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EQUITABLE VERSUS NEARLY EQUITABLE COLORING AND THE CHEN–LIH–WU CONJECTURE

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Chen, Lih, and Wu conjectured that for $r \geq 3$, the only connected graphs with maximum degree at most r that are not equitably r-colorable are $K_{r,r}$ (for odd r) and K_{r+1}. If true, this would be a strengthening of the Hajnal–Szemerédi Theorem and Brooks' Theorem. We extend their conjecture to disconnected graphs. For $r \geq 6$ the conjecture says the following: If an r-colorable graph G with maximum degree r is not equitably r-colorable then r is odd, G contains $K_{r,r}$ and $V(G)$ partitions into subsets $V_0, \ldots, V_t$ such that $G[V_0] = K_{r,r}$ and for each $1 \leq i \leq t$, $G[V_i] = K_r$. We characterize graphs satisfying the conclusion of our conjecture for all r and use the characterization to prove that the two conjectures are equivalent. This new conjecture may help to prove the Chen–Lih–Wu Conjecture by induction.

1. Introduction

In several applications of graph coloring such as the mutual exclusion scheduling problem, scheduling in communication systems, construction timetables, and round-a-clock scheduling (see [4,17,18]), there is an additional requirement that color classes be not so large or be of approximately the same size. A model imposing such a requirement is *equitable coloring* – a proper coloring such that color classes differ in size by at most one.

One of the basic results on equitable coloring is the following theorem by Hajnal and Szemerédi ([8], for a shorter proof see [10]).

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Theorem 1. *For every positive integer r, each graph with $\Delta(G) \leq r$ has an equitable $(r+1)$-coloring.*

This theorem has interesting applications in extremal combinatorial and probabilistic problems, see e.g. [1–3, 9, 11, 15, 16].

It is natural to ask which graphs G with $\Delta(G) = r \geq 3$ have equitable r-colorings; certainly such graphs are r-colorable and so do not contain K_{r+1}. Brooks' Theorem [5] can be stated as follows.

Theorem 2. *Let G be a connected graph G with $\Delta(G) \leq r$. Then G is not r-colorable if and only if $G = K_{r+1}$ or G is an odd cycle and $r = 2$.*

Note that like K_{r+1}, $K_{r,r}$ is r-regular. So if $K_{r,r}$ is contained in a graph with maximum degree r then it is a component. Moreover, if r is odd, then $K_{r,r}$ has no r-equitable coloring. Chen, Lih and Wu [6] proposed the following strengthening of Theorem 1 and Brooks' Theorem.

Conjecture 1. Let G be a connected graph with $\Delta(G) \leq r$. Then G has no equitable r-coloring if and only if either (1) $G = K_{r+1}$, or (2) $r = 2$ and G is an odd cycle, or (3) r is odd and $G = K_{r,r}$.

Some partial cases of Conjecture 1 were proved in [6, 14, 19, 20, 12]. In particular, Chen, Lih and Wu [6] proved that the conjecture holds for $r = 3$:

Theorem 3. *Let G be a connected graph with $\Delta(G) \leq 3$. Then G has no equitable 3-coloring if and only if $G = K_4$ or $G = K_{3,3}$.*

Brooks' Theorem characterizes those graphs with maximum degree r that are r-colorable, since a graph is r-colorable if and only if each of its components is. This is not the case for equitable r-coloring. For example, for an odd $r \geq 3$, the graph consisting of two disjoint copies of $K_{r,r}$ has an equitable r-coloring, but the graph consisting of disjoint copies of $K_{r,r}$ and K_r does not. This construction can be generalized. We say that a graph H is *r-equitable* if $|H|$ is divisible by r, H is r-colorable and every r-coloring of H is equitable. If G contains $K_{r,r}$ and $G - K_{r,r}$ is r-equitable, then G does not have an equitable r-coloring. This motivates the study of equitable graphs.

If an r-colorable graph G has a spanning subgraph whose components are all r-equitable, then G is also r-equitable. We say that an r-equitable graph G is *r-reducible* if $V(G)$ has a partition $\{V_1, \ldots, V_t\}$ into at least two parts such that $G[V_i]$ is r-equitable for each $i \in [t]$; otherwise G is *r-irreducible*. Clearly K_r is r-irreducible. In the next two sections (see Figures 1–3) we identify one other 5-irreducible graph F_1, three other 4-irreducible graphs F_2, F_3, F_4 and six other 3-irreducible graphs $F_5, \ldots, F_{10}$.

Together with K_r, the r-irreducible graphs from this list are the *r-basic* graphs. An *r-decomposition* of G is a partition on $V(G)$ into subsets $V_1,\dots,V_t$ such that each $G[V_i]$ is r-basic. We say that G is *r-decomposable* if it has an r-decomposition. As it was just mentioned, if r is odd and G is r-decomposable, then $G \cup K_{r,r}$ has no equitable r-coloring. We conjecture that this is the only obstacle that prevents an r-colorable graph with $\Delta(G) \le r$ from having an equitable r-coloring.

Conjecture 2. Suppose that $r \geq 3$ and G is an r-colorable graph with $\Delta(G) \le r$. Then G has no equitable r-coloring if and only if r is odd and there exists $H \subseteq G$ such that $H = K_{r,r}$ and $G - H$ is r-decomposable.

For $r \geq 6$, this conjecture means that *if an r-colorable graph G with $\Delta(G) \le r$ has no equitable r-coloring, then r is odd and $V(G)$ partitions into sets $V_0,\dots,V_t$ such that $G[V_0] = K_{r,r}$ and $G[V_i] = K_r$ for each $i = 1,\dots,r$.*

A *nearly equitable r-coloring* of a graph G is a proper r-coloring of G that has exactly one color class V^- of size $s-1$, exactly one color class V^+ of size $s+1$ and all other color classes have size s. In this case $|G|$ is divisible by r. If r is odd, G contains $K_{r,r}$ and $G - K_{r,r}$ has a nearly equitable r-coloring, then G has an equitable r-coloring, since the small class of one of the components can be combined with the large class of the other. This explains our interest in nearly equitable r-colorings.

