

## Ore-type conditions implying 2-factors consisting of short cycles

Alexandr V. Kostochka<sup>a,b,\*</sup>, Gexin Yu<sup>c</sup>

<sup>a</sup> Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

<sup>b</sup> Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

<sup>c</sup> Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

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### ABSTRACT

For every graph  $G$ , let  $\sigma_2(G) = \min\{d(x) + d(y) : xy \notin E(G)\}$ . The main result of the paper says that every  $n$ -vertex graph  $G$  with  $\sigma_2(G) \geq \frac{4n}{3} - 1$  contains each spanning subgraph  $H$  all whose components are isomorphic to graphs in  $\{K_1, K_2, C_3, K_4^-, C_5^+\}$ . This generalizes the earlier results of Justesen, Enomoto, and Wang, and is a step towards an Ore-type analogue of the Bollobás–Eldridge–Catlin Conjecture.

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### 1. Introduction

Two  $n$ -vertex graphs  $G_1$  and  $G_2$  are said to *pack* if there exist injective mappings of their vertex sets onto  $[n]$  such that the images of the edge sets do not intersect. In a similar way, one can define the packing of more than two graphs.

The study of extremal problems on packings of graphs was started in the 1970s by Bollobás and Eldridge [3], Sauer and Spencer [18], and Catlin [5].

Sauer and Spencer [18] proved that *two  $n$ -vertex graphs pack if the product of their maximum degrees is less than  $n/2$* . Kaul and Kostochka [14] characterized the pairs of  $n$ -vertex graphs with the product of maximum degrees exactly  $n/2$  that do not pack.

The following BEC-conjecture (one of the main conjectures in the area) was posed in 1978 by Bollobás and Eldridge [3], and independently by Catlin [6].

**Conjecture 1.** *Let  $G_1$  and  $G_2$  be  $n$ -vertex graphs with maximum degrees  $\Delta_1$  and  $\Delta_2$ , respectively. If  $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ , then  $G_1$  and  $G_2$  pack.*

By definition, graphs  $G_1$  and  $G_2$  pack if and only if  $G_1$  contains the complement  $\overline{G_2}$  of  $G_2$ . In the containment language, the BEC-conjecture states that every  $n$ -vertex graph  $G$  with minimum degree  $\delta$  contains each  $n$ -vertex graph  $H$  such that  $(n - \delta)(\Delta(H) + 1) \leq n + 1$ .

This conjecture is proved to be true only for some limited classes of graphs, see [1,2,8,4,15]. In particular, Aigner and Brandt [1], and independently Alon and Fisher [2] (for  $n$  sufficiently large), proved the special case  $\Delta(H) \leq 2$ :

**Theorem 1.** *If  $G$  is an  $n$ -vertex graph with  $\delta(G) \geq (2n - 1)/3$ , then  $G$  contains each  $n$ -vertex graph  $H$  with  $\Delta(H) \leq 2$ .*

\* Corresponding author at: Department of Mathematics, University of Illinois, Urbana, IL 61801, USA.

E-mail addresses: [kostochk@math.uiuc.edu](mailto:kostochk@math.uiuc.edu) (A.V. Kostochka), [gexin.yu@vanderbilt.edu](mailto:gexin.yu@vanderbilt.edu) (G. Yu).

**Theorem 1** generalizes an earlier result by Corrádi and Hajnal [7], which says that a  $3k$ -vertex graph  $G$  with minimum degree at least  $2k$  contains  $k$  disjoint triangles. Another important generalization of the Corrádi–Hajnal result is the Hajnal–Szemerédi Theorem [12] states that each  $n$ -vertex graph  $G$  with  $\delta(G) \geq (1 - 1/k)n$  contains the graph  $H(n, k)$  whose every component is  $K_k$ , given that  $k$  divides  $n$ . This theorem is the partial case of the BEC-conjecture for  $G_2$  being the disjoint union of complete graphs of the same size.

The above-mentioned results have the spirit of Dirac’s Theorem [9] (which says that every  $n$ -vertex graph with minimum degree at least  $n/2$  contains a hamiltonian cycle) in the sense that these results guarantee the existence of some subgraph if the minimum degree of the graph is large enough. Ore [17] gave a different sufficient condition for hamiltonicity: he proved that every  $n$ -vertex graph  $G$  with

$$\sigma_2(G) = \min_{xy \notin E(G)} \{\deg(x) + \deg(y)\} \geq n$$

contains a hamiltonian cycle. Justesen [13] proved an Ore-type version of the Corrádi–Hajnal Theorem by showing that every  $n$ -vertex graph  $G$  with  $\sigma_2(G) \geq 4n/3$  contains  $\lfloor n/3 \rfloor$  disjoint triangles. Enomoto [10], and Wang [19] sharpened this result. In particular, they proved the following.

**Theorem 2.** For each positive integer  $k$ , every  $3k$ -vertex graph  $G$  with  $\sigma_2(G) \geq 4k - 1$  contains  $k$  disjoint triangles.

Our main result is the following.

**Theorem 3.** Each  $n$ -vertex graph  $G$  with

$$\sigma_2(G) \geq \frac{4n}{3} - 1 \tag{1}$$

contains all spanning subgraphs whose components are isomorphic to graphs in  $\mathcal{H} = \{K_1, K_2, C_3, K_4^-, C_5^+\}$ .

Here  $C_5^+$  denotes a cycle of length five with a chord. Note that  $K_4^-$  can also be considered as  $C_4^+$ , i.e. a cycle of length four with a chord.

Condition (1) cannot be weakened. For example, for each integer  $k \geq 2$ , let  $G(k)$  denote the complement of the disjoint union  $K_k \cup K_k \cup K_{k-2}$ . Its number of vertices,  $n(k)$ , is  $3k - 2$  and  $\sigma_2(G(k)) = 4k - 4 = \frac{4n(k)-1}{3} - 1$  which is just  $1/3$  less than the lower bound in (1). However,  $G(k)$  does not contain the graph  $H(k)$  which is the disjoint union of  $k - 1$  triangles and a single vertex. In particular, **Theorem 3** generalizes and extends the above-mentioned results of Justesen, Enomoto and Wang.

**Theorem 3** is also a step towards an Ore-type analogue of the BEC-conjecture. We state and discuss this analogue in the next section in terms of graph packing. In Section 3 we present some technical results on the existence of some subgraphs in dense graphs on at most 12 vertices. The proofs in this section can be omitted at first reading. In Section 4 we prove the following weakening of **Theorem 3**.

**Theorem 4.** Each  $n$ -vertex graph  $G$  with

$$\sigma_2(G) \geq \frac{4n}{3} - 1$$

contains all spanning subgraphs whose components are isomorphic to graphs in  $\mathcal{H}_1 = \{K_1, K_2, C_3, K_4^-\}$ .

In Section 5 we prove two auxiliary statements, and in the final section we prove the main result. The idea of the proofs of most results below is as follows. We have a graph  $G$  satisfying (1) and a graph  $H$  that we want to show to be embeddable into  $G$ . We also know that  $G$  contains another graph  $H'$  that is obtained from  $H$  by replacing one (small) component, say  $F$ , with a bit ‘simpler’ component  $F'$ . Using (1), we show that there is some embedding  $f : V(H') \rightarrow V(G)$  of  $H'$  into  $G$  such that there are ‘many’ edges in  $G$  between  $f(V(F'))$  and the image of some other component  $F''$  of  $H'$ . Then we prove that under these conditions  $G[f(V(F')) \cup f(V(F''))]$  contains vertex-disjoint copies of  $F$  and  $F''$ .

The notation used is mostly from [20]. Let  $G$  be a graph. For  $W, U \subseteq V(G)$ ,  $e(W, U)$  is the number of edges connecting  $W$  with  $U$ . For  $W \subseteq V(G)$  and  $x \in V(G)$ ,  $N_W(x)$  is the set of neighbors of  $x$  in  $W$  and  $d_W(x) = |N_W(x)|$ . Also,  $G[x_1, \dots, x_k]$  (respectively,  $G[W - x_1 - \dots - x_k + y_1 + \dots + y_l]$ ) denotes the subgraph of  $G$  induced by the set  $\{x_1, \dots, x_k\}$  (respectively, by the set  $W \cup \{y_1, \dots, y_l\} \setminus \{x_1, \dots, x_k\}$ ).

