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Large Rainbow Matchings in Edge-Coloured Graphs

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Dedicated to the memory of Richard Schelp

A *rainbow subgraph* of an edge-coloured graph is a subgraph whose edges have distinct colours. The *colour degree* of a vertex v is the number of different colours on edges incident with v. Wang and Li conjectured that for $k \geqslant 4$, every edge-coloured graph with minimum colour degree k contains a rainbow matching of size at least $\lceil k/2 \rceil$. A properly edge-coloured K_4 has no such matching, which motivates the restriction $k \geqslant 4$, but Li and Xu proved the conjecture for all other properly coloured complete graphs. LeSaulnier, Stocker, Wenger and West showed that a rainbow matching of size $\lfloor k/2 \rfloor$ is guaranteed to exist, and they proved several sufficient conditions for a matching of size $\lceil k/2 \rceil$. We prove the conjecture in full.

1. Introduction

Some basic graph-theoretic problems can be stated in the language of finding in an edge-coloured graph a given subgraph with restrictions on the colours of its edges. For example, a version of Ramsey's theorem says that for any k, r and t, any huge k-edge-coloured complete r-uniform hypergraph contains a monochromatic t-vertex complete r-uniform hypergraph. In this paper, we consider conditions guaranteeing the existence of a multicoloured matching of r edges in an edge-coloured graph. We consider only *simple* graphs, that is, with no loops or multi-edges.

Let G be an edge-coloured graph (the colouring does not need to be proper). For $v \in V(G)$, $\hat{d}(v)$ is the number of distinct colours on the edges incident with v. This is called the *colour*

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degree of v. The smallest colour degree of all vertices in G is the *minimum colour degree of* G, or $\hat{\delta}(G)$.

A *rainbow matching of* G is a matching in G whose edges have distinct colours. The topic of rainbow matchings has been well studied, along with a more general topic of rainbow subgraphs (see [1] for a survey). In 2008 Wang and Li were able to bound from below the size $r(G)$ of the largest rainbow matching in G in terms of the minimum colour degree of G. They showed [5] that $r(G) \geqslant \lceil \frac{5\hat{\delta}(G)-3}{12} \rceil$ for every graph G. In another paper [3], they proved that $r(G) \geqslant \lceil \frac{2\hat{\delta}(G)}{3} \rceil$ for each bipartite G with $\hat{\delta}(G) \geqslant 3$.

Wang and Li [5] conjectured that the lower bound could be improved to $r(G) \geqslant \lceil \frac{k}{2} \rceil$ for every G with $\hat{\delta}(G) \geqslant k \geqslant 4$. The conjectured bound is sharp for properly coloured complete graphs. The motivation for the restriction $k \geqslant 4$ comes from the fact that a properly edge-coloured K_4 has no rainbow matching of size 2. However, it is an easy excercise to show that each other graph with $\hat{\delta}(G) = k \leqslant 3$ has a rainbow matching with at least $\lceil \frac{k}{2} \rceil$ edges.

Li and Xu [4] gave a result on hypergraphs that proved the conjecture for all properly coloured complete graphs with at least 6 vertices. LeSaulnier, Stocker, Wenger and West [2] proved that $r(G) \geqslant \lfloor \frac{k}{2} \rfloor$ for any edge-coloured graph, and proved several conditions sufficient for a rainbow matching of size $\lceil \frac{k}{2} \rceil$. The sufficient conditions include a bound on n, the number of vertices in G, and thus for each fixed value of k the conjecture only needed to be verified for finitely many graphs.

The aim of this paper is to prove the conjecture of Wang and Li in full.

Theorem 1.1. *If G is not a properly coloured K_4 and $\hat{\delta}(G) \geqslant k$, then $r(G) \geqslant \lceil \frac{k}{2} \rceil$.*

The only known examples for when this bound is sharp have small values for n (relative to k).

In the next section we set up the proof and cite or prove the main facts needed for it. In the last two sections we prove the theorem.

2. Preliminary results

By way of contradiction, let G with edge colouring f be a counterexample to Theorem 1.1 with the fewest edges. Let $k = \hat{\delta}(G)$ and $r := r(G)$. By [2] and [4], we may assume that k is odd, $r = \frac{k-1}{2}$, and G is not a properly coloured complete graph.