Let $\mathcal{G}(r)$ be the class of all graphs G with $\Delta(G), \chi(G) \le r$. Let $\mathcal{G}(r,n)$ be the set of graphs in $\mathcal{G}(r)$ with at most n vertices. Our main result is the following theorem.

Theorem 4. *Let $G \in \mathcal{G}(r)$ with $|G|$ divisible by r. The following are equivalent:*
(A) G is r-decomposable;
(B) G is r-equitable;
(C) G has an equitable r-coloring, but does not have a nearly equitable r-coloring.

It is easy to see that (A) $\Rightarrow$ (B) and (B) $\Rightarrow$ (C). The content of the theorem (Lemma 5) is that (C) $\Rightarrow$ (A). This is proved in Sections 2 and 3. The theorem has its own merit and the three corollaries below. In particular, it implies that the two conjectures are equivalent, even for restricted values of r and n.

Corollary 1. *For all positive integers r and $n > r$, Conjecture 1 holds for all graphs in $\mathcal{G}(r,n)$ if and only if Conjecture 2 holds for all graphs in $\mathcal{G}(r,n)$.*

Corollary 2. *Let $G \in \mathcal{G}(r)$ be r-equitable. Then G has a unique r-decomposition.*

Corollary 3. *There exists a polynomial time algorithm for deciding whether a graph $G \in \mathcal{G}(r)$ is r-equitable.*

We derive the Corollaries from the theorem in Section 4. By Corollary 1, Conjecture 2 holds for every r for which Conjecture 1 holds. In particular, by Corollary 1 and Theorem 3, our conjecture holds for $r=3$.

Most of our notation is standard; possible exceptions include the following. For a graph G, we let $|G|:=|V(G)|$, $\|G\|:=|E(G)|$. For a vertex y and set of vertices X, $N_X(y):=N(y)\cap X$ and $d_X(y):=|N_X(y)|$. The set of edges linking a vertex in A to a vertex in B is denoted by $E(A,B)$. For a set S and element x we write $S+x$ for $S\cup\{x\}$ and $S-x$ for $S\setminus\{x\}$. For a function $f: V\rightarrow Z$, the restriction of f to $W\subseteq V$ is denoted by $f|W$. Functions are viewed formally as sets of ordered pairs. If m is understood from the context then $a\oplus b$ denotes $a+b \pmod m$. For an integer n the set $\{1,\dots,n\}$ is denoted by $[n]$.

2. Core Graphs

There are two types of r-basic graphs: the r-*core* graphs and the r-*derived* graphs, which are obtained from r-core graphs. In this section we define the set $\mathcal{C}_r$ of r-core graphs and show that every graph $G\in\mathcal{G}(r)$ with $|G|$ divisible by r, which satisfies (C), contains an r-core graph. Referring to Figure 1, the 5-core graphs are K_5 and F_1; the 4-core graphs are K_4 and F_2; for all other r, the only r-core graph is K_r.

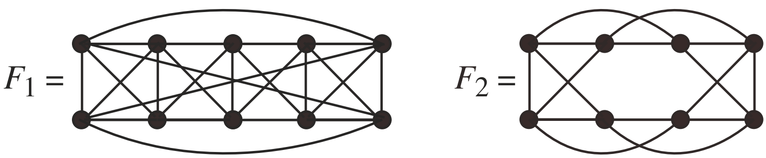

Figure 1. Core Graphs

Lemma 1. *Let $G\in\mathcal{G}(r)$ with $|G|$ divisible by r, where $r\geq 3$. If G has an equitable r-coloring, but no nearly equitable r-coloring, then G contains an r-core graph.*

Proof. Let G be a vertex minimal counterexample to the lemma. So G has an equitable r-coloring f, G does not have a nearly equitable r-coloring and also does not contain any r-core graph.

Define an auxiliary digraph $H := H(f)$ on $V(G)$ so that $xy \in E(H)$ iff $xy \in E(G)$ and no other neighbor of x has the same color as y. In the following, notation associated with digraphs, such as δ^+, applies to H, while notation associated with ordinary graphs, such as δ, applies to G.

Claim 1. *Every vertex v in H satisfies (1) $N^-[v]$ is a clique, (2) $d^+(v) = r-2 = d^-(v)$ and (3) $N(v) = N^+(v) \cup \{x,y\}$, where x and y are distinct and have the same color, not appearing on any of the vertices of $N^+[v]$. In particular, G is r-regular.*

Proof. If v has no neighbor in a color class X distinct from its own, then by moving v to X we obtain a nearly equitable r-coloring. Thus v has neighbors in all $r-1$ other classes. Since $\Delta(G) \le r$, at most one class contains more than one neighbor of v, and such a class contains exactly two.

Suppose $x,y \in N^-(v)$ and $xy \notin E(G)$. Move x and y to the class of v. If x and y came from the same class then move v to it; otherwise, v has an out-neighbor $w \in \{x,y\}$. Move y to the former class of w. In both cases, this yields a nearly equitable coloring, a contradiction. So (1) holds. Since G does not contain K_r, $d^-(v) \le r-2$. Since v was arbitrary,

$$(r-2)\,|H| \le \delta^+(H)\,|H| \le \sum_{v\in V(H)} d^+(v) = \sum_{v\in V(H)} d^-(v) \le \Delta^-(H)\,|H| \le (r-2)\,|H|\,.$$

It follows that H is $(r-2)$-regular (i.e., the out-degree and the in-degree of each vertex is $r-2$). The definition of H now implies (3). ∎

We consider two cases.

Case A. $r=3$. By Claim 1 (2), H is the union of disjoint directed cycles. If we move each vertex of one of these cycles C to the color class of its out-neighbor we obtain a new equitable 3-coloring denoted by f_C. Then $V(C)$ still induces a cycle C^* in $H(f_C)$, but its edges are reversed.

