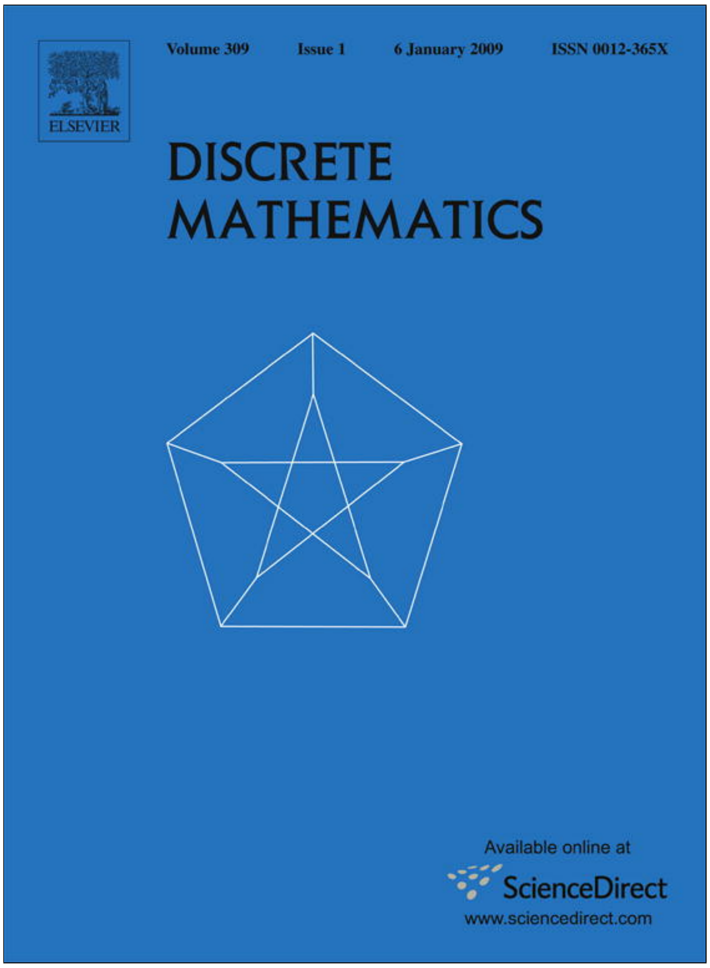
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Note

Planar graphs decomposable into a forest and a matching

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Abstract

He, Hou, Lih, Shao, Wang, and Zhu showed that a planar graph of girth 11 can be decomposed into a forest and a matching. Borodin, Kostochka, Sheikh, and Yu improved the bound on girth to 9. We give sufficient conditions for a planar graph with 3-cycles to be decomposable into a forest and a matching.

Keywords: Planar graphs; Edge decompositions; Graph decompositions

He et al. [3] proved that each planar graph with girth 11 or more can be decomposed into a forest and a matching (i.e., has an *FM-coloring*). Kleitman et al. [1] proved the same statement for planar graphs with girth at least 10. The restriction on girth was further improved to 9 by Borodin et al. [2]. Namely, the following was proved.

Theorem 1. *Every planar graph G of girth at least* 9 *has an FM-coloring.*

This implies (see [3]) that the game chromatic number and the game coloring number of every planar graph with girth at least 9 are at most 5.

The purpose of this note is to describe a broader class of sparse planar graphs whose edges can be decomposed into a forest and a matching. Due to the result in [2], we allow 3-cycles but impose some restrictions on the structure of these graphs.

By a *k-sunflower*, S_k, where $k \geq 4$, we mean the graph obtained from k-cycle C_k by putting a triangle on each edge so that one vertex of the triangle has degree 2 (there are k vertices of degree 2 and k vertices of degree 4).