## 2. A graph packing conjecture

As mentioned in the introduction, a graph  $G$  contains a graph  $H$  if and only if  $H$  packs with the complement  $\bar{G}$  of  $G$ . Ore-type conditions look more natural for packing graphs than for embedding graphs. Indeed, let  $\theta(G) = \max_{xy \in E(G)} \{\deg(x) + \deg(y)\}$ . In terms of  $\theta$ , Ore’s Theorem claims that every  $n$ -vertex graph  $G$  with  $\theta(G) \leq n - 2$  packs with the cycle  $C_n$  of length  $n$ . Note that  $\theta(G) = \Delta(L(G)) + 2$ , where  $L(G)$  is the line graph of  $G$ . By definition, for every graph  $G$ ,

$$\Delta(G) + \delta(G) \leq \theta(G) \leq 2\Delta(G). \tag{2}$$

In [16] Dirac-type packing results of Sauer and Spencer [18] and Kaul and Kostochka [14] mentioned above were extended to the following Ore-type result.

**Theorem 5.** *If two  $n$ -vertex graphs  $G_1$  and  $G_2$  satisfy  $\theta(G_1)\Delta(G_2) \leq n$ , then  $G_1$  and  $G_2$  pack, with the following exceptions:*

- (I)  $G_1$  is a perfect matching and  $G_2$  is either  $K_{n/2, n/2}$  with  $n/2$  odd or contains  $K_{n/2+1}$ ;
- (II)  $G_2$  is a perfect matching and  $G_1$  is either  $K_{r, n-r}$  with  $r$  odd or contains  $K_{n/2+1}$ .

In [15] we posed the following conjecture which by (2) extends the BEC-Conjecture.

**Conjecture 2.** *If  $G_1$  and  $G_2$  are  $n$ -vertex graphs and  $(0.5\theta(G_1) + 1)(\Delta(G_2) + 1) \leq n + 1$ , then  $G_1$  and  $G_2$  pack.*

Theorem 3 implies the partial case of Conjecture 2 when every component of  $G_2$  is a cycle of length at most five or a short path.

**Remark.** One of the referees suggested to consider an Ore-type analogue of the result by Fan and Kierstead [11] that every  $n$ -vertex graph  $G$  with  $\delta(G) \geq (2n - 1)/3$  contains the square of a hamiltonian path. That would be a challenging problem.

### 3. On small dense graphs with a $C_4$ -subgraph

In this section we present some technical facts on the existence of  $K_4^-$ -subgraphs in small (on at most 12 vertices) dense graphs. The reader can skip it at first reading.

**Lemma 1.** *Let  $V_1$  and  $V_2$  be disjoint vertex subsets of a graph  $F$  such that  $F_1 = F(V_1) = K_3$ ,  $F_2 = F(V_2)$  is the 4-cycle  $y_1y_2y_3y_4$  and  $e(V_1, V_2) \geq 9$ . If each vertex in  $V_2$  is adjacent to some vertex in  $F_1$ , then  $V_1 \cup V_2$  can be partitioned into two sets  $V'_1$  and  $V'_2$  such that  $F(V'_1)$  is  $K_3$  and  $F(V'_2)$  contains  $K_4^-$ .*

**Proof.** Let  $V_1 = \{x_1, x_2, x_3\}$ . First suppose that some  $x_i$  is adjacent to every vertex in  $V_2$ . Some vertex  $y_j$  is adjacent to at least  $\lceil 9/4 \rceil = 3$  vertices in  $V_1$ . Then  $F(V_1 - x_i + y_j) = K_3$  and  $F(V_2 - y_j + x_i) = K_4^-$ .

The only other possibility is that each vertex in  $V_1$  has exactly 3 neighbors in  $V_2$ . Suppose that  $N_{V_2}(x_1) = V_2 - y_4$ . If both  $x_2$  and  $x_3$  are neighbors of  $y_4$ , then we let  $V'_1 = V_1 - x_1 + y_4$  and  $V'_2 = V_2 - y_4 + x_1$ . So, we can assume that  $N_{V_2}(x_2) = V_2 - y_4$ . Then under conditions of the lemma,  $x_3y_4 \in E(F)$ . Vertex  $x_3$  must also be adjacent to some  $y \in \{y_1, y_3\}$ , say to  $y_1$ . Then we let  $V'_1 = \{x_3, y_1, y_4\}$  and  $V'_2 = \{x_1, x_2, y_2, y_3\}$ .  $\square$

**Lemma 2.** *Let  $V_1$  and  $V_2$  be disjoint vertex subsets of a graph  $F$  such that*

- (a)  $V_1 = \{x_1, x_2, x_3, x_4\}$ , and  $F_1 = F(V_1) = K_4$ ;
- (b)  $F_2 = F(V_2)$  is the 4-cycle  $y_1y_2y_3y_4$ ;
- (c)  $|E_F(V_1, V_2)| \geq 11$ .

*If  $F(V_1 \cup V_2)$  does not contain two vertex-disjoint copies of  $K_4^-$ , then there are  $x_4 \in V_1$  and  $y_4 \in V_2$  such that*

- (i)  $N_{F_2-y_4}(x_1) = N_{F_2-y_4}(x_2) = N_{F_2-y_4}(x_3) = \{y_1, y_2, y_3\}$ ;
- (ii)  $y_4$  has at most one neighbor in  $V_1$  and  $|E_F(V_1, V_2)| = 11$ .

**Proof.** Assume that  $V_1$  and  $V_2$  satisfy conditions (a)–(c), but  $F(V_1 \cup V_2)$  does not contain two vertex-disjoint copies of  $K_4^-$ . First we prove that

$$d_{F_2}(x_i) \leq 3 \quad \text{for each } i. \tag{3}$$

Indeed, if  $d_{F_2}(x_i) = 4$ , then we can choose some  $y \in V_2$  with  $d_{F_1}(y) \geq 3$  and let  $F'_1 = F(V_1 - x_i + y)$  and  $F'_2 = F(V_2 + x_i - y)$ .

Assume that  $y_4$  has the fewest neighbors in  $V_1$ .

CASE 1:  $d_{V_1}(y_4) = 0$ . By (c), at least 3 vertices in  $V_1$  have three neighbors in  $V_2$ , each. Thus in this case both (i) and (ii) hold.

CASE 2:  $d_{V_1}(y_4) = 1$ . Suppose  $x_4y_4 \in E(F)$ . By (c), every  $y \in V_2 - y_4$  has  $d_{V_1}(y) \geq 2$ . Hence if  $d_{V_2}(x_4) = 3$ , then we get two vertex-disjoint copies of  $K_4^-$  by switching  $x_4$  with its non-neighbor  $y \in V_2$ . If  $d_{V_2}(x_4) \leq 2$ , by (c), each of  $x_1, x_2, x_3$  has exactly 3 neighbors in  $V_2$ , i.e. (i) holds. Also there must be equality in (c), so (ii) also holds.

CASE 3:  $d_{V_1}(y_4) \geq 2$ . By (c) and (3), some  $x \in V_1$  has exactly 3 neighbors in  $V_2$ . Then switching  $x$  with its non-neighbor  $y$  in  $V_2$ , we obtain two vertex-disjoint copies of  $K_4^-$ , a contradiction. This finishes the proof.  $\square$

**Lemma 3.** *Let  $V_1$  and  $V_2$  be disjoint vertex subsets of a graph  $F$  such that*

- (a)  $V_1 = \{x_1, x_2, x_3, x_4\}$ , and  $F_1 = F(V_1) = K_4^-$  with  $x_1x_4 \notin E(F)$ ;
- (b)  $F_2 = F(V_2)$  is the 4-cycle  $y_1y_2y_3y_4$ ;
- (c)  $|E_F(V_1, V_2)| \geq 11$ .

*If  $F(V_1 \cup V_2)$  does not contain two vertex-disjoint copies of  $K_4^-$ , then either  $F(V_1 \cup V_2)$  contains a copy of  $K_4$  and disjoint from it copy of  $C_4$ , or there are  $x \in \{x_2, x_3\}$  and  $y_4 \in V_2$  such that*

- (i)  $N_{F_2-y_4}(x') = \{y_1, y_2, y_3\}$  for each  $x' \in V_1 - x$ ;
- (ii)  $y_4$  and  $y_2$  have no common neighbors in  $V_1$ .

