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An upper bound on the domination number of n-vertex connected cubic graphs

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Abstract

In 1996, Reed proved that the domination number $\gamma(G)$ of every n-vertex graph G with minimum degree at least 3 is at most $3n/8$. This bound is sharp for cubic graphs if there is no restriction on connectivity. In this paper we show that $\gamma(G) \le 4n/11$ for every n-vertex cubic connected graph G if $n > 8$. Note that Reed's conjecture that $\gamma(G) \le \lceil n/3 \rceil$ for every connected cubic n-vertex graph G is not true.

Keywords: Cubic graphs; Domination; Connected graphs

1. Introduction

A set A of vertices in a graph G *dominates* itself and the vertices at distance one from it. If a set A dominates all vertices of G, then it is called *dominating in* G. The *domination number*, $\gamma(G)$, of a graph G is the minimum size of a dominating set in G.

Graphs G with high minimum degree, $\delta(G)$, have small domination number. Ore [5] proved that $\gamma(G) \le n/2$ for every n-vertex graph without isolated vertices (i.e., with $\delta(G) \ge 1$). Blank [1] proved that $\gamma(G) \le 2n/5$ for every n-vertex graph with $\delta(G) \ge 2$ if $n \ge 8$. Reed [7] proved that $\gamma(G) \le 3n/8$ for every n-vertex graph with $\delta(G) \ge 3$. All these bounds are sharp. Reed [7] conjectured that the domination number of each connected 3-regular (cubic) n-vertex graph is at most $\lceil n/3 \rceil$. The authors [4] disproved this conjecture. They proved:

Theorem 1 (*[4]*). *There is a sequence* $\{G_k\}_{k=1}^{\infty}$ *of cubic connected graphs such that for every* k, $|V(G_k)| = 46k$ *and* $\gamma(G_k) \ge 16k$, *and thus* $\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \ge \frac{8}{23} = \frac{1}{3} + \frac{1}{69}$.

On the other hand, Kawarabayashi, Plummer, and Saito [3] proved the following upper bound for graphs without short cycles.

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Theorem 2 (*[3]*). *If G is a connected cubic n-vertex graph that has a 2-factor of girth at least $g \geq 3$, then*

$$\gamma(G) \leq n\left(\frac{1}{3} + \frac{1}{9\lfloor g/3 \rfloor + 3}\right).$$

Clearly, Reed's conjecture holds for Hamiltonian cubic graphs. Plummer [6] suggested that for such graphs on n vertices with $n > 8$, the slightly stronger bound $\gamma(G) \leq \lfloor n/3 \rfloor$ holds. In [2], this was confirmed for $n \equiv 1 \pmod 3$ and disproved for $n \equiv 2 \pmod 3$. In particular, the following holds.

Theorem 3 (*[2]*). *If G is a Hamiltonian cubic $(3k+1)$-vertex graph, then $\gamma(G) \leq k$.*

The aim of this paper is to improve Reed's upper bound of $3n/8$ for cubic graphs to $4n/11$. The main result of the paper is:

Theorem 4. *Let $n > 8$. If G is a connected cubic n-vertex graph, then*

$$\gamma(G) \leq \frac{4n}{11}.$$

We also improve the bound of Theorem 2 for graphs without short cycles as follows.

Theorem 5. *If G is a cubic connected n-vertex graph of girth g, then*

$$\gamma(G) \leq n\left(\frac{1}{3} + \frac{8}{3g^2}\right).$$

Our proofs exploit the ideas and technique of Reed's seminal paper [7]. We add to Reed's ideas a twist, a discharging counting, and consider some configurations more attentively. In the next section, we describe the setup of Reed's paper [7] with some small changes and the procedure of constructing a dominating set. In the same section we state the basic lemmas that we will prove later. In Section 3, we describe a discharging that proves the bound modulo basic lemmas. In the next three sections we prove the basic lemmas. In Section 7, we prove Theorem 5. We conclude the paper with some comments.[1]

2. The setup

We follow Reed's setup [7] with small changes. A *vdp-cover* of a graph G is a covering of $V(G)$ by vertex-disjoint paths. The *order*, $|P|$, of a path P is the number of its vertices. For $i \in \{0, 1, 2\}$, a path P is an *i-path*, if $|P| \equiv i \pmod 3$. If P is a path, $x \in V(P)$ and $P - x$ consists of an i-path and a j-path, then x is called an *(i, j)-vertex of P*.

Let G be a connected cubic graph and S be a vdp-cover of G. An endpoint x of a path $P \in S$ is an *out-endpoint* if x has a neighbor outside of P. An endpoint x of a 2-path $P \in S$ is a *(2, 2)-endpoint* if x is not an out-endpoint and is adjacent to a (2, 2)-vertex of P. By S_i we denote the set of i-paths in S.

A vdp-cover S of G is *optimal* if

(R1) $2|S_1| + |S_2|$ is minimized;
(R2) Subject to (R1), $|S_2|$ is minimized;
(R3) Subject to (R1) and (R2), $\sum_{P \in S_0} |P|$ is minimized;
(R4) Subject to (R1)–(R3), $\sum_{P \in S_1} |P|$ is minimized;
(R5) Subject to (R1)–(R4), the total number of out-endpoints of all paths in S is maximized;
(R6) Subject to (R1)–(R5), the total number of (2, 2)-endpoints of all 2-paths in S is maximized.

[1] After this paper was preliminarily accepted, we learned about two new interesting results. Lowenstein and Rautenbach [10] proved that Reed's conjecture holds for cubic graphs with girth at least 83. Kelmans [9] constructed a smaller (with 54 vertices) counter-example to Reed's conjecture and an infinite series of 2-connected examples H_k with $\frac{\gamma(H_k)}{|V(H_k)|} \geq \frac{1}{3} + \frac{1}{60}$.

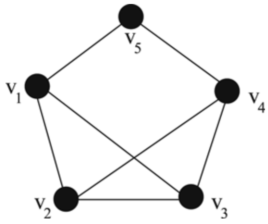

Fig. 1.

It turns out that optimal vdp-covers possess several useful properties. The next lemma summarizes Observations 1–3 on pages 280–281 and the first paragraph of the proof of Fact 11 on page 285 of [7].

Lemma 1. *Suppose that an out-endpoint x of a 1-path or a 2-path P_i in an optimal vdp-cover S is adjacent to a vertex $y \in P_j$, where $j \neq i$. Let $P_j = P'_j y P''_j$. Then*

(B1) *P_j is not a 1-path;*
(B2) *If P_j is a 0-path, then both P'_j and P''_j are 1-paths;*
(B3) *If P_j is a 2-path, then both P'_j and P''_j are 2-paths;*
(B4) *If P_j is a 2-path and z is the common endpoint of P_j and P'_j, then each neighbor of z on P''_j is a $(2, 2)$-vertex.*

A path P in a vdp-cover S will be called *special* if P has 29 vertices and none of the hamiltonian paths on $V(P)$ has an out-endpoint or a $(2, 2)$-endpoint. A special path P in a vdp-cover S will be called *very special* if there exists a path P_1 in S with $|P_1| = 1$ that is adjacent to the central vertices of three special paths one of which is P. The two other special paths in the definition of a very special path are, by definition, also very special and will be called *the siblings of P*.

Now we will repeat Reed's [7] construction of a dominating set with a slight modification regarding special paths. Let S be an optimal vdp-cover.

(C1) For each 1-path $P \in S$ that has an out-endpoint, choose a vertex $y \notin V(P)$ which is a neighbor of an out-endpoint $x(P)$ of P. Call this $y \notin V(P)$ *an acceptor for P*. If P is a single vertex and has a neighbor y that is not the central vertex of a special path, then let the acceptor of P be not the central vertex of a special path.

(C2) For each 2-path $P \in S$ that has two out-endpoints, for each of these out-endpoints choose a neighbor outside P and designate it as an acceptor corresponding to that endpoint. If x is an out-endpoint of a path with two vertices and has a neighbor y that is not the central vertex of a special path, then let the acceptor of x be not the central vertex of a special path.

(C3) For each 2-path $P \in S$ with 5 vertices that has precisely one out-endpoint and induces the graph in Fig. 1, choose the outneighbor of P as an acceptor for P.

Call a path *accepting* if at least one of its vertices was designated as an acceptor.

(C4) Construct a family $A \subseteq S$ of 2-paths as follows. Initially, let A be the set of accepting 2-paths in S. While there is any out-endpoint x of a path in A for which we have not already chosen an acceptor (because the path has only one out-endpoint), choose a neighbor y of x in $G - P$ and designate it as an acceptor for x. If y is on a previously non-accepting 2-path P', then add P' to A. Continue this process until there is an acceptor for every out-endpoint in A. In addition, for each $(2, 2)$-endpoint x of each path P in A, designate a $(2, 2)$-vertex y adjacent to x as an in-acceptor for x.

(C5) If the central vertex y of a special path $P \in S$ was designated as the acceptor for a path P_1 consisting of a single vertex y_1, then by (C1), P is very special and has two siblings P' and P'' whose central vertices are neighbors of y_1. If some vertex of $P - y$ is also an acceptor or both P' and P'' are not accepting paths, then we leave the situation as it is. But if y is the only acceptor on P and, say, P' contains an acceptor (distinct from its central vertex, y', which by the definition of very special paths is a neighbor of y_1), then we redesignate the y' as the acceptor for y_1 (and P_1). Since no special path has out-endpoints, this will not affect any other path, only P will be deleted from A.

Each accepting 2-path $P \in S$ can be written in the form $P_1 P_2 P_3$, where P_1 and P_3 are both 1-paths containing no acceptors (including in-acceptors) and are maximal with this property. By (B3), the second and the penultimate vertices of P_2 are acceptors. The paths P_1 and P_3 are called *tips of P*, and P_2 is the *central path of P*. Now a dominating set D is defined as follows.

(C6) For each 0-path $P \in S$, every (1, 1)-vertex of P is included in D.

(C7) For each accepting 2-path $P \in S$, every (2, 2)-vertex of P that is in the central path of P is included in D.

(C8) Let $P \in S$ be a 1-path with at least one out-endpoint. Then P has an out-endpoint, say $x(P)$, adjacent to the acceptor of P. Choose some $\lfloor |P|/3 \rfloor$ vertices that dominate all vertices of P except for $x(P)$, and include these $\lfloor |P|/3 \rfloor$ vertices in D.

(C9) For each 2-path $P \in S$ in which each of the ends is either an out-endpoint or a (2, 2)-endpoint, include in D all (2, 2)-vertices of P. Note that there are $\lfloor |P|/3 \rfloor$ of them and these (2, 2)-vertices dominate all vertices of P except possibly for the out-endpoints of P.

(C10) For each 2-path in S on 5 vertices whose vertices induce the graph F in Fig. 1 include vertex v_2 in Fig. 1 into D.

(C11) Let $P \in S$ be a 1-path with no out-endpoints or a non-accepting 2-path with at most one out-endpoint that does not induce the graph F on in Fig. 1. Choose a smallest dominating set in the subgraph of G induced by P and include it in D. Note that in any case, this set has at most $\lceil |P|/3 \rceil$ vertices.

(C12) Let P_1 be a tip of an accepting 2-path $P \in S$ and x be the common end of P and P_1. If x is an out-endpoint or a (2, 2)-endpoint, then include in D all (2, 2)-vertices of P that are in P_1. There are $\lfloor |P_1|/3 \rfloor$ of them and these (2, 2)-vertices dominate all vertices of P_1 except for x (which is dominated by a vertex already included in D by (C6) or (C7)). If x is neither an out-endpoint nor a (2, 2)-endpoint, then include in D a smallest dominating set in the subgraph of G induced by P_1. Similarly to (C11), this set has at most $\lceil |P_1|/3 \rceil$ vertices.

(C13) An *exceptional path* is a non-accepting 2-path $P \in S$ such that

(i) both ends of P are out-endpoints,

(ii) the acceptors of both ends are vertices of 2-paths $P' = P'_1 P'_2 P'_3$ and $P'' = P''_1 P''_2 P''_3$,

(iii) $|P'_1| \geq 13$, $|P'_3| \geq 13$, $|P''_1| \geq 13$, and $|P''_3| \geq 13$,

(iv) paths P' and P'' do not contain other acceptors, $|P'_2| = |P''_2| = 3$, and

(v) according to (C12), $|D \cap V(P')| = (|P'| + 4)/3$ and $|D \cap V(P'')| = (|P''| + 4)/3$.

