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Minors in Graphs with High Chromatic Number

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We develop lower bounds on the Hadwiger number $h(G)$ of graphs G with high chromatic number. In particular, if G has n vertices and chromatic number k then $h(G) \geqslant (4k - n)/3$.

1. Introduction

The order of the largest complete minor of a graph G is called the *Hadwiger number* of G, denoted by $h(G)$. The well-known conjecture of Hadwiger asserts that if G has chromatic number k then $h(G) \geqslant k$. Hadwiger's conjecture is straightforward for $k = 3$ and was proved for $k = 4$ by Dirac [1]. Wagner [8] proved that the case $k = 5$ is equivalent to the Four Colour Theorem, and Robertson, Seymour and Thomas [6] did the same for the case $k = 6$. But for $k \geqslant 7$ the conjecture remains unknown.

In recent times, the case of graphs of very high chromatic number has attracted more attention. In particular, graphs with $\alpha = 2$ (where $\alpha(G)$ is the independence number) have been studied: see Plummer, Stiebitz and Toft [5] for an essay on this case. It is easy to show that if G has order n and $\alpha(G) \leqslant 2$, then $h(G) \geqslant n/3$ (see Lemma 2.1 below). The main result of this paper extends this bound to take account of the chromatic number $\chi(G)$.

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Theorem 1.1. *Let G be a graph with n vertices and chromatic number k. Then $h(G) \geqslant (4k - n)/3$.*

Theorem 1.1 gives better lower bounds for $h(G)$ when k is large. For example, if $k \geqslant 2n/3$, then it yields $h(G) \geqslant 5k/6$. The inequality becomes weaker as k diminishes – indeed for $k \leqslant n/4$ it gives nothing. In Theorem 4.2 we give other lower bounds on $h(G)$ that are better than Theorem 1.1 for $k < 3n/8$. We state them in Section 4.

2. Graphs with $\alpha = 2$

Duchet and Meyniel [2] showed that a graph G of order n has a connected dominating set of order at most $2\alpha(G) - 1$, and hence $h(G) \geqslant n/(2\alpha(G) - 1)$. As mentioned above, this is easy to show if $\alpha = 2$; we state something very slightly stronger in a form that will be helpful to us.

Lemma 2.1. *If G has $n \geqslant 5$ vertices and $\alpha(G) \leqslant 2$, then $h(G) \geqslant 1 + n/3$.*

Proof. We proceed by induction on n. Since $\chi(G) \geqslant n/2$, the theorem holds for $5 \leqslant n \leqslant 8$, by Hadwiger's conjecture for graphs of chromatic number 3 or 4. Suppose $n \geqslant 9$. Now G has at most two components because $\alpha(G) \leqslant 2$. Either each is complete, in which case $h(G) \geqslant n/2 > 1 + n/3$, or G contains an induced path xyz of length two. Since $\alpha(G) \leqslant 2$, every vertex of $G - \{x, y, z\}$ is joined to at least one of x and z, and so $h(G) \geqslant 1 + h(G - \{x, y, z\}) \geqslant 1 + 1 + (n - 3)/3$ by the induction hypothesis, completing the proof. □

A graph G is called *k-critical* if $\chi(G) = k$ but $\chi(G - v) = k - 1$ for every vertex v. The following result about critical graphs is quite deep, though a simpler variant of Gallai's original proof has been given by Stehlík [7].

Theorem 2.2 (Gallai [3]). *Let G be a k-critical graph where $k \geqslant 3$. If $|V(G)| \leqslant 2k - 2$, then G has a spanning complete bipartite subgraph.*

It follows from Gallai's theorem that if G is k-critical then it is the *join* of vertex-disjoint subgraphs $G_1, \dots, G_\ell$ (meaning that any two vertices from different G_i must be adjacent), where each G_i is either a single vertex or it is k_i-critical for some $k_i \geqslant 3$, $|V(G_i)| \geqslant 2k_i - 1$, and G_i has no *apex* vertex joined to all the other vertices of G_i. Of course, $\ell = 1$ is allowed. It is important for the next lemma that $k = k_1 + \cdots + k_\ell$ and $h(G) \geqslant h(G_1) + \cdots + h(G_\ell)$.

We can now prove Theorem 1.1 for graphs with $\alpha = 2$.