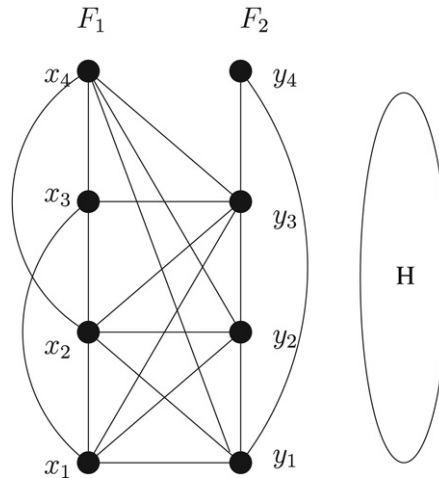


Fig. 1.

**Proof.** Assume that  $y_4$  has the fewest neighbors in  $V_1$ .

CASE 1:  $d_{V_1}(y_4) = 0$  (see Fig. 1). By (c), at least 3 vertices in  $V_1$  have three neighbors in  $V_2$ , each. So, if (i) does not hold, then we may assume that  $x_4$  has exactly two neighbors in  $V_2 - y_4$ , and every other  $x \in V_1$  is adjacent to all  $y_1, y_2$ , and  $y_3$ . If  $x_4y_2 \in E(F)$ , then switching  $y_2$  with  $x_1$  we obtain a copy of  $K_4$  and a disjoint from it 4-cycle. Otherwise,  $x_4y_1, x_4y_3 \in E(F)$  and we get a copy of  $K_4$  and a disjoint from it 4-cycle by switching  $y_2$  with  $x_4$ .

CASE 2:  $d_{V_1}(y_4) = 1$ . Suppose  $x_iy_4 \in E(F)$ .

Subcase 2.1:  $d_{V_2}(x_i) \geq 3$ . Define  $y$  to be the non-neighbor of  $x_i$  in  $V_2$  if  $d_{V_2}(x_i) = 3$  or any  $y \in V_2$  with 4 neighbors in  $V_1$  otherwise. Note that such  $y$  exists by (c) when  $d_{V_1}(y_4) = 1$ . We try to switch  $x_i$  with  $y$ . We do not get two disjoint copies of  $K_4^-$  or a copy of  $K_4$  and a copy of  $C_4$  only if  $d_{V_2}(x_i) = 3, y$  has exactly two neighbors in  $V_1$  and  $F_1 - x_i$  is not a  $K_3$ . In particular, we may assume that  $i = 2$ . Also from (c) we conclude that  $d_{V_1}(y') = 4$  for both  $y' \in V_2 - y_4 - y$ . By the symmetry between  $y_1$  and  $y_3$ , we may assume that  $y_3 \in V_2 - y_4 - y$ . By the symmetry between  $x_1$  and  $x_4$ , we may assume that  $x_1y \in E(F)$ . Then either of  $F[y_3, y_4, x_2, x_4]$  and  $F[y_1, y_2, x_1, x_3]$  contains  $K_4^-$ .

Subcase 2.2:  $d_{V_2}(x_i) \leq 2$ . By (c),  $d_{V_2}(x_i) = 2$  and each  $x \in V_1 - x_i$  is adjacent to  $y_1, y_2$ , and  $y_3$ . Thus (i) holds, unless  $i \in \{1, 4\}$ . Suppose,  $i = 4$ . Then we have  $F[x_1, x_2, y_1, y_2] = K_4$  and the 4-cycle  $(x_3, y_3, y_4, x_4)$ . So, we only need to prove (ii) in the case  $i \in \{2, 3\}$ . Suppose that  $i = 3$  and  $x_3y_2 \in E(F)$ . Then  $F[V_1 - x_1 + y_2] = K_4$  and  $F[V_2 - y_2 + x_1] = C_4$ . This proves (ii).

CASE 3:  $d_{V_1}(y_4) \geq 2$ .

Subcase 3.1:  $d_{V_2}(x_1) \geq 3$ . Switch  $x_1$  with its non-neighbor  $y$  in  $V_2$ , if  $d_{V_2}(x_1) = 3$ , and with any  $y \in V_2$ , otherwise. In both cases we obtain two  $K_4^-$ .

By the symmetry between  $x_1$  and  $x_4$ , the remaining case is the following.

Subcase 3.2:  $d_{V_2}(x_1) \leq 2$  and  $d_{V_2}(x_4) \leq 2$ . By (c), we can assume that  $d_{V_2}(x_2) = 4$ . If  $d_{V_1}(y) = 4$  for some  $y \in V_2$ , then we switch  $x_2$  with  $y$  and get two  $K_4^-$ . Otherwise, we can assume that

$$d_{V_1}(y_1) = d_{V_1}(y_2) = d_{V_1}(y_3) = 3. \tag{4}$$

By the symmetry between  $x_1$  and  $x_4$ , we can assume that  $x_1y_3 \in E(F)$ . Then  $F[x_1, x_2, y_3, y_4]$  has at least five edges. If  $F[y_1, y_2, x_3, x_4]$  also has at least five edges, then we are done. Otherwise, by (4) both  $y_1$  and  $y_2$  are adjacent to  $x_1$ , a contradiction to  $d_{V_2}(x_1) \leq 2$ .  $\square$

**Lemma 4.** Let  $V_1, V_2$ , and  $V_3$  be disjoint vertex subsets of a graph  $F$  such that

- (a)  $V_1 = \{x_1, x_2, x_3, x_4\}$ , and  $F_1 = F(V_1) \supset K_4^-$  with  $x_1x_4$  possibly not in  $E(F)$ ;
- (b)  $F_2 = F(V_2)$  is the 4-cycle  $y_1y_2y_3y_4$ ;
- (c)  $F_3 = F(V_3) \in \{K_1, K_2, C_3, K_4, K_4^-\}$ ;
- (d)  $N_{F_2}(x_1) = N_{F_2}(x_2) = N_{F_2}(x_4) = \{y_1, y_2, y_3\}$ .

If

$$d_{F_3}(y_1) + d_{F_3}(y_3) + 2(d_{F_3}(y_2) + d_{F_3}(y_4)) > 4|V_3|, \tag{5}$$

then  $V_1 \cup V_2 \cup V_3$  can be partitioned into sets  $V'_1, V'_2$  and  $V'_3$  so that  $F(V'_1)$  and  $F(V'_2)$  contain  $K_4^-$  and  $F(V'_3)$  contains  $F_3$ .

**Proof.** By the symmetry between  $y_1$  and  $y_3$ , we will assume that  $d_{F_3}(y_1) \geq d_{F_3}(y_3)$ .

If  $F_3 = K_1$  with  $V(F_3) = \{u\}$ , then  $y_1, y_2, y_4 \in N(u)$ . Then we have  $F_1, F' = \{y_3\}$  and  $K_4^- \subseteq G[u, y_1, y_2, y_4]$ .

Suppose  $F_3 = K_2$  with  $V(F_3) = \{u_1, u_2\}$ . By (5), either  $F[u_1, u_2, y_1, y_2]$  or  $F[u_1, u_2, y_3, y_4]$  has at least 5 edges. If it is  $F[u_1, u_2, y_1, y_2]$ , then we have  $F_1, F'$  with  $V(F') = \{y_3, y_4\}$  and  $K_4^- \subseteq F[u_1, u_2, y_1, y_2]$ . The other possibility is very similar.

If  $F_3$  is 3-cycle  $(z_1, z_2, z_3)$ , then  $d_{F_3}(y_2) + d_{F_3}(y_4) \geq 4$ . If  $d_{F_3}(y_4) \geq 2$ , then  $G[y_4, V(F_3)] \supseteq K_4^-$ ,  $G[y_3, x_2, x_3, x_4] = F'_1$ , and  $G[x_1, y_1, y_2] = K_3$ . Suppose now that  $d_{F_3}(y_4) \leq 1$ . Then  $d_{F_3}(y_2) = 3$  and  $d_{F_3}(y_4) = 1$ . It follows that  $d_{F_3}(y_1) = 3$  and  $d_{F_3}(y_3) \geq 2$ . We may assume that  $y_4z_1 \in E(G)$  and  $y_3z_2 \in E(G)$ . Then  $F_2 \cup F_3$  can be decomposed into 3-cycle  $G[z_2, y_2, y_3]$  and  $G[y_1, y_4, z_1, z_3] \supseteq K_4^-$ .