Suppose that H has a 2-cycle (x,y), where $f(x)=1$ and $f(y)=2$. Then the vertices in $N(\{x,y\})$ all have color 3. So switching the colors of x and y does not change the out-neighbors of any vertex not colored 3. Let $w \in N(x)-y$, z be the in-neighbor of w and z' be the out-neighbor of w. Then $f(z')=2$. First suppose that $z \ne z'$. Then $f(z)=1$. After switching the colors of x and y, vertex z is both the out-neighbor and the in-neighbor of w. So we may assume that $z=z'$. Then z has two neighbors u and u' colored 1. Thus after switching the colors of x and y it is still the case that u and u' are not in-neighbors of z. Moreover, w is no longer an in-neighbor of z. This contradicts Claim 1 (2), so we conclude that H has no 2-cycles.

Let $C=(v_1,\dots,v_k)$ be a directed cycle of H. Then $k\geq 3$. Consider any $i\in[k]$. Since $v_{i\oplus 1}$ is the unique neighbor of v_i in its color class, $v_{i\ominus 1}$ and $v_{i\oplus 1}$ must have distinct colors, and v_i must have a third color. So $f(v_i)=f(v_{i\oplus 3})$, and k is divisible by 3.

Suppose C has a chord $v_iv_j\in E(G)$. Then $f(v_i)\neq f(v_j)$. Moreover, since $v_iv_{i\oplus 1}\in E(H)$, $f(v_{i\oplus 1})\neq f(v_j)$. Since the colors appear regularly on C, $f(v_j)=f(v_{i\ominus 1})$ and $f(v_{j\oplus 1})=f(v_i)$, a contradiction. By Claim 1 (3) each vertex v of C has a unique neighbor v' in $G-C$.

Suppose that C, as above, is a smallest cycle in H. Let C' be the cycle in H that contains the neighbor v_1' of v_1 in G outside of C. Let $H':=H(f_C)$. Suppose that zw is an edge of C' with $z\notin N(C)$ and $w\in N(C)$. Then zwz is a 2-cycle in H', a contradiction. We conclude that every vertex of C' has a neighbor in C. By the minimality of C, $|C'|=|C|$. Let $C'=:(w_1,\dots,w_k)$, where $w_1:=v_1'$ and $w_i':=v_j$, if $v_j'=w_i$.

We may assume that $f(v_i)=i \pmod 3$ for all $i\in[k]$. Then $f(w_i)=4-i \pmod 3$ for all $i\in[k]$. Let $v_2'=w_j$. So $f(w_1)=3$ and $f(w_j)=1$. Let g be the coloring obtained from f by switching the colors of v_1 and v_2, replacing the color of w_1 with 1 and replacing the color of $x=w_j,\dots,w_k$ with the color $f(x)\ominus 1$ of its out-neighbor. It is easily checked that g is an equitable 3-coloring. Note that under f the unique neighbor of w_{j-1} colored with 1 is w_j, but $g(w_j)=3$. Since w_{j-1}' is neither v_1 nor v_2, it has not changed color. Thus $g(w_{j-2})$ must be 1. The only possibility for this is that $w_{j-2}=w_1$ and thus changed color from 3 to 1. We conclude that $j=3$ and $v_2w_3\in E(G)$.

The analysis above applies to any edge between C and C' and we also can switch the roles of C' and C. Thus for any edge v_iw_j, we have $v_{i\oplus 1}w_{j\oplus 2}, v_{i\oplus 2}w_{j\oplus 1}\in E(G)$. Starting from $v_1w_1\in E(G)$ we get (in order) $v_2w_3, v_3w_5, v_3w_2\in E(G)$. Since v_3 has only one neighbor on C', $w_2=w_5$ and so $k=3$. Thus $C=K_3$, a contradiction. This completes the proof of Case A.

Case B. $r\geq 4$. Our main tool is the following claim.

Claim 2. *If $r\geq 4$ then distinct $(r-1)$-cliques are disjoint.*

Proof. Suppose that $Q_1:=\{v_1,\dots,v_{r-1}\}$ and $Q_2:=\{v_{r-k},\dots,v_{2r-2-k}\}$ are distinct $(r-1)$-cliques, which share $k\geq 1$ vertices. Since v_{r-1} is adjacent to every other vertex of $Q:=Q_1\cup Q_2$, $r+1\geq |Q|=2r-2-k$. So $k\geq r-3$.

Suppose $i\in[2]$ and $v\in Q_i\setminus Q_{3-i}$ has two out-neighbors in Q_{3-i}. Then, by Claim 1 (3), every other vertex of Q_{3-i} shares an out-neighbor with v. So, by Claim 1 (1), $Q_{3-i}+v$ is an r-clique, a contradiction. Thus v has at most one out-neighbor in Q_{3-i}. Moreover, since Q_{3-i} is a clique, at most one of the other two neighbors of v is in Q_{3-i}. In particular $k \le 2$ and $4 \le r \le 5$. If $k=2$ then $N_Q(v)=Q_i - v$.