Claim 2. *No k-sunflower has an FM-coloring.*

Proof. Let $x_1, \ldots, x_k$ be the vertices of the basic cycle, C_k, in S_k (in this cyclic order), and for $i = 1, \ldots, k$, let y_i be adjacent to x_i and x_{i+1}. Suppose the edges of S_k are colored green and red so that the green edges form a forest, and the red edges, a matching. At least one edge, say x_1x_2, of C_k is red. Then x_2x_3 and x_2y_2 are green, and hence y_2x_3

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should be red. Now, for the same reasons, x_3x_4 and x_3y_3 are green, and so y_3x_4 is red, and so on. Finally, we conclude that y_kx_1 is red, a contradiction to x_1x_2 being red. □

Let G^* be obtained from a graph G by contracting triangles of G. Namely, at every step we identify the three vertices of a triangle of G into one vertex and delete the edges of this triangle, but every other edge of G survives. In particular, G^* may have parallel edges and loops. Note that by definition, no multigraph with a loop or a triple edge may have an *FM*-coloring.

Claim 3. *Suppose that G has no sunflowers, i.e. every* ≥ 4*-cycle in G has a nontriangular edge. If* G^* *has an FM-coloring then G also has an FM-coloring.*

Proof. Suppose that the sequence $G_0, \ldots, G_t$ of (multi)graphs is such that $G_0 = G$, $G_t = G^*$, and for $i = 0, \ldots, t-1$, graph G_{i+1} is obtained from G_i by contracting a triangle into a vertex. By assumption, G_t has an *FM*-coloring f_t. Suppose that for some $i \leq t-1$, G_{i+1} has an *FM*-coloring f_{i+1} and that G_{i+1} is obtained from G_i by contracting the triangle with vertices x, y, and z. For every edge $e \in E(G_i) - \{xy, yz, zx\}$, let $f_i(e) = f_{i+1}(e)$. Since f_{i+1} is an *FM*-coloring, at most one red edge is incident in G_i to the set $\{xy, yz, zx\}$. So, we may assume that no red edges are incident to y and z. In this case, color yz with red and xy and xz with green. Note that the existence of a green cycle in the new coloring f_i of G_i would imply that f_{i+1} also colors a cycle in G_{i+1} with green. Thus f_i is an *FM*-coloring of G_i. Repeating this step for $i = t-1, t-2, \ldots, 0$, we obtain an *FM*-coloring of $G_0 = G$. □

Note the following structural property of G^*:

Claim 4. *Suppose every* ≥ 4*-cycle in a graph G has at least k nontriangular edges, where* $k \geq 3$*; then the girth of* G^* *is at least k.*

Our main result is the following extension of Theorem 1.

Theorem 5. *If every* ≥ 4*-cycle in a planar graph G has at least* 9 *nontriangular edges, then G has an FM-coloring.*

It immediately follows from Claims 3 and 4 and Theorem 1 itself. As explained in [3], this implies:

Corollary 6. *The game chromatic number and game coloring number of every planar graph having no cycles with less than* 9 *nontriangular edges are at most* 5.

By $d_\Delta(G)$ denote the minimal distance between triangles in G.

Corollary 7. *Every planar graph G can be decomposed into a forest and a matching if at least one of the following holds:*

(i) $d_\Delta(G) \geq 1$ *and G has no cycles of length from* 4 *to* 16*;*
(ii) $d_\Delta(G) \geq 2$ *and G has no cycles of length from* 4 *to* 12*;*
(iii) $d_\Delta(G) \geq 4$ *and G has no cycles of length from* 4 *to* 10*;*
(iv) $d_\Delta(G) \geq 5$ *and G has no cycles of length from* 4 *to* 9.

To deduce (i) in Corollary 7 from Theorem 5, it suffices to note that if G has neither cycles of length from 4 to 16 nor two triangles with a common vertex, then G has no cycles with less than 9 nontriangular edges. (Note that the cubic graph obtained from the dodecahedron by cutting off all its corners has neither 3-cycles with a common vertex nor cycles of length 4 to 9, and it clearly has no FM-coloring. On the other hand, S_k has neither cycles of length from 4 to $k-1$, nor FM-coloring.) Similarly, we deduce (ii)–(iv).

Accordingly, the graphs in Corollary 7 have the game chromatic number and game coloring number at most 5.

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