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Note

Many disjoint dense subgraphs versus large k-connected subgraphs in large graphs with given edge density

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Abstract

It is proved that for all positive integers d, k, s, t with $t \geq k + 1$ there is a positive integer $M = M(d, k, s, t)$ such that every graph with edge density at least $d + k$ and at least M vertices contains a k-connected subgraph on at least t vertices, or s pairwise disjoint subgraphs with edge density at least d. By a classical result of Mader [W. Mader, Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend großer Kantendichte, Abh. Math. Sem Univ. Hamburg, 37 (1972) 86–97] this implies that every graph with edge density at least $3k$ and sufficiently many vertices contains a k-connected subgraph with at least r vertices, or r pairwise disjoint k-connected subgraphs. Another classical result of Mader [W. Mader, Homomorphiesätze für Graphen, Math. Ann. 178 (1968) 154–168] states that for every n there is an $l(n)$ such that every graph with edge density at least $l(n)$ contains a minor isomorphic to K_n. Recently, it was proved in [T. Böhme, K. Kawarabayashi, J. Maharry, B. Mohar, Linear connectivity forces dense minors, J. Combin. Theory Ser. B (submitted for publication)] that every $(\frac{31}{2}a + 1)$-connected graph with sufficiently many vertices either has a topological minor isomorphic to $K_{a,pq}$, or it has a minor isomorphic to the disjoint union of p copies of $K_{a,q}$. Combining these results with the result of the present note shows that every graph with edge density at least $l(a) + (\frac{31}{2}a + 1)$ and sufficiently many vertices has a topological minor isomorphic to $K_{a,pa}$, or a minor isomorphic to the disjoint union of p copies of K_a. This implies an affirmative answer to a question of Fon-der-Flaass.

Keywords: Graph minors; Edge density; Connected subgraphs

1. Introduction and results

In this note all graphs are finite and do not have loops or multiple edges. For a graph G, let $|G|$ and $\|G\|$ denote the number of its vertices and of its edges, respectively. The quotient $\|G\|/|G|$ is called the *edge density* of G. A *separation* of a graph G is a pair (G_1, G_2) of induced subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$. The *order* of a separation (G_1, G_2) is $|G_1 \cap G_2|$. A separation (G_1, G_2) is *non-trivial* if $V(G_1) \setminus V(G_2) \neq \emptyset$ and $V(G_2) \setminus V(G_1) \neq \emptyset$. We call a graph k-*separable* if it has a non-trivial separation of order at most k. Clearly, a graph is not $(k-1)$-separable

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if and only if it is either k-connected or it is a complete graph with at most k vertices. Our main result is the following Theorem 1.1.

Theorem 1.1. *For any positive integers d, k, s, t there is a number $M = M(d, k, s, t)$ such that every graph on n vertices with at least $(d + k - 1)n + M$ edges contains either a not $(k - 1)$-separable subgraph with at least t vertices or s pairwise disjoint subgraphs with edge density at least d.*

The following result mentioned in the abstract is an immediate consequence of Theorem 1.1.

Theorem 1.2. *For any positive integers d, k, s, t there is a number $M = M(d, k, s, t)$ such that every graph with edge density at least $d + k$ and at least $M(d, k, s, t)$ vertices contains a not $(k - 1)$-separable subgraph with at least t vertices, or s pairwise disjoint subgraphs with edge density at least d.*

Our first application of Theorem 1.2 deals with the existence of k-connected subgraphs in graphs with given edge density. Mader [10] (see also [3], p. 11) proved that every graph with edge density at least $2k$ contains a k-connected subgraph. Mader's result together with Theorem 1.2 immediately implies the following Theorem 1.3. (Let $d = 2k$, $s = t = r$, and $N(k, r) = M(2k, k, r, r)$.)

Theorem 1.3. *There exists a function $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for $r \geq k + 1$ every graph with edge density at least $3k$ and at least $N(k, r)$ vertices contains a k-connected subgraph with at least r vertices, or r pairwise disjoint k-connected subgraphs.*

We also use Theorem 1.2 to prove a result on the existence of certain (topological) minors in graphs with given edge density. A graph G is a *subdivision* of a graph H if G can be obtained from H by replacing the edges of H with paths between their ends, such that none of these paths has an inner vertex on another path or in H. A graph H is a *topological minor* of a graph G if G contains a subdivision of H as a subgraph. A graph H is a *minor* of another graph G if H can be obtained from a subgraph of G by contracting edges. Clearly, every topological minor of a graph is a also a minor of this graph.

Mader [8,9] proved that there exist functions $h, l : \mathbb{N} \to \mathbb{R}$ such that every graph with edge density at least $h(n)$ has a topological minor isomorphic to the complete graph K_n, and every graph with edge density at least $l(n)$ has a minor isomorphic to the complete graph K_n. It was proved independently by Bollobás and Thomason [2], and by Komlós and Szemerédi [5] that $h(n) = \Theta(n^2)$. Kostochka [7,6] and Thomason [11] independently proved that $l(n) = \Theta(n\sqrt{\log n})$. Recently, Thomason [12] found the asymptotically best possible bound of $l(n)$.

