex and for every edge xy not in C, add a vertex $w = w(xy)$ adjacent only with x and y. The new graph is our G. Again, the degree of every vertex of G' in G is five. Since every face has two edges in C, our graph G has no k-cycles for $4 \leq k \leq 7$.
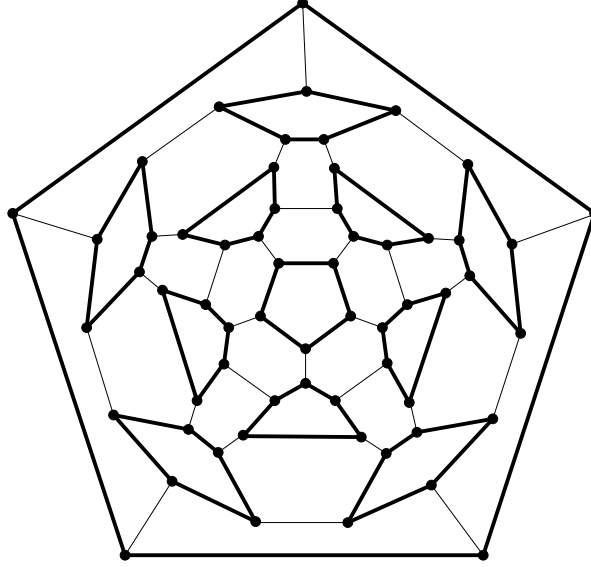

Figure 3: A preparatory graph G' to get a planar graph G with $M^*(G) = 4$ and with no cycles of length from 4 to 8.

Examples for Theorem 5. The basic graphs G' for constructing toroidal and Kleinian graphs G without cycles of length from 4 to 9 having $\delta(G) = 2$ and $M^*(G) = 4$ are the same as in the previous paragraph, satisfying (a)-(c). This time, we subdivide the edges that are not in C and for every edge xy in C, add a vertex $w = w(xy)$ adjacent only with x and y. Every vertex of G' becomes of degree four in G, and four edges of every face of G' are subdivided, by (c) our graphs G have no k-cycles for $4 \leq k \leq 9$.

To construct a planar graph, we use the truncated icosahedron with a bold 2-factor as in Fig. 3. As in the previous paragraph, subdivide every edge of the bold matching of the graph G' in this figure with a vertex, and for every thin edge xy, add a vertex $w = w(xy)$ adjacent only with x and y. Since all edges of each 5-face and three edges of each 6-face in G' are subdivided, we get a planar graph G with $\delta(G) = 2$ and $M^*(G) = 4$ that has no k-cycles for $4 \leq k \leq 8$.

Examples for Theorem 6. Let G' be either a cubic planar graph of girth $g(G') = 5$ or a toroidal or Kleinian cubic graph of girth $g(G') = 6$. We obtain G by replacing each edge xy with the graph in Fig. 4 (a path of length five with a triangle on the central edge). By construction, $M^*(G) = 3$ and a shortest cycle that is not a triangle in G has length $5g(G')$.
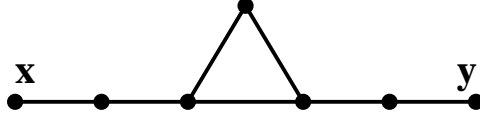
Figure 4: A replacement for each edge in an auxiliary graph G' for Theorem 6.

3 Proof of Theorem 3

Let G be a counterexample to Theorem 3 with the fewest vertices. Fix some embedding of G and let $F(G)$ be the set of faces of this embedding. Let $d(y)$ denote the degree of a vertex y or of a face y. Call a vertex v *senior*, if $d(v) \geq 6$ and *minor* otherwise. By the choice of G, $M^*(G) \geq 6$, and hence

$$\text{Each closed walk } (w_1, \ldots, w_t) \text{ encounters at least } \lceil \tfrac{t}{2} \rceil \text{ senior vertices.} \tag{1}$$

Claim 1 *G is connected.*

Proof. Suppose that G is the vertex-disjoint union of graphs G_1 and G_2, and there are no edges between $V(G_1)$ and $V(G_2)$. Note that $M^*(G) = \min\{M^*(G_1), M^*(G_2)\}$, and thus one of G_1 or G_2 has fewer vertices than G and yet has the same M^* as G. This contradicts the minimality of G. ■

Since G is connected, the boundary of every face is a closed walk.
In [9], we proved the following.