Claim 2.1. *The edges of each colour class of f form a forest of stars.*

Proof. Let F be a colour class of f. If an edge $e \in F$ connects two vertices of degree at least two in F, then the colour degrees of all vertices in G and $G-e$ are the same, and any rainbow matching in $G-e$ is a rainbow matching in G. This contradiction to the minimality of G yields the claim. □

Most of the results and notation in this section come from the paper [2] by LeSaulnier, Stocker, Wenger and West.

Let M be a maximum rainbow matching in G, with edge set $\{e_j : 1 \leqslant j \leqslant r\}$, where $e_j = u_j v_j$. Let $H = G - V(M)$. Let E_j denote the set of edges connecting $V(H)$ with $\{u_j, v_j\}$. Let E' be the

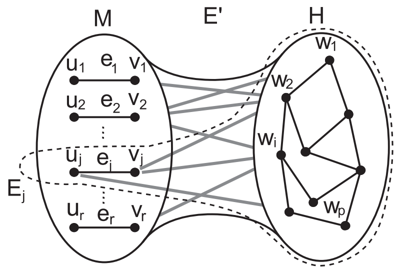

Figure 1. An example of G with notation.

set of edges connecting $V(H)$ with $V(M)$, *i.e.*, $E' = \bigcup_{j=1}^{r} E_j$. Define $p = |V(H)| = n - 2r = n - (k-1)$. Since G is not a properly coloured complete graph, $n \geqslant k+2$ and so $p \geqslant 3$. Label the vertices of H as $\{w_1, w_2, \ldots, w_p\}$.

Without loss of generality, we will assume that edge e_i is coloured i for $i = 1, \ldots, r$. A *free colour* is a colour not used on any of the edges of M. A *free edge* is an edge coloured with a free colour. If a free edge is contained in H, then M is not a maximum rainbow matching, so this is not the case.

Definition 2.2. Let $\phi : V(M) \to [k-1]$ be the ordering with

$$\phi(u_1) < \phi(v_1) < \phi(u_2) < \phi(v_2) < \cdots < \phi(u_{\frac{k-1}{2}}) < \phi(v_{\frac{k-1}{2}}).$$

A free edge wx coloured α is *important* if $x \in V(M), w \in V(H)$, and $\phi(x) = \min_y\{\phi(y) : wy \in E', wy \textit{ is coloured } \alpha\}$. All other free edges in E' are *unimportant*.

The motivation for this definition is that for each $w \in V(H)$ and each free colour α used on an edge incident with w, there is exactly one α-coloured important edge incident with w.

Lemma 2.3 ([2]). *For any $1 \leqslant j \leqslant r$, if there are three vertices in $V(H)$ incident with important edges in E_j, then only one such vertex can be incident with two important edges.*

Configuration A in the set E_j is a set A_j of important edges such that (a) it contains all p edges connecting v_j with H and one edge, say u_jw, incident with u_j; (b) the colour of u_jw (say α) is also the colour of every edge in A_j apart from the edge v_jw (which is different).

In this case, α will be called the *main colour* for E_j. Note that in our definition we are assuming that v_i is the vertex with p important edges and not u_i. This assumption will be used for the rest of the paper.

Corollary 2.4 ([2]). *If $p \geqslant 4$, then there are at most $p+1$ important edges in E_j for each j. Furthermore, if E_j has $p+1$ important edges, then E_j contains configuration A.*

Define *configuration B* to be the set of four edges $B_j = \{wu_j, w'u_j, wv_j, w'v_j\} \subseteq E_j$ such that $w, w' \in V(H)$, all four edges are important, $f(wu_j) = f(w'v_j)$ and $f(wv_j) = f(w'u_j)$.

In this case $f(wu_j)$ and $f(wv_j)$ will be called the *major colours* for E_j.

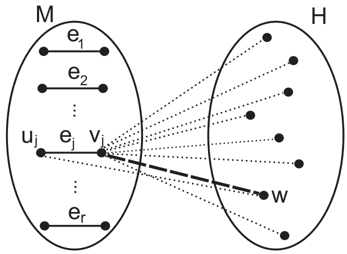

Figure 2. Configuration A.