The last case is that  $F_3$  is a 4-cycle (with at least one chord)  $(z_1, z_2, z_3, z_4)$ . Then since  $e(\{y_1, y_3\}, F_3) \leq 8$ ,  $d_{F_3}(y_2) + d_{F_3}(y_4) \geq 5$ .

If  $d_{F_3}(y_2) + d_{F_3}(y_4) = 5$ , then  $d_{F_3}(y_1) + d_{F_3}(y_3) \geq 7$ , and hence  $d_{F_3}(y_1) = 4$  and  $d_{F_3}(y_3) \geq 3$ . Let  $z \in F_3$  be a common neighbor of  $y_2$  and  $y_4$ . If  $zy_3 \notin E(G)$ , then we have three  $K_4^-$ 's:  $G[V(F_3) - z + y_3]$ ,  $G[y_1, y_2, z, y_4]$  and  $F_1$ . On the other hand, if  $zy_3 \in E(G)$ , we also have three  $K_4^-$ 's:  $G[V(F_3) - z + y_1]$ ,  $G[y_2, y_3, y_4, z]$  and  $F_1$ .

The following claim will be helpful.

**Claim 1.** *Suppose that some  $y \in \{y_1, y_3\}$  has a common neighbor, say  $z_1 \in F_3$ , with  $y_2$ . Then  $y_4$  has at most two neighbors in  $\{z_2, z_3, z_4\}$ , and if it has exactly two neighbors in  $\{z_2, z_3, z_4\}$ , then  $z_2z_4 \notin E(F_3)$  (and hence  $z_1z_3 \in E(F_3)$ ).*

**Proof.** Let  $y' \in \{y_1, y_3\} - y$ . If  $y_4$  is adjacent to each of  $z_2, z_3, z_4$ , or is adjacent to two of them and  $z_2z_4 \in E(F_3)$ , then we have three  $K_4^-$ 's:  $F[V(F_3) - z_1 + y_4]$ ,  $F[x_1, y, y_2, z_1]$  and  $F[y', x_2, x_3, x_4]$ .  $\square$

If  $d_{F_3}(y_2) + d_{F_3}(y_4) = 8$ , then  $d_{F_3}(y_1) \geq 1$ . This contradicts Claim 1.

Suppose that  $d_{F_3}(y_2) + d_{F_3}(y_4) = 7$ . In this case,  $d_{F_3}(y_1) + d_{F_3}(y_3) \geq 3$ , and therefore  $d_{F_3}(y_1) \geq 2$ . Hence  $y_2$  and  $y_1$  have a common neighbor, say,  $z_1$  in  $F_3$ . In view of Claim 1, since  $d_{F_3}(y_2) + d_{F_3}(y_4) = 7$ , we can assume that  $d_{F_3}(y_2) = 4$  and  $z_2z_4 \notin E(F_3)$ . Then  $z_1z_3 \in E(F_3)$  and  $N_{F_3}(y_1) = \{z_1, z_3\}$ . Furthermore, by symmetry, we may assume that  $y_3z_1 \in E(F)$  and that  $y_4z_2 \in E(F)$ . In this case, we replace  $F_2$  and  $F_3$  by  $F_2 - y_2 + z_1$  and  $F_3 - z_1 + y_2$ .

Finally, suppose that  $d_{F_3}(y_2) + d_{F_3}(y_4) = 6$ . Then  $d_{F_3}(y_1) + d_{F_3}(y_3) \geq 5$ . By Lemma 2 if  $F(V_2 \cup V_3)$  does not contain two vertex-disjoint copies of  $K_4^-$ , then  $F_3 \neq K_4$ . By Lemma 3, if  $F(V_2 \cup V_3)$  does not contain two vertex-disjoint copies of  $K_4^-$  or a copy of  $K_4$  and a copy of  $C_4$ , then there is a vertex, say  $z_4$ , in  $V_3$  and some  $y_i \in V_2$  such that each  $y \in V_2 - y_i$  is adjacent to each  $z \in V_3 - z_4$  and  $y_i$  has at most one neighbor in  $V_3$ . Since  $d_{F_3}(y_2) + d_{F_3}(y_4) = 6$ ,  $i = 3$ . Furthermore, in this case  $z_4z_2 \in E(F)$ . Then we have the following 3 copies of  $K_4^-$ :  $F[x_2, x_3, x_4, y_3]$ ,  $F[x_1, y_1, y_2, z_1]$ , and  $F[y_4, z_2, z_3, z_4]$ .  $\square$

The next lemma is similar.

**Lemma 5.** *Let  $V_1, V_2$ , and  $V_3$  be disjoint vertex subsets of a graph  $F$  such that*

- (a)  $V_1 = \{x_1, x_2, x_3\}$ , and  $F_1 = F(V_1) = K_3$ ;
- (b)  $F_2 = F(V_2)$  is the 4-cycle  $y_1y_2y_3y_4$ ;
- (c)  $F_3 = F(V_3) \in \{K_1, K_2, C_3, K_4, K_4^-\}$ ;
- (d)  $N_{F_2}(x_1) = N_{F_2}(x_2) = N_{F_2}(x_3) = \{y_1, y_2, y_3\}$ .

If

$$d_{F_3}(y_1) + d_{F_3}(y_3) + 2(d_{F_3}(y_2) + d_{F_3}(y_4)) > 4|V_3|, \tag{6}$$

then  $V_1 \cup V_2 \cup V_3$  can be partitioned into sets  $V'_1, V'_2$  and  $V'_3$  so that  $F(V'_1)$  is  $K_3$ ,  $F(V'_2)$  contains  $K_4^-$ , and  $F(V'_3)$  contains  $F_3$ .

The proof of this lemma mimics that of Lemma 4 but is much simpler, so we omit it.

#### 4. Packing 3- and 4-cycles

In this section, we will prove Theorem 4. Let  $\mathcal{H}_n$  be the class of  $n$ -vertex graphs whose every component is either  $K_1$ , or  $K_2$ , or  $K_3$ , or  $K_4^-$ . Let  $\mathcal{H}'_n$  consist of graphs  $H$  in  $\mathcal{H}_n$  such that at most one component of  $H$  is  $K_2$ .

It is enough to prove the theorem for graphs in  $\mathcal{H}'_n$ , since each graph  $H \in \mathcal{H}_n$  is contained in a graph  $H' \in \mathcal{H}'_n$  (we can replace two copies of  $K_2$  in  $H$  by a copy of  $K_4^-$ ). Let  $G$  satisfy the conditions of the theorem. Suppose, for a contradiction, that  $G$  does not contain some graph in  $\mathcal{H}'_n$ . Among such 'bad' graphs in  $\mathcal{H}'_n$  choose a graph  $H_0$  with fewest components that are  $K_4^-$ 's. Suppose that  $H_0$  has no  $K_4^-$ -components. The following corollary of Theorem 2 handles this case.

**Proposition 1.** *Let  $\mathcal{H}''_n$  be the class of  $n$ -vertex graphs whose every component is either  $K_1$ , or  $K_2$ , or  $K_3$ , and at most one of these components is  $K_2$ . Then every  $n$ -vertex graph  $G$  with  $\sigma_2(G) \geq 4n/3 - 1$  contains each graph in  $\mathcal{H}''_n$ .*

**Proof.** If  $n = 3k$ , then the statement directly follows from Theorem 2.

If  $n = 3k + 1$ , then  $\sigma_2(G) \geq \lceil 4(3k + 1)/3 - 1 \rceil = 4k + 1$  and hence for any vertex  $v \in V(G)$ , graph  $G - v$  satisfies the conditions of Theorem 2. Hence  $G - v$  contains each graph in  $\mathcal{H}''_{n-1}$ . On the other hand, if  $n = 3k + 1$ , then at least one component of any graph  $H \in \mathcal{H}''_n$  is  $K_1$ . This settles the case  $n = 3k + 1$ .