Case 1. $r = 5$. Then $k = 2$. All neighbors of v_3 and v_4 are in Q. Each of v_1, v_2, v_5, v_6 has three neighbors in its own clique and its remaining two neighbors outside Q. Thus, by Claim 1 (1), neither v_1 nor v_2 can have the same out-neighbor as either v_5 or v_6. So, by Claim 1 (2) they each have exactly one out-neighbor in $\{v_3, v_4\}$ and exactly one out-neighbor outside Q. We may assume that $v_1v_3, v_2v_3, v_5v_4, v_6v_4 \in E(H)$. Since $N(v_3) \subseteq Q$, we may also assume that $f(v_i) = i \pmod 5$. Thus $N^+(v_3) = \{v_2, v_4, v_5\}$ and $N^+(v_4) = \{v_2, v_3, v_5\}$. Since $v_3v_1, v_4v_1 \notin E(H)$, $v_2v_1 \in E(H)$. Similarly, $v_5v_6 \in E(H)$. Since $N^-(v_2)$ induces a K_3 in G and $v_3, v_4 \in N^-(v_2)$, we have $v_1v_2 \in E(H)$. Similarly, $v_6v_5 \in E(H)$. For $i = 5, 6$, let w_i be the out-neighbor of v_i outside Q and w_i' be the other neighbor of v_i outside Q. Then, using the definition of H, we may assume that $f(w_5) = 2 = f(w_6)$ and $f(w_5') = 3 = f(w_6')$. Since $v_3v_6, v_4v_6 \notin E(H)$, $N^-(v_6) = \{v_5, w_6, w_6'\}$ and hence by Claim 1 (1), and the fact that $d(v_5) \le 5$, we have $w_5 = w_6$, $w_5' = w_6'$, and $w_5w_5' \in E(G)$. For convenience, denote $v_7 := w_5$ and $v_8 := w_5'$. Let $Q_3 := \{v_5, v_6, v_7, v_8\}$. Then the map $v_i \mapsto v_{i+4}$ is an isomorphism from $Q_1 \cup Q_2$ to $Q_2 \cup Q_3$. Moreover, $Q_1 \cap Q_3 = \emptyset$: Using f, the only concern is whether $v_2 = v_7$. But $N^-(v_8)$ has a vertex v_5 colored 5, while $N^-(v_2) = \{v_1, v_3, v_4\}$ does not.

We can iterate this procedure until eventually $v_{10t+1} = v_1$ and $v_{10t+2} = v_2$. If $t > 1$ then after switching colors on the 2-cycles $v_2v_3v_2$ and $v_6v_8v_6$, v_9 has no neighbor colored 3, a contradiction. Thus $t = 1$ and hence G is the 5-core graph F_1.

Case 2. $r = 4$. Let t be the maximum integer such that G contains the square of a t-vertex path (with the vertex set $\{w_1, \ldots, w_t\}$).

Case 2.1. $t \ge 6$. Let $P := G[w_1 \ldots w_6]$. First suppose that $f(w_1) = f(w_6)$. Then $f(w_2) = f(w_5)$. Thus $w_1w_2, w_3w_1, w_4w_3 \in E(H)$. Either $w_2w_3 \in E(H)$ or $w_2w_4 \in E(H)$. In the first case set $C := w_1w_2w_3w_1$; otherwise set $C := w_1w_2w_3w_4w_1$. Regardless, C is a cycle in H. Then $f' := f(C)$ is an equitable 4-coloring that satisfies $f(w_1) \ne f(w_6)$. Thus we may assume that $f(w_1) \ne f(w_6)$. Let $W = \{w_2, w_3, w_4, w_5\}$ and $G' := G - W + w_1w_6$. Then $f|G'$ is a proper 4-coloring of G'.

Every 4-coloring g of G' can be extended to a 4-coloring of G such that $g|W$ is equitable: Let $\alpha := g(w_1)$, $\beta := g(w_6)$ and γ and δ be the remaining two colors. For $i \in \{2, 5\}$, $d_{G-P}(w_i) \le 1$. Thus we can color w_2 from $\{\beta, \gamma\}$ and w_5 from $\{\alpha, \delta\}$. In this way w_2 and w_5 receive different colors. Thus w_3 and w_4 can (and must) be colored with the remaining two colors.

Observe that G' has no nearly equitable 4-coloring: If G' had a nearly equitable 4-coloring, then it could be extended to a nearly equitable 4-coloring of G, a contradiction. It follows that $f|G'$ is an equitable 4-coloring.

Since G is a minimal counterexample, G' contains a 4-core graph B. Since G does not contain B, $w_1w_2 \in E(B)$ and so $w_1 \in V(B)$. Since $d_{G'}(w_1)=3$, $B \neq F_2$; so $B = K_4$. Let $X = W \cup V(B)$. Then $G-X$ has an equitable 4-coloring and $|E(X,G-X)| \leq 2$. If $G[X]$ has a nearly equitable 4-coloring then it can be extended to a nearly equitable 4-coloring of G, a contradiction. Otherwise, $W \subseteq N(\{w_1,w_6\})$ and $V(B) \subseteq N(\{w_2,w_5\})$. Since $\Delta(G)=4$, the only possibility is that $G=F_2$.

Case 2.2. $t \leq 5$. We first show that G induces a *bow tie*, i.e., two 3-cliques with one common vertex and no other edges. If $t=3$ then Q is a bow tie. So suppose $t \geq 4$. Let $W=\{w_1,w_2,w_3,w_4\}$. Note that w_1 and w_4 have distinct out-neighbors in $\{w_2,w_3\}$. Thus w_1 has another out-neighbor $u \notin W$ and u has another in-neighbor v. By Claim 1, v is adjacent to w_1. By the maximality of t, neither u nor v have neighbors in $\{w_2,w_3\}$. Thus $\{w_1,w_2,w_3,u,v\}$ induces a bow tie.

Let $N[v_3] := \{v_1,v_2,v_3,v_4,v_5\}$ induce a bow tie with $v_1v_2, v_4v_5 \in E(G)$. Since the in-neighbors of v_3 are adjacent, we may assume that $N^-(v_3) = \{v_1,v_2\}$. We also may assume that $f(v_i) = i \pmod 4$. Since $f(v_1) = f(v_5)$, $N^+(v_3)=\{v_2,v_4\}$. In particular, we can switch the colors of v_2 and v_3, since $v_2v_3v_2$ is a cycle in H.