The paths P' and P'' in the definition of an exceptional path P are called *dependents of* P.

For every exceptional path, we replace the $\lfloor |P|/3 \rfloor$ vertices of D in P (they dominated P apart from the endpoints) by a set of size $1 + \lfloor |P|/3 \rfloor$ dominating all vertices of P, but replace the $(|P'| + 4)/3 + (|P''| + 4)/3$ vertices of D in $P' \cup P''$ by $(|P'| + 1)/3 + (|P''| + 1)/3$ vertices dominating $V(P' \cup P'')$. This finishes the definition of D.

By construction (see [7, P. 283]), the set D is dominating. We will prove that $|D| \leq 4|V(G)|/11$ if $|V(G)| > 8$ and G is connected using Reed's technique. Note that a path P (or P_1) can contribute to D more than $|P|/3$ (or $|P_1|/3$) vertices only in cases (C11), (C12) or (C13). Thus the following lemmas will be helpful (and are extensions of Facts 9, 10 and 11 in [7]).

Lemma 2. *If a* 1*-path* P *in an optimal vdp-cover* S *does not have an out-endpoint and does not contain a dominating set of size at most* $|P|/3$*, then* P *has at least* 22 *vertices.*

Lemma 3. *If a* 2*-path* P *in an optimal vdp-cover is such that each of the hamiltonian paths on* $V(P)$ *has at most one out-endpoint, then* P *has at least* 11 *vertices.*

Lemma 4. *Let* $P_1 = (x_1, \ldots, x_k)$ *be a tip of an accepting* 2*-path* P *in an optimal vdp-cover. Let* $X(P_1)$ *be the set of the hamiltonian paths on* $\{x_1, \ldots, x_k\}$ *one of whose ends is* x_k*. If none of the other ends of any path in* $X(P_1)$ *is an out-endpoint of* P *or a* (2, 2)*-endpoint, then* $k \geq 13$*.*

In the next section, we will use discharging in order to prove our upper bound on $|D|$ provided that Lemmas 2–4 hold. In the subsequent sections we prove these lemmas.

3. Discharging

Consider the following discharging. Initially, every vertex in D has charge 1 and every other vertex of G has charge 0, so the total sum of charges is $|D|$. We will change the charges of vertices in such a way that

(a) the sum of charges does not decrease, and
(b) the charge of every vertex becomes at most 4/11.

The properties (a) and (b) together imply that $|D| \leq 4|V(G)|/11$. We do the discharging in several steps and at every step will check that the charge of each so far involved vertex is not greater than 4/11.

Step 1. For each 0-path P, every (1, 1)-vertex of P gives 1/3 of its charge to either of the two neighbors on P. After this step, each vertex of each 0-path P has charge 1/3.

Step 2. For each accepting 2-path P, every (2, 2)-vertex of P that is in the central path of P gives 1/3 of its charge to either of the two neighbors on P. After this step, each vertex in the central path of each accepting 2-path P has charge 1/3.

Step 3. Let P be a 1-path with at least one out-endpoint, say $x(P)$, adjacent to the acceptor of P. Distribute the charges of the $\lfloor |P|/3 \rfloor$ vertices of D in $V(P)$ evenly among the vertices in $V(P) - \{x(P)\}$. After this step, the vertex $x(P)$ has charge 0 and every other vertex of P has charge 1/3. Do this for every 1-path with at least one out-endpoint.

Step 4. Let P be a non-accepting and non-exceptional 2-path in which each of the ends is either an out-endpoint or a (2, 2)-endpoint. Distribute the charges of the $\lfloor |P|/3 \rfloor$ vertices of D in $V(P)$ evenly among the internal vertices of P. After this step, either of the ends of P has charge 0 and every other vertex of P has charge 1/3. Do this for every 2-path in which either of the ends is either an out-endpoint or a (2, 2)-endpoint.

Step 5. For each 2-path P on 5 vertices whose vertices induce the graph F in Fig. 1, vertex v_2 (the only vertex of P in D) gives 1/4 to each of its neighbors. After this step, the out-endpoint of P has charge 0 and every other vertex of P has charge 1/4.

Step 6. Let P be a 1-path with no out-endpoints. Distribute the charges of the vertices in $D \cap V(P)$ evenly among vertices of P. If $|V(P)| < 22$, then by Lemma 2, $|D \cap V(P)| < |V(P)|/3$ and each vertex of P will have charge less than 1/3. If $|V(P)| \geq 22$, then

$$|D \cap V(P)| \leq (|V(P)| + 2)/3 = (1 + 2/|V(P)|)|V(P)|/3 \leq (1 + 2/22)|V(P)|/3 = 4|V(P)|/11,$$

and, hence, each vertex of P has charge at most 4/11.

Step 7. Let P be a non-accepting 2-path with at most one out-endpoint that does not induce the graph F in Fig. 1. Since P has at most one out-endpoint, it is not exceptional. Similarly to Step 6, distribute the charges of the vertices in $D \cap V(P)$ evenly among vertices of P. If $|V(P)| < 11$, then by Lemma 3, $|D \cap V(P)| < |V(P)|/3$, and each vertex of P will have charge less than 1/3. If $|V(P)| \geq 11$, then

$$|D \cap V(P)| \leq (|V(P)| + 1)/3 = (1 + 1/|V(P)|)|V(P)|/3 \leq (1 + 1/1)|V(P)|/3 = 4|V(P)|/11,$$

and, hence, each vertex of P has charge at most 4/11.

Step 8. Let P be an exceptional path and P' and P'' be its dependents. By the definition of exceptional paths, P is non-accepting, and P' and P'' contain acceptors only for P. Distribute the charges of the vertices in $D \cap (V(P) \cup V(P') \cup V(P''))$ evenly among vertices in $V(P) \cup V(P') \cup V(P'')$. Recall that $|V(P) \cup V(P') \cup V(P'')| = |V(P)| + 58$. By (C13),

$$|D \cap (V(P) \cup V(P') \cup V(P''))| = \frac{|V(P)| + 1}{3} + 20 = \frac{|V(P)|}{3} + \frac{61}{3 \cdot 58}|V(P') \cup V(P'')|.$$

Hence, the charge of each vertex in $V(P) \cup V(P') \cup V(P'')$ is less than 4/11.

Step 9. Let P_1 be a tip of an accepting 2-path P such that the common end, $x(P_1)$, of P and P_1 is either an out-endpoint or a (2, 2)-endpoint of P. Distribute the charges of the $\lfloor |P_1|/3 \rfloor$ vertices of D in $V(P_1)$ evenly among the vertices of P_1 apart from $x(P_1)$. After this step, $x(P_1)$ has charge 0 and each other vertex of P_1 has charge 1/3.

Step 10. Let P_1 be a tip of an accepting 2-path P such that the common end, $x(P_1)$, of P and P_1 is neither an out-endpoint nor a (2, 2)-endpoint of P, and the central path of P has more than 3 vertices. Since the central path of P has more than 3 vertices, P is not a dependant of an exceptional path. Suppose that $P_1 = (x_1, \ldots, x_k)$, $P_2 = (y_1, \ldots, y_m)$, and $P_3 = (z_1, \ldots, z_l)$, so that $P = (x_1, \ldots, x_k, y_1, \ldots, y_m, z_1, \ldots, z_l)$. Recall that, by definition, y_2 is an acceptor for an out-endpoint y' of a path or for $y' = z_l$ if z_l is a (2, 2)-endpoint. Recall also that so far all out-endpoints and (2, 2)-endpoints of non-exceptional paths had charges equal to 0. If $|V(P_1)| \geq 13$, then we distribute the charges of at most $(|V(P_1)| + 2)/3$ vertices of $D \cap V(P_1)$ as follows: each vertex of P_1 gets 4/11, then we add 1/33 to the charge of each of y_1, y_2 and y_3 and give 2/11 to the vertex y' whose acceptor is y_2. The total charge that the vertices of $P_1 \cup \{y_1, y_2, y_3, y'\}$ get at this step is $4|P_1|/11 + 3/33 + 2/11$ which is at least $(|V(P_1)| + 2)/3$ when $|P_1| \geq 13$. Each of y_1, y_2 and y_3 had charge 1/3 after Step 2 and for each of them the charge changed to 4/11. Note that, since $m > 3$, the vertices y_1, y_2, y_3, and y' will not get any charge from the tip P_3.

If $|V(P_1)| < 13$, then by Lemma 4, $|D \cap V(P_1)| < |V(P_1)|/3$, and after distributing the charges of vertices of $D \cap V(P_1)$ evenly among vertices of P_1, each vertex of P_1 will have charge less than $1/3$.

Step 11. Let P be an accepting 2-path such that exactly one endpoint of P is an out-endpoint or a (2, 2)-endpoint, and the central path of P has exactly 3 vertices. If P is a dependant of an exceptional path, then the charges of its vertices are already defined on Step 8. So, below P is not a dependant of an exceptional path. Suppose that $P_1 = (x_1, \ldots, x_k)$, $P_2 = (y_1, y_2, y_3)$, and $P_3 = (z_1, \ldots, z_l)$, so that $P = (x_1, \ldots, x_k, y_1, y_2, y_3, z_1, \ldots, z_l)$. We may assume that x_1 is neither an out-endpoint nor a (2, 2)-endpoint of P. By definition, y_2 is an acceptor for an out-endpoint y' of a path P' or for $y' = z_l$ if z_l is a (2, 2)-endpoint. Since z_l is either a (2, 2)-endpoint or an out-endpoint of P, the charges of vertices in P_3 were defined on Step 9 (if the acceptor of z_l is on a 2-path, then the charge of z_l could be changed on Step 10). We define the charges of vertices in P_1 exactly as on Step 10.

Step 12. Let P be an accepting 2-path such that each of the endpoints of P is neither an out-endpoint nor a (2, 2)-endpoint, the central path of P has exactly 3 vertices, and $|D \cap V(P)| \leq (|V(P)|+1)/3$. By Lemma 3, $k+3+l \geq 11$. Hence, after distributing the charges of vertices of $D \cap V(P)$ evenly among all vertices of P, each vertex of P will have charge at most

$$\frac{|V(P)|+1}{3|V(P)|} = \frac{1}{3} + \frac{1}{3|V(P_1)|} \leq \frac{1}{3} + \frac{1}{33} = \frac{4}{11}.$$

Step 13. Let P be an accepting 2-path such that each of the endpoints of P is neither an out-endpoint nor a (2, 2)-endpoint, the central path of P has exactly 3 vertices, and $|D \cap V(P)| > (|V(P)| + 1)/3$. Again, if P is a dependant of an exceptional path, then we are done on Step 8. Suppose not. Let P_1, P_2, and P_3 be defined as at Step 11. Then $|D \cap V(P)| = (|V(P)| + 4)/3$ and this may happen only if $|D \cap V(P_1)| = (|P_1| + 2)/3$ and $|D \cap V(P_3)| = (|P_3| + 2)/3$. In this case, by Lemma 4, $k \geq 13$ and $l \geq 13$. If $k \geq 13$, $l \geq 13$ and $k + 3 + l > 29$, then $k + 3 + l \geq 32$ and $|D \cap V(P)| \leq \lceil k/3 \rfloor + 1 + \lceil l/3 \rfloor = (|V(P)| + 4)/3$. Distributing the charge evenly among the vertices of $V(P) \cup \{y'\}$, where y' is the out-endpoint of another path P' whose acceptor is y_2, we obtain that the charge of each vertex in $V(P) \cup \{y'\}$ is at most

$$\frac{|V(P)|+4}{3(|V(P)|+1)} = \frac{1}{3} + \frac{3}{3(|V(P_1)|+1)} \leq \frac{1}{3} + \frac{1}{33} \cdot \frac{4}{11}.$$

This is the only case so far that the end-vertex of a tip of a non-exceptional path gets charge greater than $2/11$. Note that it happens only when each of the tips of P has at least 13 vertices, P has no out-endpoints or (2, 2)-endpoints, $|D \cap V(P)| = (|V(P)| + 4)/3$, and P accepts only one vertex.

The only case we have not yet considered is that $k = l = 13$, in particular, P is a special path. In this case, $|D \cap V(P)| = 11$. We give every vertex of P charge $4/11$, but $29 \cdot 4/11 = 11 - 5/11$ and we need to distribute $5/11$ among some other vertices. We have the following cases for distributing this $5/11$ of charge.