If  $n = 3k - 1$ , then  $\sigma_2(G) \geq \lceil 4(3k - 1)/3 - 1 \rceil = 4k - 2$ . Adding to  $G$  a new vertex  $z$  adjacent to each other vertex, we get a graph  $G^*$  satisfying Theorem 2. Hence  $G^*$  contains  $k$  disjoint triangles. It follows that  $G$  contains the graph  $H^*_n$  that has one  $K_2$ -component and  $k - 1$   $K_3$ -components. But such an  $H^*_n$  contains each graph in  $\mathcal{H}''_n$ .  $\square$

Assume now that  $H_0$  contains some  $K_4^-$ s.

**Proposition 2.** Let  $H_0''$  be obtained from  $H_0$  by replacing one component  $K_4^-$  with  $C_4$ . Then  $G$  contains  $H_0''$ .

**Proof.** Suppose not. Let  $H_0'$  be the graph obtained from  $H_0$  by replacing one component  $K_4^-$  with the graph  $C_3 \cup K_1$ . By the choice of  $H_0$ ,  $G$  contains  $H_0'$ . Among all copies of  $H_0'$  contained in  $G$  choose a copy  $H$  with most components  $K_4^-$  embedded into  $K_4$ -subgraphs of  $G$ .

Choose in  $H$  a  $K_3$ -component with vertex set  $W = \{w_1, w_2, w_3\}$  and a  $K_1$ -component  $v$ . By the choice of  $H$ ,  $K_4^- \not\subseteq G[w_1, w_2, w_3, v]$ . Then  $v$  has at most one neighbor in  $W$ .

For every  $U \subseteq V(G)$ , define  $D(U) = 3d_U(v) + d_U(w_1) + d_U(w_2) + d_U(w_3)$ .

CASE 1.  $D(V(G)) \geq 3\sigma_2(G)$ . In this case,

$$D(V(G)) \geq 3\sigma_2(G) - 7 - 3 \geq 4n - 3 - 10 > 4(n - 4),$$

and hence there is a component of  $H$  with vertex set  $U \subset V(G)$  such that

$$D(U) > 4|U|. \tag{7}$$

If  $U = \{u\}$ , then  $u$  has at least two neighbors in  $W$  and thus  $G[W + u]$  contains  $K_4^-$ . But then  $G$  contains  $H_0$ , a contradiction.

Suppose that  $U = \{u_1, u_2\}$  and  $G[U] = K_2$ . By (7),  $v$  has a neighbor, say,  $u_1$  in  $U$ . If  $u_2$  has at least two neighbors in  $W$ , then  $G[W + u_2] \supseteq K_4^-$  and  $G[v, u_1] = K_2$ , a contradiction to the choice of  $G$ . Otherwise, again by (7),  $vu_2 \in E(G)$ . Then similarly,  $u_1$  also has at most one neighbor in  $W$ , a contradiction to (7).

If  $G[U]$  is a triangle, then  $e(W, U) \leq 9$ , and hence there are at least two edges between  $v$  and  $U$ . Thus  $G[U + v]$  contains  $K_4^-$  and  $G[W]$  contains a 3-cycle. Again  $G$  contains  $H_0$ , a contradiction.

Now suppose that  $U = \{u_1, u_2, u_3, u_4\}$  and  $G[U] \supseteq K_4^-$  with possible non-edge  $u_1u_3$ . By (7),  $3d_U(v) + e(W, U) \geq 17$ . Since  $e(W, U) \leq 12$ ,  $d_U(v) \geq 2$ . If  $d_U(v) = 2$ , then  $u_1$  or  $u_3$  (we may assume  $u_1$ ) has three neighbors in  $W$ , thus  $G[W + u_1] = K_4$  and then by the choice of  $H$ ,  $G[U]$  is also  $K_4$ . Note that if some non-neighbor of  $v$  in  $U$  has at least two neighbors in  $W$ , then we again can embed  $H_0$  into  $G$ . Thus  $16 < D(U) \leq 3d_U(v) + 4 - d_U(v) + 3d_U(v) = 4 + 5d_U(v)$ , and therefore  $d_U(v) \geq 3$ , a contradiction. So  $d_U(v) \geq 3$ . If  $u_2, u_4 \in N(v)$ , then  $G[U] = K_4$ , since  $G[N_U[v]]$  contains  $K_4$ . But then as above each vertex in  $U$  has at most one neighbor in  $W$ , and we have  $16 < D(U) \leq 3d_U(v) + 4 \leq 16$ , a contradiction. So we assume that  $u_1, u_2, u_3 \in N(v)$  and  $u_4 \notin N(v)$ . Then again, every vertex in  $U$  has at most one neighbor in  $W$ , and we have a contradiction.

CASE 2.  $D(V(G)) < 3\sigma_2(G)$ . Then  $v$  has exactly one neighbor in  $W$ , say  $w_1$ . By the definition of  $\sigma_2$ ,  $d(v) + d(w_i) \geq \sigma_2(G)$  for  $i = 2, 3$ , and hence  $2d_{G-W-v}(v) + d_{G-W-v}(w_2) + d_{G-W-v}(w_3) \geq 2\sigma_2(G) - 6 > \frac{8}{3}(n - 4)$ . Therefore there is a component of  $H$  with vertex set  $U \subset V(G)$  such that

$$2d_U(v) + d_U(w_2) + d_U(w_3) > \frac{8}{3}|U|. \tag{8}$$

On the other hand, since  $D(V(G)) < 3\sigma_2(G)$ , we have  $d(v) + d(w_1) < \sigma_2(G)$  and hence

$$d(w_1) < \min\{d(w_2), d(w_3)\}. \tag{9}$$

If  $U = \{u\}$ , then  $u$  is adjacent to  $v$  and to at least one of  $w_2$  and  $w_3$ . Thus we have a 4-cycle, a contradiction.

If  $U = \{u_1, u_2\}$  and  $G[U] = K_2$ , then by (8),  $2d_U(v) + d_U(w_2) + d_U(w_3) \geq 6$ . Similarly to Case 1, if  $u_i$  is adjacent to both  $w_2$  and  $w_3$ , and  $u_{3-i}v \in E(G)$ , then we have disjoint  $K_4^-$  and  $K_2$ , a contradiction. Hence,  $d_{W-w_1}(u_i) + 2d_{\{v\}}(u_{3-i}) \leq 3$  for  $i = 1, 2$ . It is possible only if  $vu_1, vu_2 \in E(G)$  and each  $u_i$  has exactly one neighbor in  $\{w_2, w_3\}$ . If this is the same neighbor, say  $w_2$ , then we have  $G[v, u_1, u_2, w_2] = K_4^-$  and  $G[w_3, w_1] = K_2$ . If these neighbors are distinct, say  $u_1w_2, u_2w_3 \in E(G)$ , then  $G[v, u_1, w_2, w_1] = C_4$  and  $G[w_3, u_2] = K_2$ . Both outcomes contradict the choice of  $G$ .

If  $G[U] = K_3$ , then  $2d_U(v) \geq 9 - 6 = 3$  and so  $v$  is adjacent to at least two of the vertices in  $U$ . Hence  $G[U + v]$  contains  $K_4^-$  and  $G[W]$  contains a 3-cycle, a contradiction.

If  $G[U] = K_4$ , then by (8),  $2d_U(v) \geq 11 - 8 = 3$  and so  $d_U(v) \geq 2$ . If there is  $u \in V(U)$  such that  $d_W(u) \geq 2$  and  $d_{U-u}(v) \geq 2$ , then we partition  $G[W \cup U + v]$  into two  $K_4^-$ :  $G[W + u]$  and  $G[U - u + v]$ , a contradiction to the choice of  $G$ . Thus the only possibility to satisfy (8) is that  $d_U(v) = 4$  and each vertex in  $U$  has at most one neighbor in  $W$ . Since by (8),  $d_U(w_2) + d_U(w_3) \geq 11 - 8 = 3$ , we may assume that for some  $u \in U$ ,  $uw_3 \in E(G)$  and therefore  $uw_1, uw_2 \notin E(G)$ . By (9),  $d(u) + d(w_3) \geq d(u) + d(w_1) \geq \sigma_2(G)$ , and hence  $3d(u) + d(w_1) + d(w_2) + d(w_3) \geq 3\sigma_2$ . Since  $G[U - u_1 + v] = K_4$ , we come to Case 1, which is resolved.