Lemma 2.3. *Let G be a graph with n vertices and chromatic number k, having $\alpha(G) \leqslant 2$. Then $h(G) \geqslant (4k - n)/3$.*

Proof. Suppose the lemma is false, and let G be a minimal counterexample. Certainly G is k-critical. By the remarks following Theorem 2.2 it follows that the lemma fails for some subgraph G_i, and the minimality of G thus means $\ell = 1$. But the lemma is true if $k = 1$ and so Theorem 2.2 implies that $n \geqslant 2k - 1 \geqslant 5$. Lemma 2.1 then gives $h(G) \geqslant 1 + n/3 = 1 + 2n/3 - n/3 \geqslant 1 + 2(2k - 1)/3 - n/3 > (4k - n)/3$, contradicting the choice of G. □

3. Graphs with $\alpha = 3$

The main tool we need for the proof of Theorem 1.1 is information about graphs with $\alpha = 3$, as given by the next theorem. We write $N(v)$ for the neighbourhood of a vertex v and $\overline{N}(v)$ for the set $V(G) - N(v) - \{v\}$ of non-neighbours. Given a subset $S \subset V(G)$, we write $G[S]$ for the subgraph of G induced by S.

Definition. A graph G is called *critical-like* if both the following hold:

(1) G has no vertex v for which $G[N(v)]$ is complete, and
(2) G has no vertex v with $N(v) = V(G) - \{v\}$ (that is, no apex vertex).

The term 'critical-like' is used because a k-critical graph G generally has these properties. To be precise, if G has a vertex v for which $G[N(v)]$ is complete then, because the degree of v is at least $k - 1$, G itself must be complete. Moreover G has an apex vertex v if and only if $G - v$ is $(k - 1)$-critical.

We can now state our main result about graphs with $\alpha = 3$.

Theorem 3.1. *Let G be a critical-like graph with $\alpha(G) = 3$. Then there is a set $S \subset V(G)$ with $|S| = 5$, such that $G[S]$ is connected and bipartite.*

Proof. Condition (1) of the definition means that G has no vertices of degree 0 or 1, and so G must have at least 5 vertices, because $\alpha(G) = 3$. Suppose that the theorem is false and that G is a counterexample.

Note first that G contains no induced $K_{1,3}$. For if $G[\{x, y_1, y_2, y_3\}]$ is such a subgraph with centre x, then by condition (2) there is a vertex $z \in \overline{N}(x)$, and since $\{y_1, y_2, y_3, z\}$ is not independent this means $G[\{x, y_1, y_2, y_3, z\}]$ is connected and bipartite, a contradiction.

Let $\{a, b, c\}$ be an independent set of size 3. Every other vertex has at least one neighbour in $\{a, b, c\}$ because $\alpha(G) = 3$, and no vertex has three such neighbours because G has no induced $K_{1,3}$. It follows that $V(G) - \{a, b, c\}$ is partitioned into six sets A, B, C, AB, AC, BC, where A is the set of vertices joined to precisely a in the set $\{a, b, c\}$, AB is the set of vertices joined to precisely a and b, and the other sets are defined similarly.

If $u, v \in A$ then $uv \in E(G)$, else $\{u, v, b, c\}$ would be independent. Thus $G[A]$ is complete, and likewise so are $G[B]$ and $G[C]$. Now $N(a) = A \cup AB \cup AC$ and so condition (1) implies $AB \cup AC \neq \emptyset$. Likewise, $AB \cup BC \neq \emptyset$ and $AC \cup BC \neq \emptyset$; in other words, at most one of AB, AC, BC is empty.

If $u \in AB$ and $v \in BC$ then $uv \in E(G)$, for otherwise $G[\{a, u, b, v, c\}]$ is connected and bipartite. Likewise any two vertices lying in distinct sets among AB, AC, BC are adjacent. Suppose now that $u, v \in AB$ are not adjacent. Since $AC \cup BC \neq \emptyset$, we can pick $w \in AC \cup BC$. But w is adjacent to both u and v, which means that $\{w, u, v, c\}$ induces a $K_{1,3}$, a contradiction. Hence $G[AB]$ is complete, and likewise so are $G[AC]$ and $G[BC]$. Thus we have shown that $G[AB \cup AC \cup BC]$ is complete.

Not all AB, AC, BC are empty, so we may suppose that $AB \neq \emptyset$, say $w \in AB$. Suppose there exist $u \in AC \cup BC$ and $v \in C$ with $uv \notin E(G)$. Then either $G[\{b, w, u, c, v\}]$ (if $u \in AC$) or $G[\{a, w, u, c, v\}]$ (if $u \in BC$) is connected and bipartite. This contradiction means that every vertex in $AC \cup BC$ is joined to every vertex in C. But then $G[N(c)] = G[C \cup AC \cup BC]$ is complete, contradicting (1) and finishing the proof. □

4. Lower bounds on $h(G)$

We begin by proving Theorem 1.1.

Proof of Theorem 1.1. Suppose that the theorem is false and that G is a minimal counterexample. Then G is k-critical and, by Lemma 2.3, $\alpha(G) \geqslant 3$.