Since $v_4 \notin N^-(v_3)$, it has out-neighbors colored with 1 and 2. So $v_5 \in N^+(v_4)$. The other in-neighbor v_6 of v_5 is not in P and is adjacent to v_4. Thus $P' := G[\{v_3,v_4,v_5,v_6\}]$ is a K_4-e with $v_3v_6 \notin E(G)$. The color of v_6 is either 2 or 3. After possibly switching the colors of v_3 and v_2 (this does not change $H[\{v_1,\dots,v_6\}]$), we obtain a K_4-e whose low degree vertices have the same color, and we may assume that this is color 3.

Observe that for $i \in \{4,5\}$, v_i has an out-neighbor u_i colored with 2. Since v_4 and v_5 are not in-neighbors of v_6, the in-neighbors of v_6 are other vertices u_6 and u_6' that are adjacent by Claim 1 (1). Therefore, only one of them, say, u_6, has color 2 and is thus an out-neighbor of v_6. In particular, $v_6u_6v_6$ is a cycle in H. Also $u_6 \neq v_2$, since $v_2v_3 \in E(H)$ and $u_4,u_5 \neq v_2$ by construction. If $u_4=u_5$ then u_4 shares an out-neighbor in $\{v_3,v_4\}$ with v_3 or v_6, a contradiction. Thus we may assume that $u_4 \neq u_6$. After switching the colors of v_6 and u_6 and the colors of v_3 and v_2, v_4 has no neighbor colored with 3, a contradiction. ∎

Now we finish Case B. It follows from Claim 2 that H is symmetric and transitive: Suppose $vw \in E(H)$. Then $N^-[w]$ and $N^-[v]$ are $(r-1)$-cliques with a common vertex v. So they are identical. Thus $wv \in E(H)$. Also, if

$uv \in E(H)$ then $uw \in E(H)$. In particular, every component of H induces a K_{r-1} in G.

Fix a vertex v with color α and let x and y be its two neighbors with the same color β (Claim 1 (3)). Then $x, y \notin N^+[v] = N^-[v] =: Q_1$. Let $x' \in N^-(x)$ with color $\gamma \neq \alpha$. By transitivity, $x' \notin N^+[v]$. By Claim 1 (3), $x'v \notin E(G)$. Let g be the coloring that results from switching the colors of x and x' and let $H' := H(g)$. Then $y \in N^+_{H'}[v] = N^-_{H'}[v] =: Q_2$. So y witnesses that $Q_1 \neq Q_2$, but $v \in Q_1 \cap Q_2$, contradicting Claim 2. ∎

3. The Decomposition

In this section we finish the definition of r-basic graphs by defining r-derived graphs. Then we complete the proof of Theorem 4 by showing that $(C) \Rightarrow (A)$.

Suppose that $G \in \mathcal{G}(r)$ has an equitable, but no nearly equitable, r-coloring. By Lemma 1, G contains an r-core graph Q. Since Q is equitable and G has an equitable coloring, so does G–Q. If G–Q does not have a nearly equitable r-coloring, then arguing by induction, we conclude that $G-Q$ is r-decomposable, and so G is also. If $Q \in \{F_1, F_2\}$ then Q is a component of G and so this is the case. However, if $Q = K_r$ then it could be that $G-Q$ has a nearly equitable r-coloring that cannot be extended to a nearly equitable r-coloring of G. For example, if there exists a set $S \subseteq V(G-Q)$ such that S dominates Q and S is monochromatic in every nearly equitable r-coloring of $G-H$, then no such coloring can even be extended to an r-coloring of G. This motivates the following definitions.

A subset S of the vertices of a graph $G \in \mathcal{G}(r)$ is *special* if S is monochromatic in every non-equitable r-coloring of G. A graph $D \in \mathcal{G}(r)$ is *r-derived* from a graph B' *with respect to* edge w_1w_2, denoted $B' \to D$, if, setting $Q := D - B'$ and $B = B' - w_1w_2$, we have $Q = K_r$, $D - Q = B$ and $N(Q)$ is a special subset of B.

Notice that if B' is r-equitable and D is derived from B' then D is also r-equitable. Moreover, if B is not r-equitable, then $\{w_1, w_2\}$ is special in B. We define the set of r-basic graphs $\mathcal{B}_r$ to be the smallest set containing the set $\mathcal{C}_r$ of r-core graphs subject to the condition that if $B' \in \mathcal{B}_r$ and $B' \to D$ then $D \in \mathcal{B}_r$. Let $\mathcal{D}_r := \mathcal{B}_r - \mathcal{C}_r$ be the set of r-derived graphs. Figure 2 illustrates that $K_4 \to F_3 \to F_4$ and Figure 3 illustrates that $K_3 \to F_5 \to F_6 \to F_7$ and $F_5 \to F_8 \to F_9, F_{10}$.

Let $L(H) := \{v \in V(H) \colon d(v) < \Delta(H)\}$ be the set of *low degree* vertices and $l(H) := |L(H)|$. Suppose that $B' \to D$ with respect to w_1w_2 and $Q := D - B'$. Then (1) $l(D) = |\{w_1, w_2\}| + l(B') - r < l(B')$ and (2) $r = |E(Q, B)| \leq$

$2+|N(Q)|$. It follows from (1) that $\mathcal{D}_r$ is finite. Moreover, since $l(F_1)=0=l(F_2)$, $B' \notin \{F_1,F_2\}$. Suppose $B'=K_r$. Then $N(Q)=\{w_1,w_2\}$, since $N(Q)$ is independent in B. Thus, by (2), $r\leq 4$. A finite search from K_r, $r\in\{3,4\}$, for all r-derived graphs, establishes:

Proposition 1. *If $r\geq 5$ then $\mathcal{D}_r=\emptyset$; $\mathcal{D}_4=\{F_3,F_4\}$; and $\mathcal{D}_3=\{F_5,\dots,F_{10}\}$.*