Case 1. Vertex y' is the out-endpoint of a 1-path P' of length at least 4 or of a tip P' of an accepting 2-path of length at least 4. In this case, we give $4/11$ to y' and add $1/33$ to the charge of each of the other vertices of P'. At Step 3 or Step 9, y' got charge 0 and each of the other vertices got charge $1/3$, so now each of them has charge $4/11$.

Case 2. Vertex y' is the out-endpoint of a tip P' of an accepting 2-path that consists only of y'. Then the path containing y' can be written as $P'P''P'''$, where P' and P''' are the tips, and P'' is the center. Suppose that $P'' = (w_1, w_2, \ldots, w_t)$. Note that by the definition of the center, w_2 is the acceptor for a vertex w' and the charge of w' (maybe received from P''') is at most $2/11$. We give $4/11$ to y' and $1/11$ to w'.

Case 3. Vertex y' is the out-endpoint of the (non-accepting) graph F in Fig. 1. We give $4/11$ to y' and $1/44$ to each of the remaining vertices of F. Since each of them got the charge $1/4$ on Step 5, now it will have $1/4 + 1/44 = 3/11$.

Case 4. Vertex y' is the out-endpoint of a non-accepting 2-path P' distinct from the graph F in Fig. 1. Let $P' = (w_1, \ldots, w_s)$, where $y' = w_1$. Since P' is not an exceptional path, the path P'' accepting w_s does not satisfy at least one of the conditions (i)–(v) of the definition of an exceptional path. Then the charge of w_s is at most $2/11$. In this case, we give $4/11$ to $y' = w_1$ and add $1/11$ to the charge of w_s.

Case 5. The path P' containing y' has no other vertices. Since P is special, this might happen only if P is very special and its siblings, P_1 and P_2, are non-accepting. We give $4/11$ to y' and add $1/(11 \cdot 29)$ to the charge of each vertex in P_1. Since P_1 is non-accepting and has no out-endpoints, each of its vertices had previously charge $10/29$. After adding $1/11 \cdot 29$, each will have charge $111/319 < 4/11$. This finishes the discharging.

Thus, what is left to prove Theorem 4 is to prove Lemmas 2–4. We will do it in the next three sections. In the next section we describe the approach we use and prove a number of auxiliary statements. Using these statements, we prove Lemmas 3 and 4 in Section 5. Lemma 2 has the longest proof. It will be proved in Section 6.

4. Structure of proofs and technical statements

We will need some notation. Let G' be a subgraph of a graph G and $u, v \in V(G')$. Say that u is (G', v)-*distant* if G' contains a hamiltonian v, u-path. Sometimes, if it is clear which G' we have in mind, we will simply say that u is v-*distant*.

A v-*lasso* is a graph consisting of a cycle, say C, and a path connecting v with C. In this case, C is the *loop* of this v-lasso. If $v \in V(C)$, then C itself is a v-lasso. A v-lasso with k vertices, l of whose belong to the loop, will be sometimes called a (v, k, l)-*lasso*. A typical structure used in proofs of Lemmas 2–4 will be as follows. We will consider a subpath $P_1 = (v_1, \ldots, v_k)$ of a path $P = (v_1, \ldots, v_m)$ and let $G_1 = G[V(P_1)]$. We will know that k is not large, for example, $k \le 11$. For some reasons, we will know that v_1 has no neighbors outside of P_1 and, moreover, that no (G_1, v_k)-distant vertex has a neighbor outside of P_1. If k is 2 (mod 3), then we will want to prove that some $(k-2)/3$ vertices dominate $V(P_1) - v_k$. If k is 1 (mod 3), then we will want to prove that some $(k-1)/3$ vertices dominate $V(P_1)$. We will show that we do not need to consider the case of $k = 0 \pmod 3$. Thus, we need that some $\lfloor k/3 \rfloor$ vertices dominate the first $3\lfloor k/3 \rfloor + 1$ vertices of P_1. For example, if $P_1 = P = (v_1, \ldots, v_8)$ and v_8 is the only out-endpoint of P, then we will prove that some two vertices dominate $V(P_1) - v_8$. We will do this as follows.

Since v_1 has no neighbors outside of P_1, it has two neighbors, v_i and v_j, distinct from v_2 on P_1. Path P_1 together with edge $v_1 v_i$ forms a v_k-lasso. Among all v_k-lassos on $V(P_1)$ choose a lasso L with the largest loop C. By renumbering vertices, we may assume that L consists of the cycle $C = (v_1, \ldots, v_r)$ and the path $(v_r, \ldots, v_k)$. If r is divisible by 3, then the set $D = \{v_3, v_6, \ldots, 3\lfloor k/3 \rfloor\}$ dominates what we need. So, we will need to consider only $r \ne 0 \pmod 3$. The problem of finding $\lfloor k/3 \rfloor$ vertices that dominate the first $3\lfloor k/3 \rfloor + 1$ vertices of P_1 reduces to the problem of finding $\lfloor r/3 \rfloor$ vertices that dominate $\{v_1, \ldots, v_{3\lfloor r/3 \rfloor + 1}\}$, since the remaining $3(\lfloor k/3 \rfloor - \lfloor r/3 \rfloor)$ vertices of P_1 that we need to dominate are easily dominated by the vertices $v_{3(\lfloor r/3 \rfloor)}, v_{3(\lfloor r/3 \rfloor + 1)}, \ldots, v_{3(\lfloor k/3 \rfloor)}$.

Let $G' = G[V(C)]$. By the above condition on P_1, no (G', v_r)-distant vertex has a neighbor outside of P_1. By the maximality of $|C|$, no (G', v_r)-distant vertex has a neighbor in $V(P_1) - V(C)$. Thus, no (G', v_r)-distant vertex has a neighbor outside of C. In the rest of this section we will prove that under these conditions, some $\lfloor r/3 \rfloor$ vertices dominate $\{v_1, \ldots, v_{3\lfloor r/3 \rfloor + 1}\}$ for $r = 4, 5, 7, 8, 10$ and 11. This will be heavily used later.

Lemma 5. *Let G' be the subgraph of a cubic graph G induced by vertices v_1, v_2, v_3, and v_4. If v_1 has no neighbor outside of G', then v_1 dominates $V(G')$.*

Proof. This is because the only possible neighbors of v_1 are v_2, v_3, and v_4. □

Lemma 6. *Let G' be the subgraph of a cubic graph G induced by vertices v_1, v_2, v_3, v_4, and v_5. If no (G', v_5)-distant vertex has a neighbor outside of $V(G')$, then some vertex dominates $V(G') - v_5$.*

Proof. If $v_1 v_3 \in E(G)$, then v_3 dominates $V(G') - v_5$. Suppose that $v_1 v_3 \notin E(G)$. Then $v_1 v_4, v_1 v_5 \in E(G)$. The paths $(v_3, v_2, v_1, v_4, v_5)$ and $(v_2, v_3, v_4, v_1, v_5)$ show that each of v_2 and v_3 can play the role of v_1 and thus by the above argument should be adjacent to v_5 if no vertex dominates $V(G') - v_5$. But v_5 cannot be adjacent to all of v_1, v_2, v_3, v_4. □

Lemma 7. *If a graph G' on $3k+1$ vertices has a hamiltonian path $P = (v_1, \ldots, v_{3k+1})$ and an edge $v_i v_{i+3j-1}$, where i is not divisible by 3, then G' has a dominating set of size k.*

Proof. If $i = 3m+1$, then we let $D = \{v_2, v_5, \ldots, v_{3m-1}, v_{3m+3}, v_{3m+6}, \ldots, v_{3k}\}$. Note that then $v_{i+3j-1} \in D$. Thus every $v \in D$ dominates its neighbors on P, and v_{i+3j-1} also dominates v_i.

If $i = 3m+2$, then we let $D = \{v_2, v_5, \ldots, v_{3m+3j-1}, v_{3m+3j+3}, v_{3m+3j+6}, \ldots, v_{3k}\}$. In this case $v_i \in D$, every $v \in D$ dominates its neighbors on P, and $v_i = v_{3m-2}$ also dominates $v_{i+3j-1} = v_{3m+3j+1}$. □

An immediate corollary of this lemma is the following fact.

Lemma 8. *If a graph G' on $3k+1$ vertices has a hamiltonian cycle $(v_1, \ldots, v_{3k+1})$ and an edge $v_i v_j$ with $j - i + 1$ divisible by 3, then G' has a dominating set of size k.*

Lemma 9. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \dots, v_7$. If G' contains a hamiltonian (in G') cycle $(v_1, v_2, \dots, v_7)$ and v_7 has an outneighbor, then either some two vertices dominate $V(G')$, or there are two (G', v_7)-distant vertices v_k and v_l, $k, l \neq 7$, such that each of them has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. Since the lemma does not hold, by Lemma 8,

$$\text{No edge of the form } v_i v_{i+2}, \text{ or } v_i v_{i+5} \text{ is present in } G'. \tag{1}$$

For each $i = 1, \dots, 7$, the *third neighbor* of v_i is the in-neighbor different from v_{i-1} and v_{i+1} (if it exists). Since both v_1 and v_6 are (G', v_7)-distant, under conditions of the lemma, at least one of them has no outneighbors. By symmetry, we may assume that v_1 has no outneighbors. By (1), the only possible third neighbors of v_1 are v_4 and v_5.

Case 1. $v_1v_5 \in E(G')$. In this case by (1), v_4 has no third neighbors in G'. Thus it has an outneighbor. But the path $v_4, v_3, v_2, v_1, v_5, v_6, v_7$ is hamiltonian in G'. So if the lemma does not hold, then v_6 has no outneighbors. Symmetrically to v_1, the possible third neighbors of v_6 are v_2 and v_3. If $v_6v_3 \in E(G')$, then $\{v_1, v_3\}$ dominates $V(G')$. If $v_6v_2 \in E(G')$, then symmetrically to v_4, v_3 must have an outneighbor, a contradiction to our assumptions.

Case 2. $v_1v_4 \in E(G')$. If $\{v_1, v_6\}$ dominates $V(G')$, then we are done. Suppose not. Then $v_6v_3 \notin E(G)$. Thus by (1), v_3 has an outneighbor. Since the path $v_3, v_2, v_1, v_4, v_5, v_6, v_7$ is hamiltonian in G', v_3 is v_7-distant. Hence if the lemma does not hold, then v_6 has a third neighbor in G'. By the symmetry with v_1, it should be v_2 or v_3. But we assumed that $v_6v_3 \notin E(G)$. Hence, $v_6v_2 \in E(G)$ and we have Case 1 again. □

Lemma 10. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \dots, v_8$. If G' contains a hamiltonian (in G') cycle $(v_1, v_2, \dots, v_8)$ and v_8 has an outneighbor, then either some two vertices v_i and v_j dominate $V(G') - v_8$, or some (G', v_8)-distant vertex $v_k \neq v_8$ has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. In particular, this implies that v_1 and v_7 have third neighbors. If $v_1v_7 \in E(G')$, then Lemma 9 yields our lemma. Let $v_1v_7 \notin E(G')$. By Lemma 7, $v_1v_6 \notin E(G')$ and $v_1v_3 \notin E(G')$. Hence, the only possible third neighbors for v_1 are v_4 and v_5, and by symmetry, the only possible third neighbors for v_7 are v_4 and v_3. If v_4 is not a neighbor of $\{v_1, v_7\}$, then $v_7v_3, v_1v_5 \in E(G')$ and hence $\{v_3, v_5\}$ dominates $V(G') - v_8$. Thus (by symmetry) we may assume that $v_1v_4 \in E(G')$ and hence $v_7v_3 \in E(G')$.

The existence of the path $(v_6, v_5, v_4, v_1, v_2, v_3, v_7, v_8)$ yields that v_6 has no outneighbors. The only possible third in-neighbor for v_6 is v_2. Then v_5 must have an outneighbor, but

$v_5, v_4, v_3, v_7, v_6, v_2, v_1, v_8$ is a hamiltonian v_5, v_8-path, a contradiction. □

Lemma 11. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \dots, v_{10}$. Suppose that G' contains a hamiltonian (in G') cycle $(v_1, v_2, \dots, v_{10})$ and v_{10} has an outneighbor. Then either some three vertices dominate $V(G')$, or some (G', v_{10})-distant vertex v_i has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. By Lemma 8,

$$\text{no edge of the form } v_i v_{i+2}, v_i v_{i+5}, \text{ or } v_i v_{i+8} \text{ is present in } G'. \tag{2}$$

Since the lemma does not hold, v_1 has no outneighbors and v_9 has no outneighbors. Hence either of them has a third neighbor in G'. By (2), $v_1v_9 \notin E(G')$.