Finally suppose that  $U = \{u_1, u_2, u_3, u_4\}$  and  $G[U] = K_4^-$  with non-edge  $u_2u_4$ . As in the previous paragraph,  $d_U(v) \geq 2$ . Suppose first that  $N(v) \cap U = \{u_i, u_{i+1}\}$  for some  $i$ , say, for  $i = 1$ . Then by (8),  $d_U(w_2) + d_U(w_3) \geq 7$ . In particular,  $d_{U-u_2}(w_j) \geq 2$  for  $j = 2, 3$  and we may assume that  $w_3u_2 \in E(G)$ . Then  $G[U - u_2 + w_2]$  contains  $K_4^-$  and  $G[W - w_2 + v + u_2]$  contains  $C_4$ . If  $N(v) \cap U \neq \{u_i, u_{i+1}\}$  for some  $i$ , then  $N(v) \cap U \supseteq \{u_i, u_{i+2}\}$  for some  $i \in \{1, 2\}$ . If for some  $j \neq i, i + 2$ , vertex  $u_j$  has at least two neighbors in  $W$ , then  $G[W + u_j]$  contains  $K_4^-$  and  $G[U - u_j + v]$  contains  $C_4$ . Hence for each  $j \neq i, i + 2$ , vertex  $u_j$  has at most one neighbor in  $W$ . In particular, by (8),  $d_U(v) \geq 3$ . If  $N_U(v)$  contains a triangle, say  $U - u_4$ , then  $G[U + v]$  contains  $K_4 \cup K_1$ , a contradiction to the choice of  $H$ . Otherwise, we may assume that  $N_U(v) = \{u_2, u_3, u_4\}$ . In this case, each  $u \in U$  has at most one neighbor in  $W$ , which together with (8) yields  $d_U(v) \geq 4$ . A contradiction to our last assumption finishes the proof of Proposition 2.  $\square$

Fix an embedding of  $H_0''$  into  $G$  provided by Proposition 2. Suppose that the  $C_4$ -component of  $H_0''$  is embedded into 4-cycle  $(w_1, w_2, w_3, w_4)$  in  $G$ . Let  $W = \{w_1, w_2, w_3, w_4\}$ . By the choice of  $G$ ,  $w_1w_3, w_2w_4 \notin E(G)$ . Since  $(d(w_1) + d(w_3)) + (d(w_2) + d(w_4)) \geq 2\sigma_2(G)$ , we have  $\sum_{i=1}^4 d_{G-W}(w_i) \geq 2\sigma_2(G) - 8 \geq \frac{8n}{3} - 10 > \frac{8}{3}(n - 4)$ . Hence there exists a component of  $H_0''$  mapped to a set  $U \subset V(G)$  with

$$e(W, U) > \frac{8}{3}|U|. \tag{10}$$

CASE 1.  $U = \{v\}$ . Since  $e(v, W) \geq 3$ ,  $N(v) + v$  contains  $K_4^-$  and  $G[W - N(v)]$  is  $K_1$ . This contradicts the choice of  $G$ .

CASE 2.  $U = \{u_1, u_2\}$  and  $G[U] = K_2$ . Since  $e(W, U) \geq \lceil 16/3 \rceil = 6$ , we may assume that  $e(\{w_1, w_2\}, U) \geq 3$ . Then  $G[w_1, w_2, u_1, u_2]$  contains  $K_4^-$  and  $G[w_3, w_4] = K_2$ , a contradiction to the choice of  $G$ .

CASE 3.  $U = \{u_1, u_2, u_3\}$  and  $G[U] = K_3$ . Then  $e(W, U) \geq 9$ . Suppose that we cannot decompose  $G[U \cup W]$  into  $K_4^-$  and  $K_3$ . Then by Lemma 1, there is a vertex, say  $w_4$ , in  $W$  such that  $N_W(u_i) = W - w_4$  for  $i = 1, 2, 3$ . Since  $d_{G-W}(w_1) + d_{G-W}(w_3) + 2(d_{G-W}(w_2) + d_{G-W}(w_4)) \geq 3\sigma_2(G) - 8 > 4(n - 4)$ , there exists a component of  $H_0''$  mapped to a set  $U' \subset V(G)$  with  $e(U', W) = d_{U'}(w_1) + d_{U'}(w_3) + 2(d_{U'}(w_2) + d_{U'}(w_4)) > 4|U'|$ . Since  $U$  does not satisfy this condition,  $U' \neq U$ . Applying Lemma 5 with  $F_1 = U, F_2 = U_0$  and  $F_3 = U'$ , we again get a contradiction to the choice of  $G$ .

CASE 4.  $U = \{u_1, u_2, u_3, u_4\}$  and  $G[U] = K_4^-$  with  $u_1u_4 \notin E(G)$ . Suppose that we cannot partition  $G[U \cup W]$  into two  $K_4^-$  or into  $K_4$  and  $C_4$ . Since  $e(W, U) \geq 11$ , by Lemma 3, we may assume that each of  $w_1, w_2$ , and  $w_3$  is adjacent to each of  $u_1, u_2$ , and  $u_4$  and that  $w_4$  and  $w_2$  have no common neighbors in  $U$ . Since  $d(w_1) + d(w_3) + 2(d(w_2) + d(w_4)) \geq 3\sigma_2(G)$ , there exists a component of  $H_0''$  mapped to some  $U' \subset V(G)$  with  $d_{U'}(w_1) + d_{U'}(w_3) + 2(d_{U'}(w_2) + d_{U'}(w_4)) > 4|U'|$ . Note that  $U' \neq U$ , since  $d_U(w_2) + d_U(w_4) \leq |U|$ . Then  $G[U \cup W \cup U']$  satisfies the conditions of Lemma 4, which proves this case.

CASE 5.  $U = \{u_1, u_2, u_3, u_4\}$  and  $G[U] = K_4$ . If  $G[W \cup U]$  contains two disjoint copies of  $K_4^-$ , then by Lemma 2, there are  $u_4 \in U$  and  $w_4 \in W$  such that (i)  $N_{W-w_4}(u_1) = N_{W-w_4}(u_2) = N_{W-w_4}(u_3) = W - w_4$ , and (ii)  $w_4$  has at most one neighbor in  $U$  and  $|E_G(W, U)| = 11$ . Since  $d(w_1) + d(w_3) + 2(d(w_2) + d(w_4)) \geq 3\sigma_2(G)$ , there exists a component of  $H_0''$  mapped to some  $U' \subset V(G)$  with  $d_{U'}(w_1) + d_{U'}(w_3) + 2(d_{U'}(w_2) + d_{U'}(w_4)) > 4|U'|$ . By (ii),  $U' \neq U$ . Then  $G[U \cup W \cup U']$  satisfies the conditions of Lemma 4, which finishes the proof.

### 5. Two reductions

In this section, we prove two lemmas that will help us later to find a special subgraph  $H$  in a graph satisfying (1).

Let the *microphone graph*  $M_1$  be the 5-vertex graph such that a 4-vertex subgraph of  $M_1$  is  $K_4$  and the fifth vertex has exactly one neighbor in  $M_1$ .

**Lemma 6.** *Let  $H$  be an  $n$ -vertex graph whose components are isomorphic to graphs in  $\mathcal{H} = \{K_1, K_2, C_3, K_4^-, C_5^+\}$ . Let  $H_1$  be the graph obtained from  $H$  by replacing a copy of  $C_5^+$  with a copy of the microphone graph. If an  $n$ -vertex graph  $G$  satisfying (1) contains  $H_1$ , then it contains  $H$ , as well.*

**Proof.** Suppose not. Fix an embedding of  $H_1$  into  $G$ . Suppose that the component  $M_1$  of  $H_1$  is embedded into the subset  $A = \{a_1, a_2, a_3, a_4, a_5\}$  of  $V(G)$  so that  $G[A - a_5] = K_4$ . Since  $G$  does not contain  $H$ , we may assume that the only neighbor of  $a_5$  in  $A$  is  $a_4$ . For every  $W \subseteq V(G)$ , consider the expression  $D(W) = 3d_W(a_5) + d_W(a_1) + d_W(a_2) + d_W(a_3)$ . Since  $D(V(G)) \geq 3\sigma_2$ , we have  $D(V(G) - A) \geq 3\sigma_2 - 2|E(G[A])| \geq (4n - 3) - 14 > 4(n - 5)$ , and hence there exists a component of  $H_1$  mapped to a set  $U \subset V(G)$  with  $D(U) > 4|U|$ . Let  $A_1 = \{a_1, a_2, a_3\}$ .