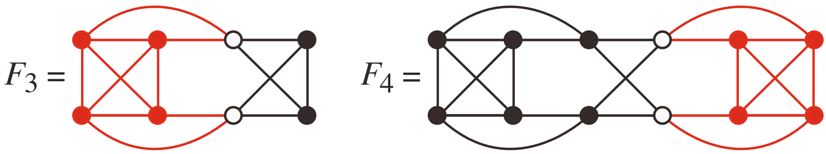

Figure 2. 4-Derived Graphs

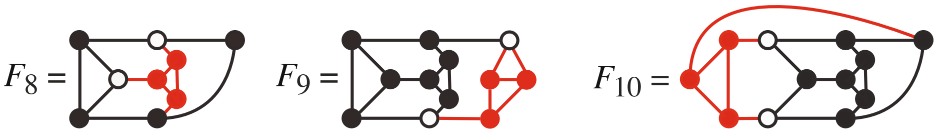

Figure 3. 3-Derived Graphs

Let G be a graph with vertex partition $\{V_1,V_2\}$. We say that an r-coloring f_1 of $G[V_1]$ is *consistent with* an r-coloring f_2 of $G[V_2]$ if there exists a permutation σ of $[r]$ such that $f_1\cup\sigma\circ f_2$ is a proper coloring of G. Define a bipartite graph $R:=R(G,f_1,f_2)$ with partite sets $V(f_1)=\{a_1,\dots,a_r\}$ and $V(f_2)=\{b_1,\dots,b_r\}$ by $a_ib_j\in E(R)$ if and only if $f_1^{-1}(i)\cup f_2^{-1}(j)$ is independent. Then f_1 is consistent with f_2 if and only if R has a 1-factor. The following lemma is an easy consequence of Hall's Theorem.

Lemma 2. *Let $\{V_1,V_2\}$ and f_1,f_2 be as above. If $|E(V_1,V_2)|\leq r$ then f_1 is consistent with f_2 unless there exists $i\in[2]$ such that some color class of f_i has neighbors in every color class of f_{3-i}. In particular, f_1 and f_2 are consistent if $|E(V_1,V_2)|<r$.*

If f is an r-coloring of a graph H and $Y\subseteq V(H)$, then Y is said to be *colorful* if every color appears on a vertex of Y.

Lemma 3. *Suppose that $H \in \mathcal{G}(r)$ and $Y \subseteq L(H)$ with $|Y| = r \geq 3$. Then H has an r-coloring for which Y is not monochromatic. Moreover, if Y is not a clique then H has an r-coloring for which Y is not colorful.*

Proof. Let $y_1, y_2, y_3 \in Y$. If $H_{i,j} := H + y_i y_j \in \mathcal{G}(r)$ for some distinct $i, j \in [3]$ then we are done. Otherwise, by Brooks' Theorem, there exists an $(r-1)$-clique Q such that $N(y_i) = Q$ for all $i \in [3]$. Then any $x \in Q$ has degree greater than r, a contradiction.

If Y is not an r-clique then add a new vertex w to H with $N(w) = Y$ to obtain $H^+ \in \mathcal{G}(r)$. So H^+ has an r-coloring h^+. The color of w does not appear on any vertex of Y. Thus Y is not colorful in $h^+|H$. ∎

Lemma 4. *Suppose that $H \in \mathcal{G}(r)$ and $\{V_1, V_2\}$ is a partition of $V(H)$ such that $H_1 := H[V_1]$ is r-equitable, $H_2 := H[V_2]$ has a nearly equitable r-coloring and $|E(V_1, V_2)| \leq r$. If either $|E(V_1, V_2)| < r$, or $N_1 := N_{V_1}(V_2)$ is not an r-clique, or $N_2 := N_{V_2}(V_1)$ is not special, then H has a nearly equitable r-coloring.*

Proof. It suffices to show that some nearly equitable r-coloring of H_2 is consistent with an (equitable) r-coloring of H_1. If $|E(V_1, V_2)| < r$ then, by Lemma 2, any r-coloring is consistent with any nearly equitable r-coloring of H_2. Otherwise $|E(V_1, V_2)| = r$. Note that $N_i \subseteq L(H_i)$, for $i \in [2]$. If N_1 is not an r-clique then by Lemma 3, H_1 has an r-coloring h_1 that is not monochromatic on N_1 and an r-coloring h_1' that is not colorful on N_1. By Lemma 2, any nearly equitable r-coloring of H_2 is consistent with one of these r-colorings of H_1. If N_2 is not special then H_2 has a nearly equitable r-coloring h_2 for which N_2 is not monochromatic. By Lemma 3, H_1 has an r-coloring h_1 for which N_1 is not monochromatic. By Lemma 2, h_1 is consistent with h_2. ∎

The next lemma completes the proof of Theorem 4.

Lemma 5. *Let $G \in \mathcal{G}(r)$ with $|G|$ divisible by r, where $r \geq 3$. If G has an equitable r-coloring, but no nearly equitable r-coloring, then G is r-decomposable.*

Proof. Suppose that G is a vertex minimum counterexample. Then G has an equitable r-coloring, but neither an r-decomposition nor a nearly equitable r-coloring.

Claim 3. *If $Q \subseteq G$ is r-basic then (1) $G - Q$ has a nearly equitable r-coloring, (2) $Q = K_r$, (3) every vertex in Q has a (unique) neighbor in $G - Q$ and (4) $N(Q)$ is special in $G - Q$.*

Proof. Since G is not r-decomposable, neither is $G-Q$. Since G has an equitable r-coloring, so does $G-Q$. By minimality, (1) holds. Note that $|E(Q,G-Q)|\leq r$ with equality only if $Q=K_r$. By Lemma 4, if any of (2–4) fail then some nearly equitable r-coloring f' of $G-Q$ can be extended to an r-coloring f of G. Since Q is r-basic, f is a nearly equitable, a contradiction. ∎

By Lemma 1 G contains an r-core graph Q and by Claim 3 (2), $Q=K_r$. Let $W:=N(Q)$ and $G_1:=G-Q$.