Case 1. $v_1v_8 \in E(G')$. Because of the hamiltonian path $(v_7, v_6, v_5, v_4, v_3, v_2, v_1, v_8, v_9, v_{10})$, vertex v_7 has a third neighbor in G'.

Case 1.1. $v_9v_2 \in E(G')$. By the symmetry between v_3 and v_7, vertex v_3 also has a third neighbor in G'. If $v_3v_7 \in E(G')$, then by (2), there is no room for the third neighbor of v_4, but there exists the hamiltonian path $(v_4, v_5, v_6, v_7, v_3, v_9, v_8, v_2, v_1, v_{10})$, a contradiction.

Since by (2), $v_5v_3 \notin E(G')$ and $v_5v_7 \notin E(G')$, the only remaining possibility is that $v_3v_6 \in E(G')$ and $v_4v_7 \in E(G')$. Now, there is no room for the third neighbor of v_5, but there exists the hamiltonian path $(v_5, v_6, v_7, v_4, v_3, v_2, v_9, v_8, v_1, v_{10})$.

Case 1.2. $v_9v_3 \in E(G')$. The set $D = \{v_1, v_3, v_6\}$ dominates $V(G')$.

Case 1.3. $v_9v_5 \in E(G')$. The third neighbor, x, of v_7 is in $\{v_2, v_3, v_4\}$. In any case, $D = \{v_1, x, v_5\}$ dominates $V(G')$.

Case 1.4. $v_9v_6 \in E(G')$. The set $D = \{v_1, v_4, v_6\}$ dominates $V(G')$.

By (2), there are no other possibilities for the third neighbor of v_9.

Case 2. $v_1v_7 \in E(G')$. By symmetry, we assume that $v_2v_9 \notin E(G')$. Because of the hamiltonian path $(v_6, v_5, v_4, v_3, v_2, v_1, v_7, v_8, v_9, v_{10})$, vertex v_6 has a third neighbor in G'.

Case 2.1. $v_9v_3 \in E(G')$. The only possible third neighbor for v_6 is v_2. Symmetrically, to v_6, vertex v_4 has a third neighbor, and its only possible third neighbor is v_8. Thus, v_5 has no third neighbor in G' but there exists the hamiltonian path $(v_5, v_4, v_3, v_9, v_8, v_7, v_6, v_2, v_1, v_{10})$, a contradiction.

Case 2.2. $v_9v_5 \in E(G')$. Then $D = \{v_3, v_7, v_9\}$ dominates $V(G')$.

Case 2.3. $v_9v_6 \in E(G')$. Then $D = \{v_1, v_4, v_9\}$ dominates $V(G')$.

By (2), there are no other possibilities for the third neighbor of v_9.

Case 3. $v_1v_5 \in E(G')$. By symmetry, we may assume that $v_9v_j \notin E(G')$ for $j = 1, 2, 3, 4$. Hence, $v_9v_6 \in E(G')$. Since there exists the hamiltonian path $(v_7, v_8, v_9, v_6, v_5, v_4, v_3, v_2, v_1, v_{10})$, vertex v_7 has a third neighbor, say x, in G'. By (2), $x \in \{v_3, v_4\}$. Then $D = \{x, v_5, v_9\}$ dominates $V(G')$.

Case 4. $v_1v_4 \in E(G')$. By symmetry, we may assume that $v_9v_6 \in E(G')$. As in Case 3, v_7 has a third neighbor, say x, in G'. By (2), $x = v_3$. Then $D = \{v_3, v_4, v_9\}$ dominates $V(G')$.

This proves the lemma. □

Lemma 12. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_{10}, v_{11}$. Suppose that G' contains a hamiltonian (in G') cycle $(v_1, v_2, \ldots, v_{11})$ and v_{11} has an outneighbor. Then either some three vertices dominate $V(G') - v_{11}$, or some (G', v_{11})-distant vertex v_i has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. By Lemma 7,

$$\text{for } i \neq 0 \pmod 3,\ \text{ no edge of the form } v_iv_{i+2}, v_iv_{i+5}, \text{ or } v_iv_{i+8} \text{ is present in } G'. \tag{3}$$

Case 1. $v_1v_{10} \in E(G')$. Since the hamiltonian path $(v_2, v_3, \ldots, v_{10}, v_1, v_{11})$ connects v_2 with v_{11}, and the lemma does not hold, v_2 has no outneighbors. Similarly, v_9 has no outneighbors. Hence either of them has a third neighbor in G'.

Because of the cycle $(v_1, v_2, \ldots, v_{10})$, (2) holds again.

Case 1.1. $v_2v_9 \in E(G')$. Since the hamiltonian path $(v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_2, v_1, v_{10}, v_{11})$ connects v_3 with v_{11}, v_3 has no outneighbors. By symmetry, v_8 has no outneighbors. Hence either of them has a third neighbor in G'. By (2), neither v_8 nor v_5 is a third neighbor of v_3.

Suppose first that $v_3v_7 \in E(G')$. Since the hamiltonian path $(v_6, v_5, v_4, v_3, v_7, v_8, v_9, v_2, v_1, v_{10}, v_{11})$ connects v_6 with v_{11}, v_6 has a third neighbor in G'. By (2), neither v_4 nor v_8 is the third neighbor of v_6. This contradicts the choice of G'. Thus, $v_3v_7 \notin E(G')$. By symmetry, $v_8v_4 \notin E(G')$.

Now, the only possible third neighbor of v_3 is v_6, and of v_8 is v_5. Then $D = \{v_{11}, v_3, v_8\}$ dominates $V(G')$.

Case 1.2. $v_2v_8 \in E(G')$. Since the hamiltonian path $(v_7, v_6, v_5, v_4, v_3, v_2, v_8, v_9, v_{10}, v_1, v_{11})$ connects v_7 with v_{11}, v_7 has a third neighbor in G'.

Suppose first that $v_3v_7 \in E(G')$. Since the hamiltonian path $(v_4, v_5, v_6, v_7, v_3, v_2, v_8, v_9, v_{10}, v_1, v_{11})$ connects v_4 with v_{11}, v_4 has a third neighbor in G'. But by (2), neither v_6 nor v_9 is the third neighbor of v_4, a contradiction.

Since by (2), v_5 is not the third neighbor of v_7, we need $v_7v_4 \in E(G')$. Since the hamiltonian path $(v_5, v_6, v_7, v_4, v_3, v_2, v_8, v_9, v_{10}, v_1, v_{11})$ connects v_5 with v_{11}, v_5 has a third neighbor in G'. By (2), this third neighbor is not v_3. Hence, $v_5v_9 \in E(G')$, and then $D = \{v_2, v_7, v_9\}$ dominates $V(G') - v_{11}$.

Case 1.3. $v_2v_7 \in E(G')$. Impossible by (2).

Case 1.4. $v_2v_6 \in E(G')$. By the symmetry between v_2 and v_9, we may assume that $v_9v_3 \notin E(G')$. By (2), $v_9v_4 \notin E(G')$ and $v_9v_7 \notin E(G')$. Hence, $v_9v_5 \in E(G')$. Since the hamiltonian path $(v_8, v_7, v_6, v_2, v_3, v_4, v_5, v_9, v_{10}, v_1, v_{11})$ connects v_8 with v_{11}, v_8 has a third neighbor in G'. By (2), $v_8v_3 \notin E(G')$. Hence $v_8v_4 \in E(G')$. Then $D = \{v_2, v_8, v_9\}$ dominates $V(G') - v_{11}$.

Case 1.5. $v_2v_5 \in E(G')$. By the symmetry between v_2 and v_9 and by (2), we need to consider only v_6 as a possible third neighbor for v_9. Since the hamiltonian path $(v_4, v_3, v_2, v_5, v_6, v_7, v_8, v_9, v_{10}, v_1, v_{11})$ connects v_4 with v_{11}, v_4 has a third neighbor, say x, in G'. This x must belong to $\{v_7, v_8\}$. Then $D = \{v_2, x, v_9\}$ dominates $V(G') - v_{11}$.

Since $v_2v_4 \notin E(G')$ by (2), this finishes Case 1.

Since Case 1 does not hold, v_1 and v_{10} have other third neighbors. By (3), $v_1v_9 \notin E(G')$ and $v_{10}v_2 \notin E(G')$.

Case 2. $v_1v_8 \in E(G')$. If $v_{10}v_3 \in E(G')$, then $D = \{v_3, v_5, v_8\}$ dominates $V(G') - v_{11}$. So, $v_{10}v_3 \notin E(G')$.

Case 2.1. $v_{10}v_4 \in E(G')$. Since the hamiltonian path $(v_7, v_6, v_5, v_4, v_3, v_2, v_1, v_8, v_9, v_{10}, v_{11})$ connects v_7 with v_{11}, v_7 has a third neighbor in G'. By (3), this third neighbor is neither v_2, nor v_5, nor v_9. Hence, $v_7v_3 \in E(G')$. Now the hamiltonian path $(v_2, v_3, v_7, v_6, v_5, v_4, v_{10}, v_9, v_8, v_1, v_{11})$ connects v_2 with v_{11}. There are three candidates for the third neighbor of v_2: v_9, v_6, and v_5. If $v_2v_9 \in E(G')$, then $D = \{v_2, v_4, v_7\}$ dominates $V(G') - v_{11}$. If $v_2v_6 \in E(G')$, then $D = \{v_2, v_4, v_8\}$ dominates $V(G') - v_{11}$. If $v_2v_5 \in E(G')$, then $D = \{v_4, v_5, v_8\}$ dominates $V(G') - v_{11}$.

Case 2.2. $v_{10}v_6 \in E(G')$. The set $D = \{v_3, v_6, v_8\}$ dominates $V(G') - v_{11}$.

Case 2.3. $v_{10}v_7 \in E(G')$. The hamiltonian path $(v_9, v_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{10}, v_{11})$ connects v_9 with v_{11}. If $v_9v_2 \in E(G')$ or $v_9v_5 \in E(G')$, then $D = \{v_2, v_5, v_7\}$ dominates $V(G') - v_{11}$. If $v_9v_3 \in E(G')$, then the hamiltonian path $(v_4, v_5, v_6, v_7, v_8, v_1, v_2, v_3, v_9, v_{10}, v_{11})$ connects v_4 with v_{11}, but there is no room for the third neighbor of v_4, since v_6 is forbidden by (3). Finally, suppose that $v_9v_6 \in E(G')$. The hamiltonian path $(v_2, v_3, v_4, v_5, v_6, v_7, v_{10}, v_9, v_8, v_1, v_{11})$ connects v_2 with v_{11}. The only possible third neighbor for v_2 is v_5. Now the hamiltonian path $(v_3, v_4, v_5, v_2, v_1, v_8, v_9, v_6, v_7, v_{10}, v_{11})$ connects v_3 with v_{11}, but there are no possible third neighbors for v_3.

By (3), Case 2 is proved.

Case 3. $v_1v_7 \in E(G')$. By the symmetry between v_1 and v_{10}, we may assume that $v_{10}v_3 \notin E(G')$.

Case 3.1. $v_{10}v_4 \in E(G')$. The hamiltonian path $(v_5, v_6, v_7, v_8, v_9, v_{10}, v_4, v_3, v_2, v_1, v_{11})$ connects v_5 with v_{11}. If $v_5v_9 \in E(G')$, then $D = \{v_3, v_7, v_9\}$ dominates $V(G') - v_{11}$.

Suppose now that $v_5v_8 \in E(G')$. The existence of the paths $(v_2, v_3, v_4, v_{10}, v_9, v_8, v_5, v_6, v_7, v_1, v_{11})$ and $(v_9, v_8, v_5, v_6, v_7, v_1, v_2, v_3, v_4, v_{10}, v_{11})$ yields that both, v_2 and v_9, have third neighbors in G'. Since both of them cannot be adjacent to v_6, either there is $x \in \{v_2, v_3\}$ adjacent to v_9, or there is $x \in \{v_8, v_9\}$ adjacent to v_2. In either case, the set $D = \{v_4, v_7, x\}$ dominates $V(G') - v_{11}$.

The last possibility for the third neighbor of v_5 is v_2. By the symmetry between v_5 and v_6, we may assume that $v_6v_9 \in E(G')$. The hamiltonian path $(v_3, v_4, v_{10}, v_9, v_8, v_7, v_6, v_5, v_2, v_1, v_{11})$ connects v_3 with v_{11}. The only possible third neighbor for v_3 is v_8. Then $D = \{v_{11}, v_3, v_6\}$ dominates $V(G')$.