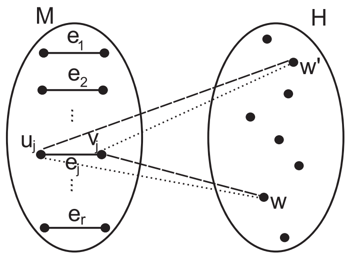

Figure 3. Configuration B.

Corollary 2.5. *If $p = 3$, then there are at most $p + 1 = 4$ important edges in E_j for each j. Furthermore, if E_j has 4 important edges, then E_j contains either configuration A or configuration B.*

Proof. If each of w_1, w_2, w_3 is incident with an important edge, then the proof of Corollary 2.4 goes through and implies that E_j contains configuration A. If only two of them, say w_1 and w_2, are incident with important edges, then, in order to have four such edges, the set of important edges in E_j must be $\{w_1u_j, w_2u_j, w_1v_j, w_2v_j\}$. Since M is a maximum matching, $f(w_1u_j) = f(w_2v_j)$ and $f(w_1v_j) = f(w_2u_j)$. □

While configuration A can occur in graphs of any order, configuration B only occurs when $p = 3$. Let J_A denote the set of indices j such that E_j contains configuration A. Let J_B denote the set of indices j such that E_j contains configuration B. By definition, $J_A \cap J_B = \emptyset$. Define $a = |J_A|$ and $b = |J_B|$. The values of a and b will depend on G, f, and the choice of M.

A colour α is *basic* for E_j if either $j \in J_A$ and α is the main colour for E_j or $j \in J_B$ and α is a major colour for E_j.

Claim 2.6. *The basic colours for distinct E_j are distinct.*

Proof. The edges of a basic colour for E_j are incident with at least $p - 1$ vertices in H. So if some colour α was basic for E_j and $E_{j'}$, then some $w \in V(H)$ would be incident with two edges of colour α, and so one of them would be unimportant, a contradiction. □

Definition 2.7. Let $d_I(e_j)$ denote the number of important edges that are incident with u_j or v_j. Let $d_I(w_j)$ be the number of important edges incident with w_j.

There are only $r = \frac{k-1}{2}$ non-free colours, and each vertex is incident with at least k distinct colours, therefore:

$$\text{Each vertex in } H \text{ is incident with at least } \tfrac{k+1}{2} \text{ important edges.} \tag{2.1}$$

The number of important edges coming out of $V(M)$ equals the number of important edges coming out of H, which gives the inequality

$$\sum_{j=1}^{\frac{k-1}{2}} d_I(e_j) = \sum_{i=1}^{p} d_I(w_i) \geqslant p\,\frac{k+1}{2} = pr + p. \tag{2.2}$$

Since $d_I(e_j) \leqslant p + 1$ for each j, in order to satisfy (2.2):

$$\text{There are at least } p \text{ distinct values of } j \text{ such that } d_I(e_j) = p + 1, \textit{ i.e.}, a + b \geqslant p \geqslant 3. \tag{2.3}$$

Lemma 2.8. *Let i be such that all of the free edges in E_i are important. Let ϕ be the ordering of $V(H)$ described in Definition 2.2. If $j < i$ is fixed, then in the ordering ϕ' of $V(H)$, where*

$$\begin{aligned}\phi'(u_1) < \phi'(v_1) < \phi'(u_2) < \phi'(v_2) < \cdots < \phi'(u_{j-1}) < \phi'(v_{j-1})\\ < \phi'(u_i) < \phi'(v_i) < \phi'(u_j) < \phi'(v_j) < \cdots < \phi'(u_{i-1}) < \phi'(v_{i-1})\\ < \phi'(u_{i+1}) < \phi'(v_{i+1}) < \cdots < \phi'(u_{\frac{k-1}{2}}) < \phi'(v_{\frac{k-1}{2}}),\end{aligned}$$

the set of edges that are important is the same for ϕ and ϕ'.