Claim 2 *Suppose a graph has neither adjacent triangles nor adjacent vertices of degree 2. Then for every vertex v, the total number of triangular faces and 2-vertices incident with v is at most $d(v)$.*

We will use discharging to obtain a contradiction with the assumption $M^*(G) \geq 6$.
Let the *initial charge* of every $x \in V(G) \cup F(G)$ be $\mu(x) = d(x) - 4$.
By Euler's formula $|V(G)| - |E(G)| + |F(F)| \geq N(S)$, we have

$$\sum_{x \in V(G) \cup F(G)} (d(x) - 4) = \sum_{x \in V(G) \cup F(G)} \mu(x) \leq -4N(S). \tag{2}$$

A 2-vertex on the boundary of a triangular face will be called *triangular*. The vertices and faces of G discharge their initial charge by the following rules:
Rule 1. Every senior vertex gives $\frac{1}{3}$ to each adjacent 2-vertex and each incident triangular face.
Rule 2. Every non-triangular face gives $\frac{1}{3}$ to each incident edge and $\frac{2}{3}$ to each incident triangular vertex of degree 2.

Rule 3. Every edge incident with a minor vertex v gives its charge to v. If both ends of edge xy are senior vertices and xy is on the boundary of a triangular face f, then xy gives its charge to f.

Clearly, the resulting charge of every edge is nonnegative. It remains to show that the *final charge* $\mu^*(y)$ is nonnegative for each $y \in V(G) \cup F(G)$, and that the final charge of every face of length at least 7 is strictly positive. This yields a contradiction to (2), since the total charge does not change, and hence should be strictly negative when G is projective-planar, and non-positive when G is a toroidal or Kleinian graph. Recall that every quasiplane graph G with $M^*(G) \geq 5$ and no 4-cycles has non-triangular faces.

If y is a triangular face, then $\mu^*(y) \geq 3 - 4 + 3 \times \frac{1}{3} = 0$ by Rules 1 and 3.

Suppose y is a face with $d(y) \geq 6$. If y is not incident with triangular 2-vertices, then $\mu^*(y) \geq d(y) - 4 - \frac{d(y)}{3} = \frac{2(d(y)-6)}{3}$. This is nonnegative when $d(y) = 6$ and strictly positive when $d(y) \geq 7$. Now suppose y is incident with $k \geq 1$ triangular 2-vertices. We delete all k vertices and as a result obtain the new face y' of size at least 6 in the case of projective planar graphs and at least 7 otherwise. Observe that $d(y) = d(y') + k$ and, on the other hand, the expenditure of y is also precisely by k greater than that of y'. Indeed, every triangular 2-vertex contributes 1 to the size of y and increases its expenditure by $2 \times \frac{1}{3} - \frac{1}{3} + \frac{2}{3} = 1$ by Rule 2.

Suppose now that y is a vertex.

If $d(y) = 2$, then by Rule 1 it gets $\frac{2}{3}$ from the adjacent vertices and exactly $\frac{4}{3}$ more: either all from the incident edges, if y has no incident triangular faces, or $\frac{2}{3}$ from the incident edges and $\frac{2}{3}$ from the incident non-triangular face by Rules 2 and 3. Hence, $\mu^*(y) = 0$.

Let $d(y) = 3$. Since G has no 4-cycles and hence at most one face incident with y is a triangle, by Rule 3, y gets at least $\frac{4}{3}$ from the incident edges. Hence, $\mu^*(y) > 0$.

If $4 \leq d(y) \leq 5$, then v gives out nothing, and $\mu^*(y) = d(y) - 4 \geq 0$.

If $d(y) \geq 6$, then $\mu^*(y) \geq d(y) - 4 - \frac{d(y)}{3} \geq 0$ by Claim 2.

This completes the proof of Theorem 3.

4   Proof of Theorem 4

Let G be a counterexample to the main statement of Theorem 4. This time, call a vertex v *senior*, if $d(v) \geq 5$ and *minor* otherwise. As in the previous section, let the *initial charge* of every $x \in V(G) \cup F(G)$ be $\mu(x) = d(x) - 4$.

The vertices and faces of G discharge their initial charge by the following rules:

Rule 1. Every senior vertex gives $\frac{1}{2}$ to each incident triangular face.

Rule 2. Every non-triangular face gives $\frac{1}{2}$ to each incident edge and $\frac{1}{2}$ to each incident triangular 2-vertex.

Rule 3. Every edge incident with a minor vertex v gives its charge to v. If both ends of edge xy are senior vertices and xy is on the boundary of a triangular face f whose third vertex is a 2-vertex, then xy gives its charge to this 2-vertex.

Clearly, every edge has a nonnegative resulting charge. It remains to show that the *final charge* $\mu^*(y)$ is nonnegative for each $y \in V(G) \cup F(G)$, and that the final charge of every face of length at least 9 is strictly positive. This yields a contradiction to (2), as in the proof of Theorem 3. Recall that every quasiplane graph G with $M^*(G) \geq 5$ and no 4-cycles has non-triangular faces.