Case 3.2. $v_{10}v_6 \in E(G')$. The hamiltonian path $(v_2, v_3, v_4, v_5, v_6, v_{10}, v_9, v_8, v_7, v_1, v_{11})$ connects v_2 with v_{11}. If $v_2v_9 \in E(G')$, then $D = \{v_4, v_7, v_9\}$ dominates $V(G') - v_{11}$.

Suppose now that $v_2v_8 \in E(G')$. Consider the cycle $(v_{11}, v_{10}, v_9, v_8, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$ as the cycle $(v'_{11}, v'_1, v'_2, \ldots, v'_{10})$, where $v'_{11} = v_{11}$, $v'_1 = v_{10}$, $v'_2 = v_9$, and so on. In this cycle, $v'_1v'_8 \in E(G')$, i.e., we have Case 2.

The last possibility for the third neighbor of v_2 is v_5. The hamiltonian path $(v_4, v_3, v_2, v_5, v_6, v_{10}, v_9, v_8, v_7, v_1, v_{11})$ connects v_4 with v_{11}. By (3), the third neighbor of v_4 should be v_8. Then $D = \{v_2, v_6, v_8\}$ dominates $V(G') - v_{11}$. By (3), Case 3 is proved.

Since Cases 1–3 do not hold and in view of (3), we assume below that $v_1v_i \notin E(G')$ for $i = 10, 9, 8, 7, 6, 3$ and, by symmetry, that $v_{10}v_j \notin E(G')$ for $j = 1, 2, 3, 4, 5, 8$.

Case 4. $v_1v_5 \in E(G')$. The hamiltonian path $(v_4, v_3, v_2, v_1, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11})$ connects v_4 with v_{11}. By (3),

$$\text{the third neighbor of } v_4 \text{ is in the set } \{v_7, v_8\}. \tag{4}$$

Case 4.1. $v_{10}v_6 \in E(G')$. If $v_4v_8 \in E(G')$, then $D = \{v_2, v_6, v_8\}$ dominates $V(G') - v_{11}$.

Let $v_4v_7 \in E(G')$. Because of the hamiltonian path $(v_2, v_3, v_4, v_7, v_8, v_9, v_{10}, v_6, v_5, v_1, v_{11})$, vertex v_2 has a third neighbor. By symmetry, v_9 also has a third neighbor. If $v_2v_9 \in E(G')$, then $D = \{v_2, v_6, v_7\}$ dominates $V(G') - v_{11}$. If $v_2v_9 \notin E(G')$, then $v_2v_8 \in E(G')$ and $v_3v_9 \in E(G')$. In this case, $D = \{v_2, v_6, v_3\}$ dominates $V(G') - v_{11}$.

Case 4.2. $v_{10}v_7 \in E(G')$. By (4), $v_4v_8 \in E(G')$. The hamiltonian path $(v_9, v_8, v_4, v_3, v_2, v_1, v_5, v_6, v_7, v_{10}, v_{11})$ connects v_9 with v_{11}. If the third neighbor, say x, of v_9 is in $\{v_2, v_3\}$, then $D = \{x, v_5, v_7\}$ dominates $V(G') - v_{11}$. So, let $v_6v_9 \in E(G')$. Then there is no room for the third neighbor of v_3, but the hamiltonian path $(v_3, v_2, v_1, v_5, v_4, v_8, v_7, v_6, v_9, v_{10}, v_{11})$ connects v_3 with v_{11}. This contradiction finishes Case 4.

Case 5. $v_1v_4 \in E(G')$. By the symmetry between v_1 and v_{10}, we assume that $v_{10}v_7 \in E(G')$. Because of the hamiltonian path $(v_3, v_2, v_1, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11})$, vertex v_3 has a third neighbor. By symmetry, v_8 also has a third neighbor. If $v_3v_9 \in E(G')$, then $D = \{v_3, v_4, v_6\}$ dominates $V(G') - v_{11}$.

Suppose that $v_3v_8 \in E(G')$. Because of the hamiltonian path $(v_2, v_3, v_8, v_9, v_{10}, v_7, v_6, v_5, v_4, v_1, v_{11})$, vertex v_2 has a third neighbor. If $v_2v_9 \in E(G')$, then $D = \{v_2, v_4, v_6\}$ dominates $V(G') - v_{11}$. If $v_2v_6 \in E(G')$, then $D = \{v_9, v_4, v_6\}$ dominates $V(G') - v_{11}$. Finally, if $v_2v_5 \in E(G')$, then $D = \{v_{11}, v_5, v_8\}$ dominates $V(G')$.

The only possibility remaining for the third neighbor of v_3 is v_6. By symmetry, we assume that $v_8v_5 \in E(G')$. Now $D = \{v_{11}, v_3, v_8\}$ dominates $V(G')$. This finishes the last case of the proof. □

5. Proofs of Lemmas 3 and 4

For convenience, we restate Lemma 3 here.

Lemma 13. *Let G be connected and have more than 8 vertices. If a 2-path P in an optimal vdp-cover is such that each of the hamiltonian paths on $V(P)$ has at most one out-endpoint, then either some $(|P|-2)/3$ vertices dominate all vertices of P apart from an out-endpoint or P has at least 11 vertices.*

Proof. If a 2-path $P = (v_1, v_2, \ldots, v_k)$ has less than 11 vertices, then $k \in \{2, 5, 8\}$. If $k = 2$, then clearly both vertices of P are out-endpoints. The case $k = 5$ was considered in Reed's paper [7], but we also outline this case using the ideas of Section 4 as follows. If neither of v_1 and v_5 is an out-endpoint, then each of them has three neighbors in $V(P)$. Furthermore, in this case $v_1v_5 \in E(G)$, since otherwise there is no room for their neighbors. But G is connected and has more than 5 vertices, so some vertex of the cycle $(v_1, v_2, v_3, v_4, v_5)$ has an outneighbor, a contradiction to the choice of P. Thus, we may assume that v_5 is an out-endpoint of P. Then by Lemma 6, some vertex dominates $V(P) - v_5$.

Now, let $k = 8$. If one of v_1 and v_8 is an out-endpoint, then we may assume that it is v_8. Consider a v_8-lasso on $V(P)$ with a largest loop. As described in Section 4, we may assume that this loop is the cycle $C = (v_1, \ldots, v_r)$. Let G' be the subgraph of G induced by this loop.

Case 1. Vertex v_8 is an out-endpoint of P. If $r = 8$, then by Lemma 10, some two vertices dominate $V(P) - v_8$.

Let $r = 7$. Since v_8 is an out-endpoint of P, it has at most two neighbors in $V(G')$ (one of which is v_7), and we are done by Lemma 9. By Lemma 7, $r \neq 6$ and $v_1v_3 \notin E(G)$. Therefore, $r \leq 5$ and $v_1v_4, v_1v_5 \in E(G)$. Then the path $P_1 = (v_2, v_3, v_4, v_1, v_5, v_6, v_7, v_8)$ shows that v_2 is (G', v_8)-distant. Hence, v_2 should have a neighbor in G' distinct from v_1 and v_3. This neighbor is not in $\{v_4, v_5\}$, since $v_1v_4, v_1v_5 \in E(G)$. This contradicts the maximality of r.

Case 2. P has no out-endpoints. If $r = 8$, then since G has more than 8 vertices, some vertex of the cycle C should have an outneighbor, and we have Case 1.

Let $r = 7$. By the maximality of the loop, no two of the three neighbors of v_8 in $V(G')$ are consecutive on the loop $(v_1, v_2, \ldots, v_7)$. Thus, by symmetry, we may assume that the neighbors of v_8 are v_7, v_5 and v_3. It is easy to check that in this case each of v_1, v_2, v_4 and v_6 is (G', v_8)-distant. For example, path $(v_4, v_5, v_6, v_7, v_1, v_2, v_3, v_8)$ shows that v_3 is (G', v_8)-distant. Again, this contradicts the fact that $V(P)$ has a neighbor in $G - P$.

Let $r \leq 6$. By the symmetry between v_1 and v_8, we may assume that $v_8v_2 \notin E(G)$. Then no vertex in G' is adjacent to both, v_1 and v_8. Hence, $\{v_1, v_8\}$ dominates $V(P)$. □

For convenience, we also restate Lemma 4.

Lemma 14. *If a tip $P_1 = (v_1, v_2, \ldots, v_{3t+1})$ of an accepting 2-path P has no out-endpoint and no $(2, 2)$-endpoint and $t \leq 3$, then some set D of t vertices of G dominates all vertices of P_1.*

Proof. For $t \leq 2$, it was proved in [7] (Fact 11) and also will be clear from the proof for $t = 3$. So, suppose that a tip $P_1 = (v_1, v_2, \ldots, v_{10})$ of an accepting 2-path P has no out-endpoint and no $(2, 2)$-endpoint. Let v_{11} be the neighbor of v_{10} on $P - P_1$ and let G' be the subgraph of G induced by $V(P_1) + v_{11}$. Since our system of paths was chosen so to maximize the number of out-endpoints and $(2, 2)$-endpoints and taking into account (B4) of Lemma 1, no (G', v_{11})-distant vertex in G' has an outneighbor (with respect to $V(G')$). We choose a (G', v_{11})-distant vertex in G' and an edge incident to this vertex so as to maximize the length of the loop of a v_{11}-lasso in G'. We renumber the vertices in G' so that this vertex is v_1 and this loop is $(v_1, v_2, \ldots, v_r)$. If $r = 11$, then we are done by Lemma 12.

Let $r < 11$. Then v_1 has two neighbors in $G' - v_2$. By Lemma 7,

$$v_1v_{3j} \notin E(G) \quad \text{for } j = 1, 2, 3. \tag{5}$$

Case 1. $r = 10$. By Lemma 11, either some 3 vertices dominate $V(P_1)$ (and then we are done), or some (G', v_{11})-distant vertex v_j has an outneighbor, x (with respect to P_1). By the choice of our vdp-cover, x should be in P. By Lemma 1, it cannot be in $P - P_1 - v_{11}$. Thus, $x = v_{11}$, a contradiction to $r < 11$.

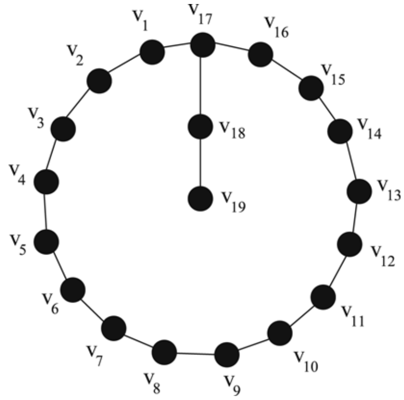

Fig. 2.

Case 2. $r = 8$. Let $G'' = G' - \{v_9, v_{10}, v_{11}\}$. By Lemma 10, either some two vertices v_i and v_j dominate $V(G'')$, or some (G'', v_8)-distant vertex v_k, $1 \le k \le 7$, has an outneighbor, x (with respect to $V(G'')$). In the first case, $D = \{v_i, v_j, v_9\}$ dominates P_1. In the second case, by Lemmas 1 and 12, we have a contradiction to the maximality of r.

Case 3. $r = 7$. Let $G''' = G' - \{v_8, v_9, v_{10}, v_{11}\}$. By Lemma 9, either some two vertices v_i and v_j dominate $V(G''')$, or some (G''', v_7)-distant vertex v_k, $1 \le k \le 6$, has an outneighbor, x (with respect to $V(G'''))$. In the first case, $D = \{v_i, v_j, v_9\}$ dominates P_1. In the second case, by Lemmas 1 and 12, we have a contradiction to the maximality of r.

By (5) and the cases above, $r = 5$ and the three neighbors of v_1 are v_2, v_4, and v_5. Since there is the path $(v_3, v_2, v_1, v_4, v_5, \ldots, v_{10})$, vertex v_3 has no neighbors outside of P. By Lemmas 1 and 12, it has no neighbors in $P - P_1$. Hence $v_3 v_j \in E(G)$ for some $6 \le j \le 10$. This contradicts the maximality of r. $\square$

6. Proof of Lemma 2

Recall that Lemma 2 states that each 1-path P in an optimal vdp-cover S that does not have an out-endpoint and does not contain a dominating set of size at most $|P|/3$, has at least 22 vertices. Fact 9 in [7] states that such a path must have at least 16 vertices. Hence we need to prove that such a path cannot have 19 vertices and cannot have 16 vertices. We will prove this in two big lemmas.