Assume first that $\alpha(G) = 3$. Now G cannot have a vertex v for which $G[N(v)]$ is complete because otherwise, as remarked earlier, $G = K_k$ and K_k is not a counterexample. Nor can G have an apex vertex w, for then $\chi(G - w) = k - 1$ and $h(G) = 1 + h(G - w)$, so $h(G) = 1 + h(G - w) \geqslant 1 + (4(k-1) - (n-1))/3 = (4k - n)/3$. Thus G is critical-like and so, by Theorem 3.1, G contains a set S of 5 vertices such that $G[S]$ is connected and bipartite. Now $G[S]$ contains an independent set of size (at least) 3 and so this independent set dominates $G - S$. Hence $h(G) \geqslant h(G - S) + 1$. But $\chi(G - S) \geqslant k - 2$ since $G[S]$ is bipartite. Hence $h(G) \geqslant 1 + h(G - S) \geqslant 1 + 4(k-2)/3 - (n-5)/3 = 4k/3 - n/3$, contradicting G being a counterexample.

Thus $\alpha(G) \geqslant 4$. Let I be an independent set of size 4. Then $\chi(G - I) \geqslant k - 1$. So $h(G) \geqslant h(G - I) \geqslant 4(k-1)/3 - (n-4)/3 = 4k/3 - n/3$, a final contradiction. □

Theorem 1.1 is weak if k is small relative to n. In such cases we can get a somewhat better bound by making use of the theorem of Duchet and Meyniel [2] cited above. In fact we use the improvement obtained by Kawarabayashi and Song [4].

Theorem 4.1 (Kawarabayashi and Song). *Let G be a graph with n vertices and $\alpha(G) \geqslant 3$. Then $h(G) \geqslant n/(2\alpha(G) - 2)$.*

The bounds we obtain on $h(G)$ can be understood more clearly if we write $y = h(G)/n$ and $x = \chi(G)/n = k/n$. Then Theorem 1.1 states that $y \geqslant (4x - 1)/3$, and Hadwiger's conjecture corresponds to $y \geqslant x$. We are interested only in the ranges $0 \leqslant x, y \leqslant 1$. We shall define a sequence of straight lines L_r for integers $r \geqslant 4$ and prove that the points $(x, y) = (k/n, h(G)/n)$ lie above each of these lines.

The line L_4 is the line $y = (4x - 1)/3$ and in general the line L_r is of the form $y = (x - 1/r)/a_r$, where $a_4 = 3/4$ and the other values of a_r are determined recursively. Observe that the line L_r meets the x-axis at $(1/r, 0)$. Let L_r meet the horizontal line $y = 1/(2r - 2)$ at the point $(x_r, 1/(2r - 2))$. Then $x_r = 1/r + a_r/(2r - 2)$; in particular $x_4 = 3/8$. Given a_r, we choose a_{r+1} so that L_{r+1} passes through this same point $(x_r, 1/(2r - 2))$. This means $x_r - 1/(r + 1) = a_{r+1}/(2r - 2)$, which implies $a_{r+1} = a_r + (2r - 2)/r(r + 1)$. This can be

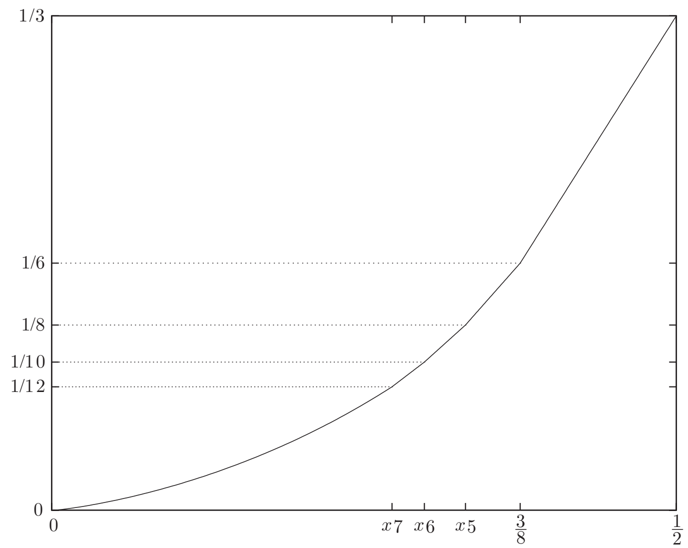

Figure 1. A sketch of the function $g(x)$ for $0 \leqslant x \leqslant 1/2$.

rewritten as $a_{r+1} = a_r - 2/r + 4/(r+1)$, and so

$$a_r = 2H_r + \frac{2}{r} - \frac{47}{12} \quad \text{where } H_r = \sum_{i=1}^{r} \frac{1}{i}.$$

Thus, for example, L_5 is the line $y = 20(x - 1/5)/21$. The slopes of the lines L_r decrease with r. Thus, if we define $x_3 = 1$, we see that the envelope formed by the lines is given by the function

$$g(x) = \frac{x - 1/r}{a_r} \quad \text{for} \quad x_r \leqslant x \leqslant x_{r-1}, \quad \text{where} \quad x_r = \frac{1}{r} + \frac{a_r}{2r-2} \quad \text{for} \quad r \geqslant 4.$$

A sketch of the function $g(x)$ is shown in Figure 1.