Proof. The only change from ϕ to ϕ' is that u_i and v_i come earlier. Thus, we will consider the effect of moving one pair of vertices to another spot in the ordering. Note that the number of important edges is not affected by the order of the vertices, only the selection of the set of important edges. Thus, for every edge that is changed from important to unimportant, there must be an edge that changes from unimportant to important. Therefore, since the relative order among all other vertices does not change, it suffices to show that if the status of the edges incident with u_i and v_i does not change, then the set of important edges in the whole graph does not change.

Let e be an edge incident with $u_i \in V(M)$ and $w \in V(H)$ (the case when e is incident with v_i is symmetric). Since e is already important by the hypothesis, it can not *change* into an important edge. By the definition of an important edge, e can turn from important to unimportant if and only if u_i is the earliest edge with its colour incident with w and then moved after another edge with the same colour. And since u_i is being moved earlier by hypothesis, it can not change into an unimportant edge. □

Because M is a maximum rainbow matching, if E_j contains configuration A or B, then E_j contains exaclty $p + 1$ free edges. That is, if $j \in J_A \cup J_B$, then every free edge of E_j is important.

3. Proof of Theorem 1.1: Case $a > 0$

Definition 3.1. A *special vertex* v is a vertex with $d(v) = n - 1$ and $\hat{d}_G(v) = k$ such that one colour appears on $n - k = p - 1$ distinct edges incident with v (each other colour appears exactly once). For a special vertex v, the colour that appears $n - k$ times is called the *main colour of* v.

If a colour is on $n-k$ different edges incident with v, then v is special. This proves that if $j \in J_A$ then v_j is a special vertex.

We call an edge xy a *main edge* if x is special and xy is coloured with the main colour of x. Let M have the most main edges among all rainbow matchings in G with r edges. This implies that:

$$\text{If } i \in J_A \text{ then } u_i \text{ is special and } i \text{ is the main colour of } u_i. \tag{3.1}$$

This is because v_i is special and its main colour is free, and therefore not i. Since e_i could be replaced by one of the main edges of v_i, the choice of M shows that e_i is already a main edge. This shows that e_i is a main edge of u_i.

We will use a fixed index $i \in J_A$. By Lemma 2.8 and the remark immediately after, we may assume $i = 1$.

Consider edges u_1u_j for $j \in J_A \cup J_B$. These edges exist for $j \neq 1$ because u_1 is a special vertex.

Case 1: $f(u_1u_j)$ is the main colour of v_1, or the main colour of v_j (if $j \in J_A$). Without loss of generality, we will assume that u_1u_j is the main colour of v_1. By the definition of configuration A, the main colour of v_1 is free and it is on an edge that is incident with u_1 and a vertex in H. Thus there are two different edges incident with u_1 with the main colour of v_1, but only the main colour may be repeated at special vertex u_1. This creates a contradiction.

Case 2: $f(u_1u_j)$ is 1, j, or free, and neither the main colour of v_j nor a major colour for E_j. In this case, a larger rainbow matching can be obtained by replacing e_1 and e_j with three edges: u_1u_j, a main edge of v_1 (we have $p-1$ choices for such an edge), and either a main edge of v_j (if $i \in J_A$) or a major edge of E_j (if $j \in J_B$).

Case 3: $j \in J_B$ and $f(u_1u_j)$ is a major colour of E_j. If $f(u_1v_j)$ is not a major colour of E_j, then we may swap u_j and v_j (because configuration B is symmetric) and get Case 2. So suppose each of $f(u_1u_j)$ and $f(u_1v_j)$ is a major colour of E_j. Then each of u_j and v_j has a free colour repeated on edges incident with it. Configuration B only occurs only when $p = 3$, so a vertex is special when a colour is repeated $n-k = p-1 = 2$ times. Therefore both u_j and v_j are special with free main colours. This implies that e_j is not a main edge. But this is a contradiction because M could have contained more main edges by replacing edge e_j with a main edge of v_j.