Lemma 15. *If a* 1*-path P in an optimal vdp-cover S does not have an out-endpoint and does not contain a dominating set of size at most* $|P|/3$*, then P cannot have* 19 *vertices.*

Proof. Let $P = (v_1, v_2, \ldots, v_{19})$ be a counter-example to the lemma and $G' = G[V(P)]$. Consider a v_{19}-lasso on $V(P)$ with the largest loop. We may assume that it is a $(v_{19}, 19, r)$-lasso. If $r = 19$, then a vertex of the 19-cycle has an outneighbor, a contradiction to the fact that P has no out-endpoints. Thus $r \le 18$. By Lemma 7, r is not divisible by 3.

Case 1. $r = 17$. (See Fig. 2.)

Since v_{19} is an endpoint of P, it has two neighbors on the loop $C = (v_1, \ldots, v_{17})$. Then v_{18} is an endpoint of a hamiltonian path in G' and hence, has two neighbors on C. By Lemma 7, we may assume that the distance on C between a neighbor of v_{19} and a neighbor of v_{18} on C is not 0 or 2 (mod 3). Also, by the maximality of r, this distance must be greater than 2. Since v_{18} is adjacent to v_{17}, the only possible neighbors of v_{19} are v_4, v_8, v_{10}, and v_{13}. If v_4 and v_{13} are the neighbors of v_{19}, then we have no room on C for the third neighbor of v_{18}. By symmetry, this leaves us with the following two cases.

Case 1.1: v_{18} is adjacent to v_{17} and v_3, and v_{19} is adjacent to v_7 and v_{13}.

Case 1.2: v_{18} is adjacent to v_{17} and v_3, and v_{19} is adjacent to v_7 and v_{10}.

In both cases, since v_1 is an endpoint of a hamiltonian path in G', it has a third neighbor on C. By Lemma 7, v_1 is not adjacent to v_6, v_9, v_{12} and v_{15}. By the same lemma applied to the path $(v_2, v_1, v_{17}, v_{16}, \ldots, v_3, v_{18}, v_{19})$, v_1 is

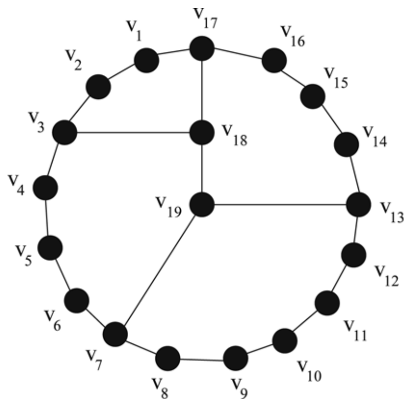

Fig. 3.

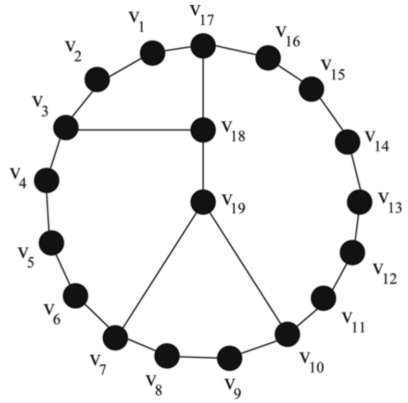

Fig. 4.

not adjacent to v_4, v_{10}, v_{13}, and v_{16}. By the symmetry between v_1 and v_2,

the third neighbor of v_2 on C is in $\{v_6, v_9, v_{12}, v_{15}\}$. (6)

If $v_1v_8 \in E(G)$, then G' contains the 19-cycle $(v_1, v_2, \ldots, v_7, v_{19}, v_{18}, v_{17}, \ldots, v_8, v_1)$, contrary to the maximality of r. Hence, we may assume that the third neighbor of v_1 on C is in $\{v_5, v_{11}, v_{14}\}$.

Case 1.1 is shown in Fig. 3. In this case, if $v_1, v_{14} \in E(G)$, then G' contains the 19-cycle $(v_1, v_2, \ldots, v_{13}, v_{19}, v_{18}, v_{17}, \ldots, v_{14}, v_1)$. The remaining possible neighbors of v_1 are v_5, and v_{11}. By the symmetry between v_1 and v_2, we conclude that the only possible neighbors of v_2 are v_9, and v_{15}. Then $\{v_{18}, v_{19}, v_5, v_9, v_{11}, v_{15}\}$ is a dominating set in G'.

Fig. 4 shows Case 1.2. In this case, if $v_1v_{11} \in E(G)$, then G' contains the 19-cycle $(v_1, v_2, \ldots, v_{10}, v_{19}, v_{18}, v_{17}, \ldots, v_{11}, v_1)$. Hence, the possible neighbors of v_1 are v_5, and v_{14}. By the symmetry between v_1 and v_9, the possible neighbors of v_9 are v_5, and v_{13}. If $v_2v_6 \in E(G)$, then G' contains the 19-cycle $(v_2, v_1, v_{17}, v_{16}, \ldots, v_7, v_{19}, v_{18}, v_3, v_4, v_5, v_6, v_2)$, contrary to the maximality of r. From this, (6), and the fact that $v_9v_2 \notin E(G)$ we derive that the third neighbor of v_2 is either v_{12} or v_{15}. By the symmetry between v_2 and v_8, the possible neighbors of v_8 also are v_{12}, and v_{15}. If $v_1v_5 \in E(G)$, then the set $\{v_{19}, v_{18}, v_9, v_5, v_{12}, v_{15}\}$ dominates $V(G')$. Thus, we may assume that $v_1v_{14} \in E(G)$ and, by symmetry, $v_9v_{13} \in E(G)$. In this case G' contains the 19-cycle $(v_1, v_2, \ldots, v_9, v_{13}, v_{12}, v_{11}, v_{10}, v_{19}, v_{18}, v_{17}, v_{16}, v_{15}, v_{14}, v_1)$, a contradiction.

Case 2. $r = 16$ (see Fig. 5). By the maximality of r, v_{19} is not adjacent to v_1, v_2, v_3, v_{15}, v_{14}, and v_{13}. By Lemma 7, v_{19} is also not adjacent to v_{17}, v_{11}, v_8, and v_5. In particular, v_{19} has two neighbors on C and these neighbors are not consecutive on C by the maximality of r. Since v_{19} has neighbors on C, by symmetry v_{17} also has two neighbors on C that are non-consecutive on C. Let v_x be the other neighbor of v_{17} on C. By symmetry, we may assume that $8 \le x \le 14$.

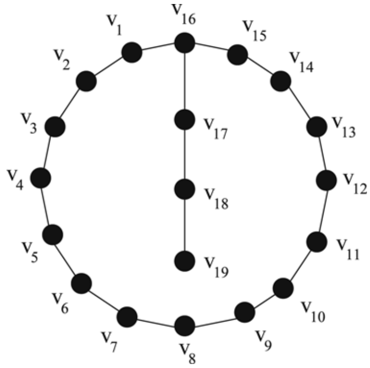

Fig. 5.

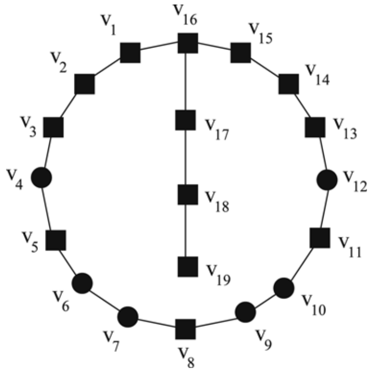

Fig. 6.

In Fig. 6, the possible neighbors of v_{19} (namely, v_4, v_6, v_7, v_9, v_{10}, and v_{12}) are drawn as shaded circles, while the forbidden neighbors are drawn as shaded squares. If $x = 12$, then only one vertex, v_{10}, from these shaded circles is a possible neighbor of v_{19}. Thus, $x \neq 12$. By the symmetry between v_{19} and v_{17}, the set of the two neighbors of v_{19} on C is not $\{v_6, v_{10}\}$.

We claim that *the distance on C between some neighbor of v_{19} and some neighbor of v_{17} is exactly* 4. Indeed, if v_4 and v_{12} (that are at distance 4 from v_{16} on C) are not neighbors of v_{19}, then some neighbor v_y of v_{19} is in the set $\{v_9, v_{10}\}$. Then $x \geq 13$ and to avoid being at distance 4 from v_y on C, we need $x = 14$ and $y = 9$. But this contradicts Lemma 7 applied to the path $(v_{19}, v_{18}, v_{17}, v_{14}, v_{13}, \ldots, v_1, v_{16}, v_{15})$. Thus the claim holds.

Without loss of generality we assume that v_{19} is adjacent to v_4. Similarly to the paragraph above, Lemma 7 implies that the distance on C between any neighbor of v_{19} and any neighbor of v_{17} is at least 4 and is not 2 (mod 3). By these properties and by symmetry, it is enough to consider the following cases.

Case 2.1: v_{17} is adjacent to v_{16} and v_8, and v_{19} is adjacent to v_4 and v_{12}.

Case 2.2: v_{17} is adjacent to v_{16} and v_{13}, and v_{19} is adjacent to v_4 and v_6.

Case 2.3: v_{17} is adjacent to v_{16} and v_{10}, and v_{19} is adjacent to v_4 and v_6.

Case 2.4: v_{17} is adjacent to v_{16} and v_{13}, and v_{19} is adjacent to v_4 and v_7.

Case 2.5: v_{17} is adjacent to v_{16} and v_{11}, and v_{19} is adjacent to v_4 and v_7.

Fig. 7 shows Case 2.1 with a dominating set of 6 vertices. In Case 2.2, G' contains a lasso with the loop $(v_{16}, v_{15}, \ldots, v_4, v_{19}, v_{18}, v_{17}, v_{16})$. Then vertices v_1, v_2, and v_3 play the roles of v_{17}, v_{18}, and v_{19}, respectively. Fig. 8 (left) shows Case 2.2 with a new labeling of the vertices (in all remaining cases we will give new labellings of the vertices). Here v_1^*, v_2^*, and v_3^* were our original v_{17}, v_{18}, and v_{19} and the vertices labeled $v_{17}{}^*$, $v_{18}{}^*$, and $v_{19}{}^*$ (former v_1, v_2, and v_3) behave the same way as the original v_{17}, v_{18}, v_{19} in terms of which vertices are forbidden as their neighbors. In Fig. 8 (right) we redraw G' and consider possible neighbors of $v_{19}{}^*$, which already has v_4 as a neighbor. We will always consider the neighbors of $v_{19}{}^*$ in the remaining cases; possible neighbors are represented by shaded circles, while forbidden neighbors are represented by shaded squares. The third neighbor of $v_{19}{}^*$ could be v_7, v_9, v_{10} or

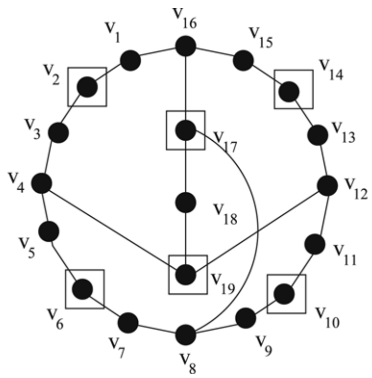

Fig. 7.

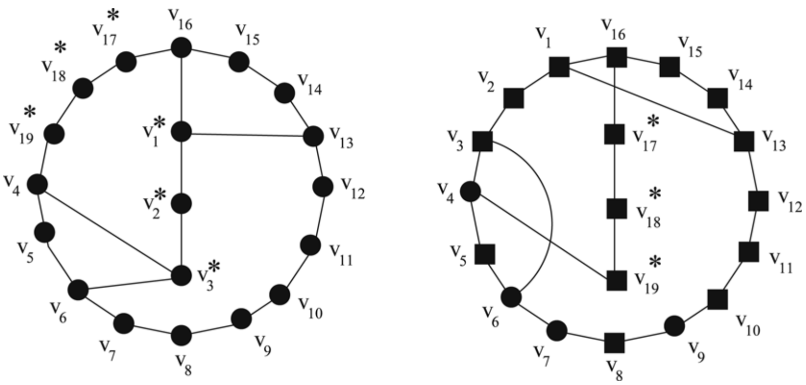

Fig. 8.