Theorem 4.2. *Let G be a graph with n vertices and chromatic number k. Then $h(G) \geqslant ng(k/n)$. In other words, $h(G) \geqslant (k - n/r)/a_r$ for every $r \geqslant 4$.*

Proof. Since $a_4 = 3/4$, the case $r = 4$ is just Theorem 1.1. Proceeding by induction, we suppose the theorem true for some $r \geqslant 4$, and prove it for $r + 1$.

Suppose instead that the theorem fails for $r + 1$ and let G be a smallest counterexample. The slope of the line L_{r+1} is less than the slope of L_r, and both these lines pass through the point $(x_r, 1/(2r-2))$, so $(x - 1/(r+1))/a_{r+1} \leqslant (x - 1/r)/a_r$ for $x \geqslant x_r$. Since G satisfies the theorem for r but not for $r + 1$, it must be that $k/n < x_r$.

As $k/n < x_r$, $(k - n/(r+1))/a_{r+1} < (x_r - 1/(r+1))n/a_{r+1}/(2r-2)$. Since G fails the theorem for $r + 1$, $h(G) < n/(2r-2)$, so by Theorem 4.1 we have $\alpha(G) \geqslant r + 1$. Let I be an independent set of size $r + 1$ in G. By the minimality of G, $h(G) \geqslant h(G - I) \geqslant ((k-1) - (n - (r+1))/(r+1))/a_{r+1} = (k - n/(r+1))/a_{r+1}$, contradicting the choice of G. □

Our interest in this paper has been in graphs of high chromatic number, interpreted as meaning graphs G where $\chi(G)/|G|$ is substantially greater than zero. All the same, we might ask what the bound given by Theorem 4.2 looks like when k/n is small.

The theorem states that $h(G) \geqslant (k - n/r)/a_r$ for $x_{r+1} \leqslant k/n \leqslant x_r$. In this range, the value of $(k - n/r)/a_r$ lies between $n/2r$ and $n/(2r-2)$ (indeed this is how the line L_{r+1} was constructed). If k/n is small then r is large. In this case it is well known that $H_r = \log r + O(1)$, so $a_r = 2\log r + O(1)$ and $x_r = a_r/(2r-2) = \log r/r + O(1/r)$. Consequently $x_{r+1} = \log r/r + O(1/r)$ too, and since $x_{r+1} \leqslant k/n \leqslant x_r$ we have $k/n = \log r/r + O(1/r)$ as well. To express r in terms of k and n, write $x = k/n$; then $x = \log r/r + O(1/r)$ so $r \approx -(\log x)/x$. Hence Theorem 4.2 yields $h(G) \geqslant n/(2r-2) \approx -nx/2\log x$.

This bound can be compared with that given by Theorem 4.1 for a graph G with independence number $\alpha = \alpha(G)$. Because $k \geqslant n/\alpha$ we have $\alpha \geqslant 1/x$. If G is a graph for which k is close to n/α then α is close to $1/x$ and the bound $h(G) \geqslant n/(2\alpha - 2)$ given by Theorem 4.1 is close to $nx/2$, which is better than that given by Theorem 4.2. On the other hand, if k is much larger than $n(\log\alpha)/\alpha$, then α is much larger than $-(\log x)/x$, and the bound given by Theorem 4.2 is better.

References

[1] Dirac, G. A. (1964) Homomorphism theorems for graphs. *Math. Ann.* **153** 69–80.

[2] Duchet, P. and Meyniel, H. (1982) On Hadwiger's number and the stability number. In *Graph Theory* (B. Bollobás, ed.), Vol. 13 of *Annals of Discrete Mathematics*, North-Holland, pp. 71–73.

[3] Gallai, T. (1963) Kritische Graphen II. *Publ. Math. Inst. Hungar. Acad. Sci.* **8** 373–395.

[4] Kawarabayashi, K. and Song, Z. (2007) Independence numbers and clique minors. *J. Graph Theory* **56** 219–226.

[5] Plummer, M. D., Stiebitz, M. and Toft, B. (2003) On a special case of Hadwiger's conjecture. *Discussiones Mathematicae Graph Theory* **23** 333–363.

[6] Robertson, N., Seymour, P. D. and Thomas, R. (1993) Hadwiger's conjecture for K_6-free graphs. *Combinatorica* **13** 279–362.

[7] Stehlík, M. (2009) Critical graphs with connected complements. *J. Combin. Theory Ser. B* **89** 189–194.

[8] Wagner, K. (1937) Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.* **114** 570–590.