If y is a triangular face, then $\mu^*(y) \geq 3 - 4 + 2 \times \frac{1}{2} = 0$ by Rule 1.

Suppose y is a face with $d(y) \geq 8$. If y is not incident with triangular 2-vertices, then $\mu^*(y) \geq d(y) - 4 - \frac{d(y)}{2} = \frac{d(y)-8}{2}$. This is strictly positive when $d(y) \geq 9$. Now suppose y is incident with $k \geq 1$ triangular 2-vertices. When we delete all k vertices, we obtain a new face y' of size at least 8 in the case of projective planar graphs and at least 9, otherwise. Observe that $d(y) = d(y') + k$ and, on the other hand, the expenditure of y is also precisely by k greater than that of y'. Indeed, every triangular 2-vertex contributes 1 to the size of y and increases its expenditure by $\frac{1}{2} - \frac{1}{2} + 2 \times \frac{1}{2} = 1$ by Rule 2.

Now let y be a vertex.

First suppose $d(y) = 2$. If y is not triangular, then by Rule 2 each of the incident edges gets 1 from the two incident faces, and by Rule 3 y gets 2×1 from these edges. If y is triangular, then by Rule 3 it gets $2 \times \frac{1}{2}$ from the incident edges and $\frac{1}{2}$ from the opposite edge on the incident triangular face. Apart from it, by Rule 2, y gets $\frac{1}{2}$ from the incident non-triangular face. Hence, $\mu^*(y) = 0$.

If $d(y) = 3$, then since G has no 4-cycles, by Rule 3, it gets at least 2 from the incident edges. Hence, $\mu^*(y) > 0$.

If $d(y) = 4$, then v gives out nothing, and $\mu^*(y) = d(y) - 4 = 0$.

If $d(y) = 5$, then $\mu^*(y) \geq 5 - 4 - 2 \times \frac{1}{2} = 0$ by Rule 1.

If $d(y) \geq 6$, then $\mu^*(y) \geq d(y) - 4 - \frac{d(y)}{4} \geq 0$ by Rule 1.

This completes the proof of Theorem 4.

5 Proof of Theorem 5

Let G' be a counterexample to the main statement of Theorem 5. Now call a vertex v *senior*, if $d(v) \geq 4$ and *minor* otherwise.

Let G be obtained from G' by subdividing with a vertex every edge xy that does not belong to any 3-cycle and is incident only with senior vertices. By construction, $M^*(G) \geq 4$ and no cycles of intermediate length arise. Thus, G is also a counterexample to the main statement of Theorem 5, and has the additional property that

$$\text{senior vertices are adjacent only if they belong to a common triangle.} \qquad (3)$$

As in the two previous proofs, let the *initial charge* of every $x \in V(G) \cup F(G)$ be $\mu(x) = d(x) - 4$. The vertices and faces of G discharge their initial charge in two stages. First, every non-triangular face gives 1 to every incident edge that is also incident with a triangle face, and $\frac{1}{2}$ to every other edge. The resulting charge of every edge becomes exactly 1. The next claim is crucial.

Claim 3 *The new charge $\mu'(y)$ is nonnegative for every non-triangular face. Furthermore, if G has no 10-cycles, then the new charge of some non-triangular face is positive.*

Let f be a face of length $l \geq 10$. An edge incident with f is *f-easy* if it gets $\frac{1}{2}$ from f and is *f-tough* if it gets 1 from f. By (3), the f-easy edges can be partitioned into pairs such that each pair has a common end that is a minor vertex. If we have at least four such pairs, then the first half of the claim is proved. Suppose that there are at most three such pairs. Deleting these at most six edges breaks the remaining at least $l - 6$ edges into some $j \leq 3$ intervals of f-tough edges. By (3), each such interval has at most two edges. Moreover, if an interval has two edges $x_{i-1}x_i$ and x_ix_{i+1}, then $x_{i-1}x_{i+1}$ is also an edge in G. Thus if exactly h of the intervals have length two, then $l \geq 10 + h$. Therefore, $j + h \geq (10 + h) - 6$, a contradiction to $j \leq 3$.

To prove the second statement of the claim, suppose that G has no cycles of length k for $4 \leq k \leq 10$. Suppose that each non-triangular face f has exactly eight f-easy edges. Let f_0 be a face of length $l \geq 11$. By the same argument as in the previous paragraph, after deleting f_0-easy edges from the boundary of f_0, the remaining edges break into at least 3 intervals. This means that some interval of f_0-easy edges on the boundary of f_0 consists of exactly two edges, say v_1v_2 and v_2v_3. This, in turn, means that edges v_lv_1 and v_3v_4 belong to triangular faces, and vertices v_1 and v_3 are senior. Let f_1 be the face that shares edges v_1v_2 and v_2v_3 with f_0. Since G has no butterflies, the edge incident with f_1 containing v_1 and other than v_1v_2 does not belong to a triangular face, and hence is f_1-easy. Similarly, the edge incident with f_1 containing v_3 and other than v_2v_3 also is f_1-easy. By (3), f_1 contains at least six consecutive f_1-easy edges. Then the new charge of f_1 is positive. This proves the claim.