CASE 1:  $U = \{u\}$ . By the choice of  $U$ ,  $D(U) \geq 5$ . Then  $u$  is adjacent to  $a_5$  and at least two vertices in  $A_1$ . Hence  $G[A - a_5 + u]$  contains  $C_5^+$ .

CASE 2:  $U = \{u_1, u_2\}$  and  $G[U] = K_2$ . By the choice of  $U$ ,  $D(U) \geq 9$ . Then some vertex of  $U$ , say  $u_1$ , has at least two neighbors in  $A_1$ . If  $a_5u_2 \in E(G)$ , then  $G[A - a_5 + u_1]$  contains  $C_5^+$  and  $G[u_2, a_5] = K_2$ . If  $a_5u_2 \notin E(G)$ , then the only way to have  $D(U) \geq 9$  is that  $u_1$  is adjacent to all vertices in  $A - a_4$  and  $u_2$  is adjacent to all vertices in  $A_1$ . In this case, after switching the roles of  $u_2$  and  $u_1$ , the previous argument works.

Observe that in order to have  $D(U) > 4|U|$  for any  $U$  with  $|U| \geq 3$ , we need

$$d_U(a_5) \geq 2. \tag{11}$$

CASE 3:  $U = \{u_1, u_2, u_3\}$  and  $G[U] = K_3$ . By the choice of  $U$ ,  $D(U) \geq 13$ . Some vertex in  $U$ , say  $u_1$ , has at least two neighbors in  $A_1$ . If  $a_5u_2, a_5u_3 \in E(G)$ , then  $G[A - a_5 + u_1]$  contains  $C_5^+$  and  $G[a_5, u_2, u_3] = K_3$ . Hence we may assume that  $N(a_5) \cap U = \{u_1, u_2\}$ . Then by the above  $d_{A_1}(u_3) \leq 1$ . So, to have  $D(U) \geq 13$ , we need  $N(u_2) \cap A_1 = N(u_1) \cap A_1 = A_1$  and  $d_{A_1}(u_3) = 1$ . Let  $a \in A_1$  be the neighbor of  $u_3$  in  $A_1$ . Then  $G[a_4, a_5, u_1, u_3, a]$  contains  $C_5^+$  and  $G[A_1 - a + u_1]$  is a triangle, a contradiction.

CASE 4:  $U = \{u_1, u_2, u_3, u_4\}$  and  $G[U] \supset K_4^-$  with  $u_1u_3$  as the only possible non-edge. By the choice of  $U$ ,  $D(U) \geq 17$ . If  $G[U + a_5]$  contains  $C_5^+$ , we are done. By (11),  $G[C + a_5]$  does not contain  $C_5^+$  only if  $N(a_5) \cap U = \{u_2, u_4\}$  and  $u_1u_3 \notin E(G)$ . Therefore, there are at least 11 edges between  $A_1$  and  $U$ , and we can find  $a \in A_1$  and  $u \in \{u_1, u_3\}$  such that  $U - u \subseteq N(a)$  and  $A_1 - a \in N(u)$ . Then  $G[U - u + a + a_5]$  contains  $C_5^+$  and  $G[A_1 - a_5 - a + u]$  contains  $K_4^-$ .

CASE 5:  $U = \{u_1, u_2, u_3, u_4, u_5\}$  and  $G[U]$  contains cycle  $(u_1, u_2, u_3, u_4, u_5)$  and edge  $u_2u_5$ . By the choice of  $U$ ,  $D(U) \geq 21$ . We may assume that  $a_1$  has the most neighbors in  $U$  among vertices in  $A_1$ . Since  $D(U) \geq 21$ ,

$$d_U(a_5) + d_U(a_1) \geq 7. \tag{12}$$

Subcase 5.1:  $d_U(a_1) = 5$ . If some neighbor  $u$  of  $a_5$  in  $U$  is adjacent to  $a_2$  or  $a_3$ , then either of  $G[A - a_1 + u]$  and  $G[U - u + a_1]$  contains  $C_5^+$ , a contradiction. Otherwise,  $e(A_1, U) \leq 15 - 2d_U(a_5)$  and hence  $D(U) \leq 15 + d_U(a_5) \leq 20$ , a contradiction again.

Subcase 5.2:  $d_U(a_5) = 3$ . Since  $e(A_1, U) \leq 21 - 9 = 12$  and  $d_U(a_1) \leq 4$ , we have  $d_U(a) = 4$ , for each  $a \in A_1$ . Then  $a_5$  has at least two common neighbors in  $U$  with  $a_1$ , say,  $u$  and  $u'$ . If  $G[U - u + a_2] \supseteq C_5^+$  or  $G[U - u' + a_2] \supseteq C_5^+$ , then we are done, since in this case either of  $G[A - a_2 + u]$  and  $G[A - a_2 + u']$  also contains  $C_5^+$ . Otherwise,  $u$  and  $u'$  are the two vertices next on the cycle  $(u_1, u_2, u_3, u_4, u_5)$  to the non-neighbor of  $a_2$  in  $U$ , and the third neighbor of  $a_5$  in  $U$  is the non-neighbor of  $a_1$ . By the symmetry between  $a_1, a_2$  and  $a_3$ , we conclude that  $u$  and  $u'$  are adjacent to all vertices in  $A_1$ , and the third neighbor of  $a_5$  is adjacent to none of them. Let the non-neighbors of  $a_5$  in  $U$  be  $u_i$  and  $u_{i+1}$ . Then either of  $G[U - a_3 + u_i]$  and  $G[U - u_i + a_5 + a_3]$  contains  $C_5^+$ .

Subcase 5.3:  $d_U(a_5) = 4$ . Let  $u_i$  be the non-neighbor of  $a_5$  in  $U$ . If some  $u \in U - u_{i-1} - u_{i+1}$  has at least two neighbors in  $A_1$ , then either of  $G[A - a_5 + u]$  and  $G[U - u + a_5]$  contains  $C_5^+$ . Otherwise, to have  $D(U) \geq 21$ , we need  $d_{A_1}(u_{i-1}) = d_{A_1}(u_{i+1}) = 3$  and  $d_{A_1}(u_i) = d_{A_1}(u_{i-2}) = d_{A_1}(u_{i+2}) = 1$ . Since  $d_U(a_1) \leq 4$ , no vertex in  $A_1$  is a common neighbor of  $u_{i-2}, u_i$ , and  $u_{i+2}$ . By the symmetry between  $u_{i-2}$  and  $u_{i+2}$ , we may assume that for some distinct  $a, a' \in A_1, u_{i-2}a, u_i a' \in E(G)$ . Let  $a''$  be the third vertex in  $A_1$ . Then either of  $G[A - a - a' + u_{i-2} + u_{i+2}]$  and  $G[U + a + a' - u_{i-2} - u_{i+2}]$  contains  $C_5^+$ .

Subcase 5.4:  $d_U(a_5) = 5$ . Since  $D(U) \geq 21$ , some  $u \in U$  has at least 2 neighbors in  $A_1$ . Then either of  $G[A - a_5 + u]$  and  $G[U - u + a_5]$  contains  $C_5^+$ .  $\square$

The  $T$ -graph is the 5-vertex graph obtained from  $K_{2,3}$  by adding an edge connecting the two vertices of degree 3. Equivalently, the  $T$ -graph is the 5-vertex graph obtained from  $K_5$  by deleting the edges of a triangle. Sometimes, the  $T$ -graph is also called the *book with 3 pages*.

**Lemma 7.** Let  $H$  be an  $n$ -vertex graph whose components are isomorphic to graphs in  $\mathcal{H} = \{K_1, K_2, C_3, K_4^-, C_5^+\}$ . Let  $H_2$  be the graph obtained from  $H$  by replacing a copy of  $C_5^+$  with a copy of the  $T$ -graph. If an  $n$ -vertex graph  $G$  satisfying (1) contains  $H_2$ , then it contains  $H$ , as well.