Case 4: $f(u_1u_j) = h$, where $2 \leqslant h \leqslant r$, and $j \in J_A$. Consider an important edge $e \in E_h$. It cannot be coloured with the main colour of v_1 or v_j, or else some vertex in H will be incident with two important edges with the same colour, which is a contradiction. We will attempt to replace e_h, e_1, and e_j with edges e, v_1w_s, u_1u_j, and v_jw_t for some $s \neq t$ that give the main colours of v_1 and v_j. The only way for this to not be possible is if $p = 3$, and the two main edges of v_1 and v_j form a C_4 that is incident with e. But in this case, the important edge that is incident with v_1 and is not a main edge of v_1 is incident with the important edge of v_j that is not a main edge, and they must have different colours. Then we can replace e_h, e_1, and e_j with edges e, v_1w_s, u_1u_j, and v_jw_t for some $s \neq t$ that give the main colour of v_1 or v_j, and a free colour that is not the main colour of either v_1 or v_j and not the colour of e. Therefore E_h has no important edges.

Case 5: $f(u_1u_j) = h$, where $2 \leqslant h \leqslant r$, and $j \in J_B$. Since $p = 3$, $V(H) = \{w_1, w_2, w_3\}$. Without loss of generality, assume that the major edges of E_j are incident with w_1 and w_2.

Consider an important edge $e \in E_h$. It cannot have the main colour of v_1, or have a major colour of E_j and be incident with w_1 or w_2. Suppose first that the edges with the main colour of v_1 incident with v_1 go to w_1 and w_2. If $f(e)$ is a major colour of E_j and e is incident with w_3 (without loss of generality, assume that $f(e) = f(v_j w_1)$), then replace e_1, e_j, and e_h with u_1u_j, e, v_jw_2, and v_1w_1. This will also work if e has any other free colour and is incident with w_3. If e is incident with w_1 and $f(e) \neq v_1w_3$, then replace e_1, e_j, and e_h with u_1u_j, e, v_1w_3, and v_jw_2. This works symmetrically if e is incident with w_2. This leaves only the case when $f(e) = f(v_1w_3)$ and e is incident with w_1 or w_2. By the minimality of G, only two such edges may exist.

Suppose now that the edges with the main colour of v_1 go to w_1 and w_3 (w_2 and w_3 is a symmetric situation). If e is incident with w_1, then replace e_1, e_j, and e_h with u_1u_j, e, v_1w_3, and v_jw_2. If e is incident with w_2, then replace e_1, e_j, and e_h with u_1u_j, e, v_1w_3, and v_jw_1. This leaves only the case when e is incident with w_3. Since G is a simple graph, only two such edges may exist.

Cases 1, 2 and 3 all led to contradictions. The vertex u_1 is special with main colour 1. Therefore, there must be $a-1$ instances of Case 4 and b instances of Case 5. This creates $a-1$ values of i where E_i has no important edges and b other values of i where E_i has at most 2 important edges. By definition, for all $i \notin J_A \cup J_B$, the set E_i has at most p important edges. Then

$$\begin{aligned}\sum_{i=1}^{r} d_I(e_i) &= \sum_{i\in J_A\cup J_B} d_I(e_i) + \sum_{i\notin J_A\cup J_B} d_I(e_i)\\ &\leqslant (p+1)(a+b) + ((a+b-1)2 + p(r-(a+b)-(a+b-1)))\\ &= pr + (a+b) - (p-2)(a+b-1).\end{aligned}$$

Recall that by (2.3), $a+b \geqslant 3$. Thus, since $p \geqslant 3$ and $a \geqslant 1$,

$$\sum_{i=1}^{r} d_I(e_i) < pr + p, \tag{3.2}$$

a contradiction to (2.2).

4. Proof of Theorem 1.1: Case $a = 0$

If $a = 0$, then $b \geqslant 3$ by (2.3). This also implies that $p = 3$ and $V(H) = \{w_1, w_2, w_3\}$.

We will partition J_B into three sets: J_B^1 will be the set of indices i such that the free edges of E_i are incident with w_1 and w_2; J_B^2 will be the set of indices i such that the free edges of E_i are incident with w_1 and w_3; and J_B^3 will be the set of indices i such that the free edges of E_i are incident with w_2 and w_3. We define $b_1 = |J_B^1|$, $b_2 = |J_B^2|$, and $b_3 = |J_B^3|$, so that $b_1 + b_2 + b_3 = b$.

Subsection 4.1 will cover the situation when at least two of the values b_1, b_2, and b_3 are positive. The vertices w_1, w_2, and w_3 can be reordered, so that b_1 is the smallest positive value of the three. Then $0 < b_1 \leqslant b_2 + b_3$.