At the second stage of the discharging, each edge incident with a minor vertex gives all its charge (equal to 1) to this minor vertex. Each edge not incident with a minor vertex gives all its charge to the incident triangular face.

Note that each triangular face gets at least 1 and each minor vertex at least 2. Senior vertices and non-triangular faces do not change their charge at this stage. This completes the proof of Theorem 5.

6 Proof of Theorem 6

Let G be a counterexample to the main statement of Theorem 6. In particular, $M^*(G) \geq 3$ and hence

$$\text{each path connecting any two triangular faces has an internal} \geq 3\text{-vertex.} \qquad (4)$$

Let G' be obtained from G by contracting every triangular face of G into a vertex. Since each 3-cycle can have at most one 2-vertex, $\delta(G') \geq 2$. Then we replace each inclusion maximal path in G' whose internal vertices all have degree 2 with a single edge to obtain a graph G'' with $\delta(G'') \geq 3$.

Every graph G'' with $\delta(G'') \geq 3$ embedded into the plane or projective plane has a face of size at most 5, and every such graph embedded into the torus or the Klein bottle has a face of size at most 6. Let such a face be $f'' = v_1 \ldots v_k$.

Let f be the pre-image of f'' in graph G embedded into the corresponding surface. Since G has no adjacent 2-vertices, our f is not a triangular face (because all triangular faces of G were contracted) and hence has size at least 26 when G is projective-plane and 31 when G is embedded into the torus or the Klein bottle. By (4), the pre-image of each edge of f'' contains at most one edge of a triangular face, and hence contains at most five edges in total. This contradicts our choice of f''.

This contradiction completes the proof of Theorem 6.

7   Proof of Theorem 7

We give a proof only for the most difficult part (iv) of Theorem 7, because the argument for other cases is simpler.

If G has no butterflies and cycles of length from 4 to 10, then as follows from the proof of Theorem 5, all vertices, edges and faces have nonnegative new charges μ^*. Moreover, if f is a nontriangular face with $\mu^*(f) = 0$, then $d(f) \leq 16$. Since in the discharging in Theorem 5 we deal only with charges that are multiples of $\frac{1}{2}$, it follows that $\mu^*(f) \geq \frac{1}{2}$ unless $\mu^*(f) = 0$. Furthermore, it is not hard to check that $\mu^*(f) \geq k - 4$, provided that $d(f) \geq 4k - 3$, when $k \geq 5$.

Since G is embedded in S, we have

$$\sum_{x \in V \cup E \cup F} \mu^*(x) = \sum_{x \in V \cup F} \mu(x) = \sum_{x \in V \cup F} (d(x) - 4) = -4N(S).$$

Recall that $\mu^*(e) = 0$ for every edge of G, $\mu^*(v) \geq 0$ for every $v \in V$, and $\mu^*(T) = 0$ for every triangle T in G. This implies that

$$\sum_{f \in F, f \neq T} \mu^*(f) \leq -4N(S).$$

We want to redistribute the charges of the nontriangular faces of G to the edges of G so that every edge has charge at least 2^{-9}. (This will give the result.)

Note that every edge, e, is incident with a nontriangular face, f_0. However, f_0 may have $\mu^*(f_0) = 0$. As mentioned in the course of the proof of Theorem 5, such f_0 is adjacent to a face f^+ with $\mu^*(f^+) > 0$. In this situation we say that f_0 is *dependent* on f^+. Our redistribution consists in dividing $\mu^*(f^+)$ evenly among the edges incident with f^+ and all faces dependent on f^+.

Clearly, $\mu^*(f^+)$ is divided into at most $16d(f^*)$ portions, p. If $d(f^*) \leq 16$ then $p \geq \frac{1}{2} \times \frac{1}{16} \times \frac{1}{16} = 2^{-9}$; if $d(f) \geq 4k-3$, where $k \geq 5$, then $p \geq (k-4) \times \frac{1}{4k} \times \frac{1}{16} \geq 1 \times \frac{1}{20} \times \frac{1}{16} > 2^{-9}$. Hence, for each edge e, its final charge, $\mu^{**}(e)$ is at least 2^{-9}. It follows that $|E| \geq -4N(S) \times 2^{-9}$, a contradiction.

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