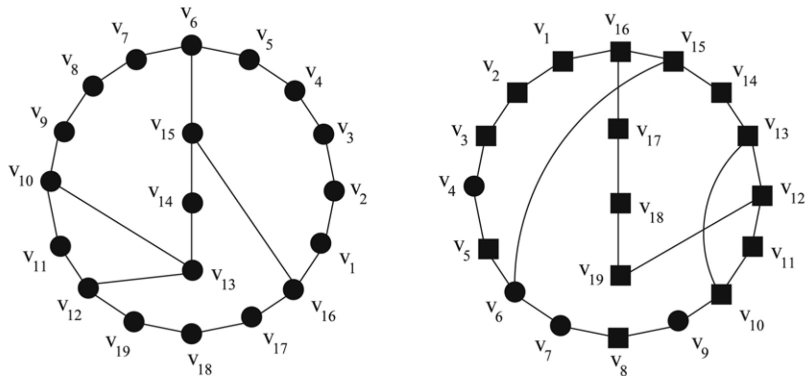

Fig. 9.

v_{12}. If $v_{19}{}^*v_{12} \in E(G)$, then $v_{17}{}^*$ must be adjacent to v_8, and we have Case 2.1. If $v_{19}{}^*v_{10} \in E(G)$, then G' contains a $(v_{11}, 19, 17)$-lasso with the loop $(v_1^*, v_2^*, \ldots, v_{10}, v_{19}^*, v_{18}^*, \ldots, v_{13}, v_1^*)$ contradicting the assumption that $r = 16$. If $v_{19}{}^*v_9 \in E(G)$, then $v_{17}{}^*$ has no possible third neighbor. If $v_{19}{}^*v_7 \in E(G)$, then G' contains a $(v_{17}^*, 19, 17)$-lasso with the loop $(v_7, v_8, \ldots, v_{16}, v_1^*, v_2^*, v_3^*, v_6, v_5, v_4, v_{19}^*, v_7)$.

Fig. 9 (left) shows Case 2.3, with the relabelling of the vertices so that the new v_{17}, v_{18}, and v_{19} are the former v_7, v_8, and v_9. Vertex v_{19} has v_{12} as a neighbor and the third neighbor of v_{19} could be v_4, v_7 or v_9. If $v_{19}v_4 \in E(G)$ (see Fig. 9 (right)), then v_{17} must be adjacent to v_8, and we have Case 2.1. If $v_{19}v_7 \in E(G)$, then G' contains cycle $(v_1, v_2, \ldots, v_6, v_{15}, v_{14}, \ldots, v_7, v_{19}, v_{18}, v_{17}, v_{16}, v_1)$ on 19 vertices. If $v_{19}v_9 \in E(G)$, then G' contains a $(v_{17}^*, 19, 17)$-lasso with the loop $(v_1, v_2, \ldots, v_6, v_{15}, v_{14}, \ldots, v_9, v_{19}, v_{18}, v_{17}, v_{16}, v_1)$.

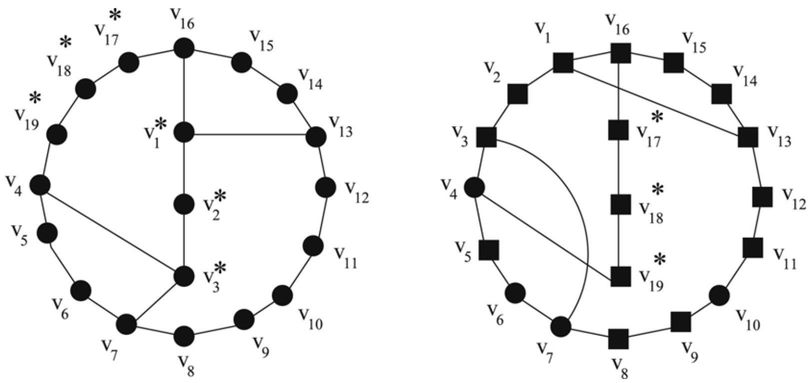

Fig. 10.

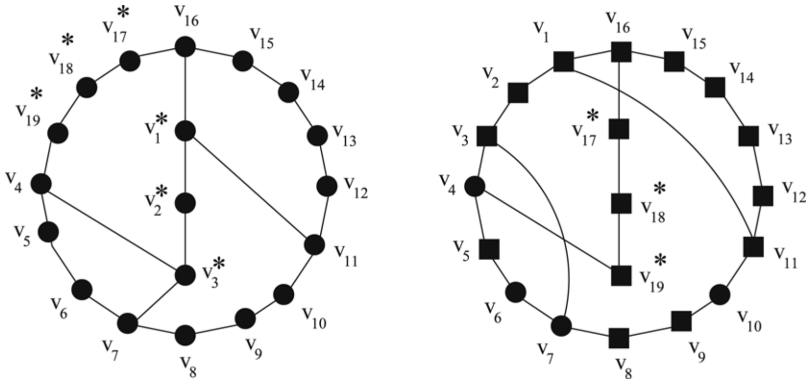

Fig. 11.

Fig. 10 shows Case 2.4 with a redrawing of G' as in Case 2.2. One of the neighbors of $v_{19}{}^*$ on C is v_4 and the other could be v_6, v_9, v_{10} or v_{12}. If it is v_9, v_{10} or v_{12}, then the argument is exactly as in Case 2.2. If $v_{19}{}^*v_6 \in E(G)$, then G' contains a $(v_1^*, 19, 17)$-lasso with the loop $(v_6, v_5, v_4, v_3^*, v_7, v_8, \ldots, v_{19}^*, v_6)$.

Fig. 11 shows Case 2.5 with a redrawing of G'. One of the neighbors of $v_{19}{}^*$ on C is v_4 and the other could be v_6, v_9, v_{10} or v_{12}. If $v_{19}{}^*v_{12} \in E(G)$, then $v_{17}{}^*$ must be adjacent to v_8, and we have Case 2.1. If $v_{19}{}^*v_{10} \in E(G)$ (respectively, $v_{19}{}^*v_9 \in E(G)$), then G' contains cycle $C_1 = (v_{10}, v_9, \ldots, v_1^*, v_{11}, v_{12}, v_{13}, \ldots, v_{19}^*, v_{10})$ of length 19 (respectively, contains cycle $C_2 = C_1 - v_{10}$ of length 18). Finally, if $v_{19}{}^*v_6 \in E(G)$, then G' contains a $(v_1^*, 19, 17)$-lasso with the loop $(v_6, v_5, v_4, v_3^*, v_7, v_8, \ldots, v_{19}^*, v_6)$. This concludes Case 2.

Case 3. $r = 14$ (see Fig. 12). By Lemma 6 applied to the subgraph G'' of G induced by the vertex set $\{v_{15}, v_{16}, \ldots, v_{19}\}$, either some (G'', v_{15})-distant vertex has a neighbor outside of $V(G'')$, or some vertex x dominates $V(G'') - v_{15}$. In the latter case, the set $\{x, v_2, v_5, v_8, v_{11}, v_{14}\}$ dominates G', so we assume the former. This means that we can rename the vertices $v_{16}, v_{17}, v_{18}, v_{19}$ so that $(v_{15}, v_{16}, v_{17}, v_{18}, v_{19})$ is a path and v_{19} has a neighbor, y, outside of G''. By the optimality of S, y should be on C.

Since G' has no lasso with loop size greater than 14, y is not in $\{v_1, v_2, v_3, v_4, v_5, v_9, v_{10}, v_{11}, v_{12}, v_{14}\}$. By Lemma 7, $y \neq v_8$. By symmetry, $y \neq v_6$. Hence, $y = v_7$ and the third neighbor, v_i, of v_{19} should be in G''. Then the vertex v_{i+1} is (G'', v_{15})-distant. By the above argument, v_{i+1} should be adjacent to v_7, but v_7 already has three neighbors, a contradiction.

Case 4. $r = 13$ (see Fig. 13).

Similarly to Case 3, by Lemma 6 applied to the subgraph G'' of G induced by the vertex set $\{v_{15}, v_{16}, \ldots, v_{19}\}$, either some (G'', v_{15})-distant vertex has a neighbor outside of $V(G'')$, or some vertex x dominates $V(G'') - v_{15}$. In the latter case, the set $\{x, v_2, v_5, v_8, v_{11}, v_{14}\}$ dominates G', so we again assume the former. As in Case 3, we can assume that $(v_{15}, v_{16}, v_{17}, v_{18}, v_{19})$ is a path and that v_{19} has a neighbor, y, outside of G''. By Lemma 7, $v_{19}v_{14} \notin E(G)$, so

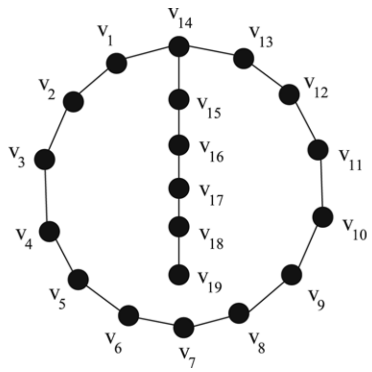

Fig. 12.

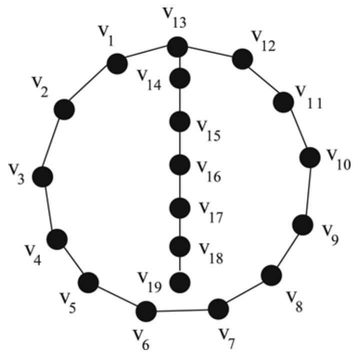

Fig. 13.

by the optimality of S, y should be on C. But since G' has no lasso with loop size greater than 13, there is no place on C for y.

Case 5. $4 \le r \le 11$. Recall that r is not divisible by 3. Let G'' be the subgraph of G' induced by the set $\{v_1, v_2, \ldots, v_r\}$. By the maximality of r, no (G'', v_r)-distant vertex has a neighbor outside of $V(G'')$. Thus the cases of $r = 11, 10, 8, 7, 5$, and 4 follow from Lemmas 12, 11, 10, 9, 6 and 5, respectively. Thus, Lemma 15 is proved. $\square$

The proof of our second statement that finishes the proof of Lemma 2 mimics the proof of the above lemma but is much simpler.

Lemma 16. *If a* 1*-path* P *in an optimal vdp-cover* S *does not have an out-endpoint and does not contain a dominating set of size at most* $|P|/3$*, then* P *cannot have* 16 *vertices.*

Proof. Let $P = (v_1, v_2, \ldots, v_{16})$ be a counter-example to the lemma and $G' = G[V(P)]$. Consider a v_{16}-lasso on $V(P)$ with a largest loop. We may assume that it is a $(v_{16}, 16, r)$-lasso. If $r = 16$, then either a vertex of the 16-cycle has an outneighbor, or G has exactly 16 vertices. The latter cannot happen by Theorem 3. The former contradicts the fact that P has no out-endpoints. Thus, $r \le 15$. By Lemma 7, r is not divisible by 3.

Case 1. $r = 14$ (see Fig. 14). Similarly to Case 1 of Lemma 15, v_{16} has two neighbors on the loop $C = (v_1, \ldots, v_{14})$ and v_{15} also has two neighbors on C. By Lemma 7, we may assume that the distance on C between a neighbor of v_{16} and a neighbor of v_{15} on C is not 0 or 2 (mod 3) and this distance must be greater than 2. Since $v_{15}v_{14} \in E(G)$, the possible neighbors of v_{16} are only v_4, v_7, and v_{10}. If v_4 and v_{10} are the neighbors of v_{16}, then we have no room on C for the third neighbor of v_{15}. Thus, $v_{16}v_7 \in E(G)$. By symmetry, we can assume that v_{16} is adjacent to v_7 and v_{10}, and v_{15} is adjacent to v_3 and v_{14}.

Since v_1 is an endpoint of a hamiltonian path in G', it has a third neighbor on C. By Lemma 7, v_1 is not adjacent to any of v_6, v_9 and v_{12}. By the same lemma applied to the path $(v_2, v_1, v_{14}, v_{13}, \ldots, v_3, v_{15}, v_{16})$, v_1 is not adjacent to v_4 and v_{13}. Thus, the possible third neighbors of v_1 are v_5, v_8, and v_{11}. If $v_1v_8 \in E(G)$, then G' contains cycle

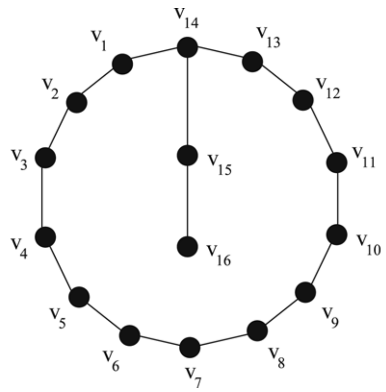

Fig. 14.