Claim 4. *G_1 has an equitable or nearly equitable r-coloring that is not monochromatic on W.*

Proof. By hypothesis G has an equitable r-coloring h. Then $h_1:=h|V(G_1)$ is an equitable r-coloring of G_1. If W is not monochromatic in h_1 then we are done. So suppose that $W\subseteq h_1^{-1}(\alpha)$. If some $w\in W$ has no neighbor in a class distinct from $h_1^{-1}(\alpha)$ then after moving w there, we have a nearly equitable r-coloring that is not monochromatic on W. Otherwise, $|W|=r>2$ and each vertex in W has exactly one neighbor in each class $h_1^{-1}(\beta)$ with $\alpha\neq\beta$. Fix $\beta\neq\alpha$. Let $P:=x_1x_2\dots x_t$ be a maximal path in $X:=h_1^{-1}(\alpha)\cup h_1^{-1}(\beta)$ starting with $x_1:=w_1$ and subject to the condition that $d_X(x_i)\leq 2$ for each $i\in[t]$. Then no vertex of W is an internal vertex of P, and so there exists a vertex $w\in W-V(P)$. If x_t has a neighbor $x_{t+1}\in X$ with $d_X(x_{t+1})>2$ then there exists a class $h_1^{-1}(\gamma)$ disjoint from X in which x_{t+1} has no neighbor. Then move x_{t+1} there. Regardless, switch the colors of the vertices of P between α and β. The result f_1 is an equitable or nearly equitable r-coloring of G_1 for which W is not monochromatic. ∎

Let f_1 be as in Claim 4. By Claim 3 (4), G_1 has no nearly equitable r-coloring for which W is not monochromatic. So f_1 is equitable. Choose $w_1,w_2\in W$ so that $f_1(w_1)\neq f_1(w_2)$ and let $G':=G_1+w_1w_2$. So f_1 is also an equitable r-coloring of G' and G' has no nearly equitable r-coloring. By minimality, G' is r-decomposable. Since G_1 is not r-decomposable, there exists $B'\subseteq G'$ such that B' is r-basic, $w_1w_2\in B'$ and $G'-B'$ is r-equitable. Let $B:=B'-w_1w_2$, $G_2:=G_1-B$ and $G_3:=G-G_2=G[Q\cup B]$. So G_2 is r-equitable.

Let $L:=L(B)$ and $l:=l(B)$. We shall need the following inequalities:

$$2\leq|E(B,Q)|\leq\min(l+2,r), \tag{1}$$

$$|E(G_2,G_3)|\leq r+l+2-2\,|E(B,Q)|\,. \tag{2}$$

Case 1. $B'=K_r$. If $W\subseteq B$ then by (2) $|E(G_2,G_3)|\leq l+2-r<r$. Thus every r-coloring of G_2 is consistent with every r-coloring of G_3. It then follows from Claim 3 (4) that W is special in B. So G_3 is r-derived from

B' and thus r-basic, a contradiction. Otherwise there exists $v \in V(Q)$ such that $U := \{v, w_1, w_2\}$ is an independent set. Also $|E(U,G_2)| \le 3$. If $r = 3$ and $G_2[N(U)] = K_3$ then $G[G_3 \cup N(U)] = F_8$, a contradiction. Otherwise, by Lemma 3 we can choose an r-coloring f_2 of G_2 and a color α such that α does not appear on $N(U)$. Extend f_2 to f by coloring the vertices of U with α, and then coloring $G_3 - U$ greedily, ending with v_1. This is possible, since at the time any vertex is colored either it has an uncolored neighbor or it is adjacent to two vertices in U (with the same color). Since $f|G_3$ is a nearly equitable r-coloring and G_2 is r-equitable, f is a nearly equitable r-coloring of G.

Case 2. $B' \ne K_r$. Then $l \le 2$. By (1) and the fact that G_3 is not derived from B', and using Lemma 4, we see that G_3 has a nearly equitable r-coloring f_3. By (2), $|E(G_2,G_3)| \le r$. If $|E(G_2,G_3)| < r$ then f_3 is consistent with any (equitable) r-coloring of G_2, a contradiction. Otherwise, setting $N_2 := N_{V(G_2)}(V(G_3))$ and $N_3 := N_{V(G_3)}(V(G_2))$, we have $N_3 \subseteq (Q \setminus N(L)) \cup L$. Also $|N_3| = r$. If $N_2 \ne K_r$ or N_3 is not special then by Lemma 4, G has a nearly equitable r-coloring. Otherwise, N_3 is independent and so has only one vertex in Q. It follows that $r = 3$, $l = 2$ and $L \subseteq N_3$. So $B' = F_5$. Since the two vertices of L are adjacent in B', but not B, $L = \{w_1, w_2\}$. It follows that $G[V(G_3) \cup N_2] = F_9$, a contradiction. ∎

4. Proofs of Corollaries

In this section, we prove the corollaries of Theorem 4.

Lemma 6. *Suppose Conjecture 1 holds for graphs in $\mathcal{G}(r,n)$, and $G \in \mathcal{G}(r,n)$. If $|G|$ is not divisible by r, then G has an equitable r-coloring.*

Proof. Arguing by contradiction, let G be a minimal counterexample. If every component of G has an equitable r-coloring, then so does G. Thus some component G' does not. Since Conjecture 1 holds for G', r is odd and there exists $G_1 \subseteq G' \subseteq G$ with $G_1 = K_{r,r}$. Since r is odd, G_1 has a nearly equitable r-coloring g. By the minimality of G, the graph $G_2 := G - G_1$ has an equitable r-coloring f. Since $|G_2|$ is not divisible by r, f has two color classes X and Y with $|X| + 1 = |Y|$. Combining the small class of g with Y and the large class of g with X yields an equitable coloring of G', a contradiction. ∎

Lemma 7. *If Conjecture 2 holds for graphs in $\mathcal{G}(r,n)$, then every graph $G \in \mathcal{G}(r,n)$ has either an equitable or nearly equitable r-coloring.*