Subsection 4.2 will cover the situation when two of the values are zero. Without loss of generality, we will assume $b_3 = b_2 = 0$ and $b_1 = b$.

In both subsections, $b_1 > 0$. We will use a fixed index $i \in J_B^1$. By Lemma 2.8 and the remark immediately after, we may assume $i = 1$. We will show that:

$$\text{There are } b-1 \text{ values for } j \text{ such that } E_j \text{ has 2 or fewer important edges.} \tag{4.1}$$

If (4.1) holds, then it generates a contradiction to (2.2) exactly as in (3.2).

4.1. Subcase: $a = 0$ and $1 \leqslant b_1 \leqslant b_2 + b_3$.

Since $k = n-2$, $\hat{d}(u_1) \geqslant n-2$. Thus the number of distinct colours on the edges connecting u_1 with $\bigcup_{i \in J_B^2 \cup J_B^3} \{u_i, v_i\}$ is at least $2(b_3 + b_2) - 1 \geqslant b - 1$.

Case A: $i \in J_B^3$, and the edge u_1u_i exists. This is symmetric to the case when $i \in J_B^2$.

If $f(u_1u_i) = f(v_1w_1)$ (Case $f(u_1u_i) = f(v_iw_3)$ is symmetric), then we replace edges e_1 and e_i in M with edges v_1w_2, v_iw_3, and u_1u_i. If $f(u_1u_i)$ is equal to a free colour other than $f(v_1w_1)$ or $f(v_iw_3)$, then replace edges e_1 and e_i in M with edges v_1w_1, v_iw_3, and u_1u_i.

It follows that $f(u_1u_i)$ is not free. We will consider what important edges may be in E_h for $f(u_1u_i) = h$. Suppose $e \in E_h$ is an important edge. First, assume that e is incident with w_2. Since w_2 is incident with at most one important edge of each colour, $f(e) \neq f(u_1w_2)$ and $f(e) \neq f(u_iw_2)$. So, since $f(u_1w_2) = f(v_1w_1)$ and $f(u_iw_2) = f(v_iw_3)$, we can replace edges e_1, e_i, and e_h in M with edges u_1u_i, e, v_1w_1 and v_iw_3. Thus e is not incident with w_2. Second, assume that e is incident with w_3. Since w_3 is incident with at most one important edge of colour $f(e)$, we have $f(e) \neq f(u_iw_3) = f(v_iw_2)$. If also $f(e) \neq f(v_1w_1)$, then we replace in M edges e_1, e_i, and e_h with u_1u_i, e, v_1w_1 and v_iw_2. Finally, assume that $f(e) = f(v_1w_1)$. Again, since w_3 is incident with at most one important edge of colour $f(v_1w_1)$, only one edge incident with w_3 in E_h can be important. So, altogether E_h has at most two important edges.

Case B: $i \in J_B^2 \cup J_B^3$, and the edge u_1v_i exists. By the symmetry of configuration B, the proof is exactly the same as in case A.

This implies that there are $b-1$ values for j such that E_j has 2 or fewer important edges. Thus (4.1) holds.

4.2. Subcase: $a = 0$ and $b_3 = b_2 = 0$.

Let $i \in J_B^1 = J_B$. Suppose that edge w_3v_i exists. Since E_i has configuration B, edge w_3v_i cannot be free. Let $f(w_3v_i) = h$. Suppose e is a free edge in E_h. Assume first that e is incident with w_1. Since w_1 is incident with at most one important edge of colour $f(e)$, $f(e) \neq f(v_iw_1) = f(u_iw_2)$. So we can replace edges e_i and e_h in M with edges v_iw_3, u_iw_2, and e, a contradiction. Hence e is not incident with w_1 and similarly is not incident with w_2. Thus all important edges in E_h are incident with w_3. It follows that E_h has at most two such edges.

Similarly to the start of Subsection 4.1, since $\hat{d}(w_3) \geqslant k = n-2$, at least $b_1 - 1 = b - 1$ distinct colours were used on the edges in the set $\{w_3v_i : i \in J_B^1\}$. This implies that there are $b-1$ values for j such that E_j has 2 or fewer important edges. So (4.1) holds again.

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