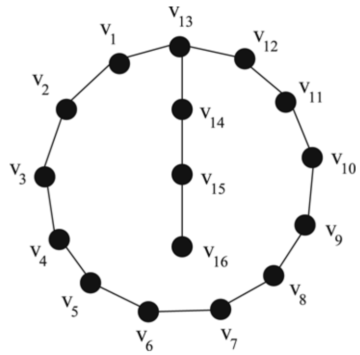

Fig. 15.

$(v_1, v_2, \dots, v_7, v_{16}, v_{15}, \dots, v_8, v_1)$ contrary to the maximality of r. Similarly, if $v_1v_{11} \in E(G)$, then G' contains cycle $(v_1, v_2, \dots, v_{10}, v_{16}, v_{15}, \dots, v_{11}, v_1)$. Thus, $v_1v_5 \in E(G)$. By the symmetry between v_1 and v_9, we conclude that $v_9v_5 \in E(G)$. But then v_5 has degree at least 4, a contradiction.

Case 2. $r = 13$ (see Fig. 15). Similarly to Case 2 of Lemma 15, v_{16} is not adjacent to v_{14}, v_{11}, v_8, v_5, and v_2, and the two neighbors of v_{16} on C are not consecutive on C. By symmetry, v_{14} also has two non-consecutive neighbors on C. Let v_x be the other neighbor of v_{14} on C. By symmetry, we may assume that $7 \le x \le 11$.

By the maximality of r, the distance on C from each of the two neighbors of v_{16} to v_{13} and to v_x is at least 4. Since $7 \le x \le 11$, only v_4, v_6, and v_7 qualify as possible neighbors of v_{16} on C. Furthermore, if $v_{16}v_7 \in E(G)$, then $x \ge 7 + 4 = 11$. The case when $v_{14}v_{11} \in E(G)$ and $v_{16}v_6 \in E(G)$ is impossible by Lemma 7, applied to the path $(v_{12}, v_{11}, \dots, v_1, v_{13}, v_{14}, v_{15}, v_{16}, v_{12})$. Thus, up to the symmetry between v_{14} and v_{16}, there is only one case: v_{14} is adjacent to v_{13} and v_{11}, and v_{16} is adjacent to v_4 and v_7.

In Fig. 16 (left) we give a new labeling of the vertices of G'. Here v_1^*, v_2^*, and v_3^* were our original v_{14}, v_{15}, and v_{16}, respectively, and the vertices labeled $v_{14}{}^*$, $v_{15}{}^*$, and $v_{16}{}^*$ (the former v_1, v_2, and v_3) behave the same way as the original v_{14}, v_{15}, v_{16}. In Fig. 16 (right) we redraw G' and consider possible neighbors of $v_{16}{}^*$, which already has v_4 as a neighbor. Since v_7 already has degree 3, the third neighbor of $v_{16}{}^*$ is v_6. Then the 14-cycle $(v_7, v_8, \dots, v_{16}^*, v_6, v_5, v_4, v_3^*, v_7)$ forms the loop of a v_2^*-lasso contrary to the maximality of r.

Case 3. $4 \le r \le 11$. The proof repeats the argument of Case 5 of Lemma 15. □

7. Proof of Theorem 5

We will need the following simple fact.

Lemma 17. *Let* $\pi = (i_1, i_2, \dots, i_k)$ *be a permutation of* $\{1, 2, \dots, k\}$ *and* $f(\pi) = |i_k - i_1| + \sum_{j=1}^{k-1} |i_{j+1} - i_j|$. *Then* $f(\pi) \le k^2/2$.

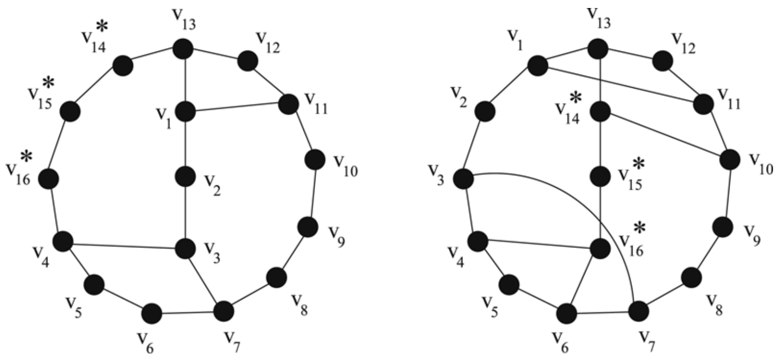

Fig. 16.

Proof. Clearly, $f(\pi) = \max\{i_k - i_1, i_1 - i_k\} + \sum_{j=1}^{k-1} \max\{i_j - i_{j+1}, i_{j+1} - i_j\}$.

For any choice of the maxima, we have a sum of $2k$ numbers, where each of the numbers $1, 2, \ldots, k$ occurs exactly twice (with pluses or minuses) and exactly half of the $2k$ numbers are with minuses. Hence the maximum is attained when every $i \geq k/2 + 1$ appears twice with plus and every $i \leq k/2$ appears twice with minus. If k is odd, then $(k+1)/2$ should appear once with plus and once with minus.

Thus, for even k,

$$f(\pi) \leq 2\left([k + (k-1) + \cdots + (k/2+1)] - [k/2 + (k/2-1) + \cdots + 1]\right) = \frac{k^2}{2}.$$

Similarly, for odd k,

$$f(\pi) \leq 2\left(\left[k + (k-1) + \cdots + \frac{k+3}{2}\right] - \left[\frac{k-1}{2} + \left(\frac{k-1}{2} - 1\right) + \cdots + 1\right]\right) = 2\frac{k+1}{2}\frac{k-1}{2}. \quad \square$$

Lemma 18. *If in an optimal vdp-cover of a cubic connected graph G of girth g, a path P of the vdp-cover has at most one out-endpoint, then $|P| > 1 + g^2/4$.*

Proof. Let $P = (v_1, \ldots, v_k)$ and v_1 be not an out-endpoint. Let $G_1 = G[V(P)]$. By the optimality of our vdp-cover, no (G_1, v_k)-distant vertex is an out-endpoint. As in Section 4, we may assume that a (G_1, v_k)-lasso with a largest loop consists of the cycle $C = (v_1, \ldots, v_r)$ and the path $(v_r, \ldots, v_k)$. Let $V' = \{v_1, \ldots, v_r\}$ and $G' = G[V']$. It could be that $r = k$. By the choice of P, v_k and r, no (G', v_r)-distant vertex has a neighbor outside of V'. Hence, each (G', v_r)-distant vertex has a third neighbor in V'.

By construction, v_1 and v_{r-1} are v_r-distant vertices. Let $t = \lfloor g/2 \rfloor$. We will construct $2t$ distinct vertices $a_1, b_1, a_2, b_2, \ldots, a_t, b_t$ in V' with the following properties:

(p1) a_i and b_i are neighbors on the cycle C for all $i = 1, \ldots, t$;
(p2) for all $i = 1, \ldots, t$, G' contains a hamiltonian b_i, v_r-path that uses only edges of C and some of the edges $b_1a_2, b_2a_3, \ldots, b_{i-1}a_i$;
(p3) a_i is the third neighbor of b_{i-1} for all $i = 2, \ldots, t$;
(p4) the distance on the cycle C between a_i and a_j is at least $g - 2|j - i|$ for all $i \neq j$, $i, j = 1, \ldots, t$.

We define these vertices inductively as follows. First let $a_1 = v_r$ and $b_1 = v_{r-1}$. Properties (p1) and (p2) hold for a_1 and b_1 and (p3) and (p4) are not applicable. Suppose that for some $1 \leq j \leq t-1$, we have found $a_1, b_1, a_2, b_2, \ldots, a_j, b_j$ that satisfy (p1)–(p4). In particular, (p4), (p3), and (p1) together imply that all $a_1, b_1, a_2, b_2, \ldots, a_j, b_j$ are distinct and that for $i < j$ no a_i or b_i is a neighbor of b_j. Define a_{j+1} to be the third neighbor of b_j. By (p2) for b_j, G' contains a hamiltonian v_r, b_j-path P_{b_j} that uses only edges of C and some of the edges $b_1a_2, b_2a_3, \ldots, b_{j-1}a_j$. Then the edge b_ja_{j+1} is not an edge of this path and hence is a chord of P_{b_j} creating a v_r-lasso. One of the other two neighbors of a_{j+1} (the one farther from v_r on P_{b_j}) is v_r-distant. Define this vertex to be b_{j+1}. Observe that a hamiltonian path in G' connecting b_{j+1} with v_r uses only edges of P_{b_j} and the edge b_ja_{j+1}. Hence, (p2) holds $i = j+1$. Since b_j is the third neighbor of a_{j+1}, the edge $a_{j+1}b_{j+1}$ belongs to C. Thus, (p1) and (p3) also hold for $i = j+1$. To prove (p4), observe that $(a_1, b_1, a_2, b_2 \ldots, a_{j+1})$ is a path every second edge of which is a chord of C. Hence if the distance between some a_s and a_{j+1} on C is less than $g - 2(j + 1 - s)$, then we have a closed walk that uses path $(a_s, b_s, a_{s+1}, b_{s+1} \dots, a_{j+1})$ and has length less than g. Since edge $b_j a_{j+1}$ is passed in this walk only once, it contains a simple cycle of length less than g, a contradiction. This proves the induction step.

Now we have t distinct vertices $a_1, \dots, a_t$ on C satisfying (p4). Suppose that their cyclic order on C is $(j_1, j_2, \dots, j_t)$. Then assuming that $j_{t+1} = j_1$, we have

$$r \geq \sum_{i=1}^{t} (g - 2|j_{i+1} - j_i|) = tg - 2\sum_{i=1}^{t} |j_{i+1} - j_i|.$$

By Lemma 17, the last expression is at least

$$tg - t^2 = t(g - t) = \left\lfloor \frac{g}{2} \right\rfloor \left\lceil \frac{g}{2} \right\rceil = \left\lfloor \frac{g^2}{4} \right\rfloor.$$

Let a_l be the closest in $\{a_2, \dots, a_t\}$ vertex to $a_1 = v_r$ on C if we go from v_r to v_{r-1} and so on. Let P'' be the part of $(v_1, v_2, \dots, v_r)$ from a_l to v_r. Then the closed walk consisting of paths P'' and $(a_1, b_1, a_2, b_2 \dots, a_l)$ passes the edge $a_1 b_1$ twice. Hence, to avoid a cycle shorter than g, the length of P'' has to be at least $g + 2 - 2(l - 1)$. This improves the lower bound on r above to $2 + \lfloor g^2/4 \rfloor > 1 + g^2/4$. □

Lemma 19. *If in an optimal vdp-cover of a cubic connected graph G of girth g, a tip P of a path P_1 in the cover has no out-endpoint and no $(2, 2)$-endpoint, then $|P| > g^2/4$.*

Proof. Let $P = (v_1, \dots, v_{k-1})$, where v_1 is an endpoint of P_1, and let v_k be the neighbor in $P_1 - P$ of v_{k-1} on path P_1. By Lemma 1, v_1 has no neighbor in $P_1 - P - v_k$. Thus, we can repeat the proof of Lemma 18 practically word by word and prove our lemma. □

Theorem 6. *If G is a cubic connected n-vertex graph of girth g, then $\gamma(G) \leq \frac{n}{3}(1 + 8/g^2)$.*

Proof. Consider an optimal vdp-cover S of G. The paths P in S that need more than $|P|/3$ vertices to dominate them could be only either 1-paths without out-endpoints, or 2-paths with at most one out-endpoint, or tips of 2-paths with no out-endpoints. By Lemmas 18 and 19, each such path has at least $g^2/4$ vertices. Therefore, we have at most $4n/g^2$ such paths. For each such path P, we spend at most $(|P| + 2)/3$ dominating vertices. Thus, altogether we spend at most

$$\frac{n}{3} + \frac{2}{3}\frac{4n}{g^2} = \frac{n}{3}\left(1 + \frac{8}{g^2}\right)$$

dominating vertices. □

8. Concluding remarks

We think that both the upper and lower known bounds on the maximal domination number of cubic connected n-vertex graphs could be improved. It is also interesting whether Reed's conjecture holds for 3-connected cubic n-vertex graphs and for cubic n-vertex bipartite graphs. Recently, the second author constructed a 2-connected counterexample to this conjecture [8].

Acknowledgements

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