Proof. If G has no equitable r-coloring then G contains $G_1=K_{r,r}$ and $G–G_1$ is equitable by Conjecture 2. Combining a nearly equitable coloring of G_1 with any (equitable) r-coloring of $G-G_1$ yields a nearly equitable coloring of G. ∎

Proof of Corollary 1. Fix $r\geq 3$ and assume that Conjecture 1 holds for graphs in $G\in\mathcal{G}(r,n)$. Arguing by contradiction, let $G\in\mathcal{G}(r,n)$ be a graph with the fewest vertices such that Conjecture 2 fails for G. Then G has no equitable r-coloring, r is odd, G contains $G_1=K_{r,r}$ and $G-G_1$ is not r-decomposable. So by Theorem 4, $G-G_1$ is not equitable. By Lemma 6, r divides $|G|$. Any nearly equitable r-coloring of $G-K_{r,r}$ can be extended to an equitable r-coloring of G. Thus $G-G_1$ has no such coloring. By the minimality of G, Conjecture 2 holds for $G-G_1$. Thus by Lemma 7, $G-G_1$ has an equitable coloring. So by Theorem 4, $G-K_{r,r}$ is r-decomposable. ∎

Lemma 8. *Let $\{V_1,\dots,V_t\}$ be an r-decomposition of a graph $G\in\mathcal{G}(r)$. If $B\subseteq G$ is r-basic then $B\subseteq G[V_i]:=G_i$ for some $i\in[t]$.*

Proof. For all $i\in[t]$ and $v\in V(B)\cap V_i$, every $y\in N_B(v)$ is in V_i, except possibly one. It follows that every 3-clique and every K_4-e of B is contained in some part V_i. Let $\{v_1,v_2,v_3\}\subseteq V(B)\cap V_i$ be a 3-clique. If $B\in\{K_r,F_1,\dots,F_4\}$ then $V(B)$ can be ordered as $v_1,\dots,v_{|B|}$ so that v_{i+2} has at least two neighbors in $\{v_1,\dots,v_i\}$ for all $i\in[|B|-2]$. It follows that $B\subseteq G_i$. Otherwise $r=3$ and $B\neq K_3$. Let $v\in V(B\cap G_i)$.

Suppose v has two adjacent neighbors. Since they form a K_3 with v, they are both in G_i. Examining the 3-basic graphs shows that in this case, if $G_i\neq K_3$ then $d_{G_i}(v)=3$, and so $N(v)\subseteq G_i$. Suppose $B\in\{F_5,F_6,F_7\}$. Each $Q_j=K_4-e$ in B is contained in a part V_j of the decomposition. Thus $N[Q_j]\subseteq V_j$. Since the various $N[Q_j]$ cover B and are overlapping, $B\subseteq G_j$.

If $B\in\{F_8,F_9\}$ then B contains two disjoint, but adjacent 3-cliques Q_1 and Q_2. Moreover there are two vertices $v_1,v_2\notin Q_1\cup Q_2$ that have neighbors in both. It follows that v_1 is in the same part V_j as one of them. Then $G_j\neq K_3$. So $Q_1,Q_2\subseteq V_j$. Thus $v_1,v_2\in V_j$. Depending on whether $B=F_8$ or $B=F_9$, either the last vertex of B is adjacent to v_1 and v_2 or the remaining vertices induce a K_4-e that is adjacent to v_1 and v_2. Regardless, $B\subseteq G_j$.

Finally, suppose that $B=F_{10}$. Then B consists of two 3-cycles $C^1=x_1x_3x_5x_1$ and $C^2=x_2x_4x_6x_2$ and a 6-cycle $C^3=y_1\dots y_6y_1$ with $x_iy_i\in E(B)$ for all $i\in[6]$. Each of C^1 and C^2 is contained in (possibly identical) parts of the partition; say $C^1\subseteq G^1$ and $C^2\subseteq G^2$. First suppose that some $G^i\neq K_3$. By our observation above, $N(C^i)\subseteq G^i$. Then every vertex in $N(C^{3-i})$ has two neighbors in G^i and hence is in G^i. So $G^{3-i}=G^i$, since otherwise $l(G^i)=3$.

So suppose that $G^1 = K_3 = G^2$. Then there exists a part G^3 that contains C^3, since every vertex of C_3 has at least two neighbors in the same part. But then $G^3 = C_6$, a contradiction. ∎

It is worth remarking that the only r-basic graphs that are properly contained in any other r-basic graphs are K_3 and K_4.

Proof of Corollary 2. Suppose that $\mathcal{D}$ and $\mathcal{D}'$ are r-decompositions of G and $U \in \mathcal{D}$. By Lemma 8, there exists $U' \in \mathcal{D}'$ and $U'' \in \mathcal{D}$ such that $U \subseteq U' \subseteq U'' = U$. Thus $\mathcal{D} \subseteq \mathcal{D}'$ and by symmetry $\mathcal{D} = \mathcal{D}'$. ∎

Proof of Corollary 3. By Theorem 4 it suffices to determine whether G is r-decomposable. We will sketch a recursive polynomial time algorithm for obtaining an r-decomposition of a graph G or deciding that G is not r-decomposable, without worrying about optimizing the degree of the polynomial. First determine whether G has an r-basic subgraph $B \neq K_r$. This can be done in polynomial time since such graphs have at most 12 vertices. If such a B is found then, by Lemma 8 and our remark, $V(B)$ is a part of every r-decomposition of G. So apply the algorithm recursively to $G - B$.

Otherwise, determine whether G contains $B = K_r$. Since $\Delta(G) \leq r$, we can do this in polynomial time: Every K_r is contained in the closed neighborhood of some vertex and it is easy to check in $O(r^2)$ steps whether the closed neighborhood of a vertex contains K_r. If no such B is found, then G has no r-basic subgraph, and hence is not r-decomposable. Otherwise, since B is not properly contained in any r-basic subgraph, it is a part of every r-decomposition of G. So apply the algorithm recursively to $G - B$. ∎

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