Applying Theorem 1.2 with $d = h(n)$ (resp., $d = l(n)$) shows that for $t \geq k + 1$ every graph with edge density at least $h(d) + k$ (resp. $l(d) + k$) and at least $M(h(d), k, s, t)$ (resp., $M(l(d), k, s, t)$) vertices contains a k-connected subgraph with at least t vertices, or it has a topological minor (resp. a minor) isomorphic to the disjoint union of s copies of the complete graph K_n. We will combine this result with a recent theorem of Böhme, Kawarabayashi, Maharry, and Mohar [1].

Theorem 1.4 ([1]). *There exists a function $L : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that every $(\frac{31}{2}a + 1)$-connected graph with at least $L(p, q)$ vertices either has a topological minor isomorphic to $K_{a,pq}$, or it has a minor isomorphic to the disjoint union of p copies of $K_{a,q}$.*

We let $P(a, p) = M(l(a), \lceil\frac{31}{2}a + 1\rceil, p, L(p, a))$ and obtain the following Theorem 1.5. (Note that K_a is a minor of $K_{a,a}$.)

Theorem 1.5. *Every graph with edge density at least $l(a) + \frac{31}{2}a + 1$ and at least $P(a, p)$ vertices has a topological minor isomorphic to $K_{a,pa}$, or a minor isomorphic to the disjoint union of p copies of K_a.*

D. Fon-der-Flaass [4] has asked whether there are functions $r : \mathbb{N} \to \mathbb{N}$ and $R : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $a, p \in \mathbb{N}$ every $r(a)$-regular graph on at least $R(a, p)$ vertices has a minor isomorphic to the disjoint union of p copies of K_a. Since $K_{a,pa}$ contains vertices of degree pa, Theorem 1.5 implies the following Corollary that answers the question of D. Fon-der-Flaass.

Corollary 1.6. *Every graph with edge density at least $l(a) + \frac{31}{2}a + 1$, maximum degree Δ and at least $P(a, \lceil\frac{\Delta+1}{a}\rceil)$ vertices has a minor isomorphic to the disjoint union of p copies of K_a.*

2. Proof of Theorem 1.1

Let $M(d,k,s,t) = 2^{s-1}\binom{t}{2} + (2^{s-1}-1)(d+k-1)(k-1)$. If $\|G\| \geq (d+k-1)|G| + M(d,k,1,t)$, then the graph G itself has edge density greater than d, and so the statement of the theorem is trivially true for $s=1$.

Suppose there is an $s>1$ such that the statement of the theorem is not true, and let G be a graph such that:

(a) $\|G\| \geq (d+k-1)|G| + 2^{s-1}\binom{t}{2} + (2^{s-1}-1)(d+k-1)(k-1)$;
(b) G contains neither a not $(k-1)$-separable subgraph on at least t vertices nor s pairwise disjoint subgraphs with edge density at least d;
(c) s is minimum with respect to (a) and (b);
(d) $|G|$ is minimum with respect to (a), (b) and (c).

Since G has more than $\binom{t}{2}$ edges, $|G| > t$. Hence it follows from (b) that G has a non-trivial separation (G_1, G_2) of order at most $k-1$.

Claim 1. *There is an* $i \in \{1,2\}$ *such that*

$$\|G_i\| \geq (d+k-1)|G_i| + 2^{s-2}\binom{t}{2} + (2^{s-2}-1)(d+k-1)(k-1).$$

Proof of Claim 1. Suppose that for $i \in \{1,2\}$,

$$\|G_i\| < (d+k-1)|G_i| + 2^{s-2}\binom{t}{2} + (2^{s-2}-1)(d+k-1)(k-1).$$

Then, since $|G_1| + |G_2| \leq |G| + (k-1)$, we have

$$\begin{aligned}\|G\| &\leq \|G_1\| + \|G_2\| < (d+k-1)(|G_1|+|G_2|) + 2^{s-1}\binom{t}{2} + (2^{s-1}-2)(d+k-1)(k-1)\\ &\leq (d+k-1)|G| + 2^{s-1}\binom{t}{2} + (2^{s-1}-1)(d+k-1)(k-1) \text{ - a contradiction.} \quad \square\end{aligned}$$

Assume w.l.o.g. that

$$\|G_2\| \geq (d+k-1)|G_2| + 2^{s-2}\binom{t}{2} + (2^{s-2}-1)(d+k-1)(k-1),$$

and let $G_1^- = G_1 - V(G_2)$.

Claim 2. $\|G_1^-\| \geq d|G_1^-|$.

Proof of Claim 2. Suppose that $\|G_1^-\| < d|G_1^-|$. Then

$$\begin{aligned}\|G_2\| &\geq \|G\| - \|G_1^-\| - (k-1)|G_1^-|\\ &> (d+k-1)|G| - d|G_1^-| - (k-1)|G_1^-| + 2^{s-1}\binom{t}{2} + (2^{s-1}-1)(d+k-1)(k-1)\\ &= (d+k-1)|G_2| + 2^{s-1}\binom{t}{2} + (2^{s-1}-1)(d+k-1)(k-1).\end{aligned}$$

Since $|G_2| < |G|$, this contradicts (d). $\square$

Since G_2 is a subgraph of G it follows from (b) that G_2 does not contain a not $(k-1)$-separable subgraph on at least t vertices. This implies by (c) and Claim 1 that G_2 contains $s-1$ pairwise disjoint subgraphs with edge density at least d. Claim 2 implies that these subgraphs together with G_1^- form a family of s pairwise disjoint subgraphs of G with edge density at least d. This contradicts (b), and the theorem is proved. $\square$

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