**Proof.** Suppose not. Fix an embedding of  $H_2$  into  $G$ . Suppose that the  $T$ -graph component of  $H_2$  is mapped to a subset  $A = \{a_1, a_2, a_3, a_4, a_5\}$  of  $V(G)$  so that  $d_A(a_4) = d_A(a_5) = 4$ . Since  $G$  does not contain  $H$ , the set  $A_1 = \{a_1, a_2, a_3\}$  is independent in  $G$ . For every  $W \subseteq V(G)$ , consider the expression  $D(W) = d_W(a_1) + d_W(a_2) + d_W(a_3)$ . Since  $D(V(G)) \geq \frac{3}{2}\sigma_2$ , we have  $D(V(G) - A) \geq \frac{3}{2}\sigma_2 - 6 \geq (4n - 3)/2 - 6 > 2(n - 5)$ , and hence there exists a component of  $H_2$  mapped to a set  $U \subset V(G)$  with  $D(U) > 2|U|$ . By symmetry, we may assume that

$$d_U(a_1) \geq d_U(a_2) \geq d_U(a_3). \tag{13}$$

CASE 1:  $U = \{u\}$ . By the choice of  $U$ ,  $D(U) \geq 3$ . In particular,  $a_1u, a_2u \in E(G)$ . Then  $G[A - a_3 + u]$  contains  $C_5^+$ .

CASE 2:  $U = \{u_1, u_2\}$  and  $G[U] = K_2$ . By the choice of  $U$ ,  $D(U) \geq 5$ . So, we may assume that among the edges connecting  $A_1$  with  $U$  only  $a_3u_2$  is missing. Then  $G[a_3, u_1] = K_2$  and  $G[A + u_2 - a_3]$  contains  $C_5^+$ .

CASE 3:  $U = \{u_1, u_2, u_3\}$  and  $G[U] = K_3$ . By the choice of  $U$ ,  $D(U) \geq 7$ . So, by (13),  $d_U(a_1) = 3$  and  $d_U(a_2) \geq 2$ . Then  $G[U \cup \{a_1, a_2\}]$  contains  $C_5^+$  and  $G[A - a_1 - a_2] = K_3$ .

CASE 4:  $U = \{u_1, u_2, u_3, u_4\}$  and  $G[U] \supset K_4^-$ . By the choice of  $U$ ,  $D(U) \geq 9$ . Hence by (13),  $d_U(a_1) \geq D(U)/3 \geq 3$ . Then  $G[U + a_1]$  contains  $C_5^+$  and  $G[A - a_1] = K_4^-$ .

CASE 5:  $U = \{u_1, u_2, u_3, u_4, u_5\}$  and  $G[U]$  contains cycle  $(u_1, u_2, u_3, u_4, u_5)$  and edge  $u_2u_5$ . By the choice of  $U$ ,  $D(U) \geq 11$ . By (13),  $d_U(a_1) \geq 4$ .

Subcase 5.1:  $d_U(a_1) = 5$ . Since  $d_U(a_2) + d_U(a_3) \geq 6$ , some  $u \in U$  is adjacent to both of them. Then either of  $G[A - a_1 + u]$  and  $G[U - u + a_1]$  contains  $C_5^+$ .

Subcase 5.2:  $d_U(a_1) = 4$ . Then  $d_U(a_2) = 4$  and  $d_U(a_3) \geq 3$  also. Let  $u_i$  be the only non-neighbor of  $a_1$  in  $U$ . If any  $u \in U - u_{i-1} - u_{i+1}$  is adjacent to both  $a_2$  and  $a_3$ , then again either of  $G[A - a_1 + u]$  and  $G[U - u + a_1]$  contains  $C_5^+$ . Otherwise, the only possibility to have  $D(U) \geq 11$  is that  $d_{A_1}(u_{i-1}) = d_{A_1}(u_{i+1}) = 3, d_{A_1}(u_{i-2}) = d_{A_1}(u_{i+2}) = 2$ , and  $d_{A_1}(u_i) = 1$ .

Let  $a_j = a_1$  when  $d_{A_1}(u_1) = 3$ , and let  $a_j$  be the only non-neighbor of  $u_1$  in  $A_1$  when  $d_{A_1}(u_1) = 2$ . So, if  $d_{A_1}(u_1) \geq 2$ , then  $G[A - a_j + u_1]$  contains a  $C_5^+$ . Furthermore, since  $d_{U-u_1}(a_j) \geq 3$  and  $u_2u_5 \in E(G)$ ,  $G[U + a_j - u_1]$  also contains  $C_5^+$ . Thus, the only remaining possibility is that  $i = 1$ . By the symmetry between  $a_1$  and  $a_2$ , the only non-neighbor of  $a_2$  in  $U$  is also  $u_1$ . Hence  $N_U(a_3) = \{u_5, u_1, u_2\}$ . Then  $G[A - a_3 + u_4]$  contains  $C_5^+$  and  $G[U - u_4 + a_3]$  is the microphone graph. This means that  $G$  contains the graph  $H_1$  obtained from  $H$  by replacing a copy of  $C_5^+$  by a copy of the microphone graph. Hence by Lemma 6,  $G$  contains  $H$ , a contradiction.  $\square$



## 6. Proof of Theorem 3

Similarly to Section 4, let  $\mathcal{H}_n$  be the class of  $n$ -vertex graphs whose every component is either  $K_1$ , or  $K_2$ , or  $K_3$ , or  $K_4^-$ , or  $C_5^+$ . Let  $G$  satisfy the conditions of the theorem. Suppose, for a contradiction, that  $G$  does not contain some graph in  $\mathcal{H}_n$ . Among such ‘bad’ graphs in  $\mathcal{H}_n$  choose a graph  $H_0$  with fewest components that are  $C_5^+$ . By Theorem 4,  $H_0$  has a  $C_5^+$ -component. Let  $H'_0$  be obtained from  $H_0$  by replacing a  $C_5^+$ -component with  $K_4^-$  and an isolated vertex. By the minimality of  $H_0$ , there exists an embedding of  $H'_0$  in  $G$ . Among embeddings of  $H'_0$  in  $G$ , choose and fix one such that

(\*) it has the largest total number of edges in subgraphs of  $G$  induced by the components of  $H'_0$ .

Suppose that the isolated vertex of  $H'_0$  is mapped to a vertex  $v \in V(G)$  and a  $K_4^-$ -component of  $H'_0$  is mapped to a set  $W = \{w_1, w_2, w_3, w_4\} \subset V(G)$ , where only  $w_1w_3$  can be a non-edge of  $G[W]$ . Since  $G[W + v]$  does not contain  $C_5^+$ , only the three cases below are possible up to symmetry.

CASE 1.  $N(v) \cap W \subseteq \{w_1\}$ . For every  $Y \subseteq V(G)$ , consider the expression  $D(Y) = 3d_Y(v) + d_Y(w_2) + d_Y(w_3) + d_Y(w_4)$ . Since  $D(V(G)) \geq 3\sigma_2$ , we have  $D(V(G) - W - v) \geq 3\sigma_2 - (3 + 3 + 3 + 3) \geq (4n - 3) - 12 > 4(n - 5)$ , and hence there exists a component of  $H'_0$  mapped to a set  $U \subset V(G)$  with  $D(U) > 4|U|$ . Denote  $W_1 = W - w_1$ .

Case 1.1:  $U = \{u\}$ . By the choice of  $U$ ,  $D(U) \geq 5$ . Then  $u$  is adjacent to  $v$  and to at least two vertices in  $W_1$ . Hence  $G[W + u]$  contains either  $C_5^+$  or the  $T$ -graph. By Lemma 7, this contradicts the choice of  $G$ .

Case 1.2:  $U = \{u_1, u_2\}$  and  $G[U] = K_2$ . By the choice of  $U$ ,  $D(U) \geq 9$ . Then some vertex of  $U$ , say  $u_1$ , has at least two neighbors in  $W_1$ . If  $vu_2 \in E(G)$ , then  $G[u_2, v] = K_2$  and  $G[W + u_1]$  contains either  $C_5^+$  or the  $T$ -graph. If  $vu_2 \notin E(G)$ , then the only way to have  $D(U) \geq 9$ , is that both  $u_1$  and  $u_2$  are adjacent to all vertices in  $W_1$  and  $u_1v \in E(G)$ . But then after switching the roles of  $u_2$  and  $u_1$ , the previous argument works.

Observe that in order to have  $D(U) > 4|U|$  for a  $U$  with  $|U| \geq 3$ , we need

$$d_U(v) \geq 2. \tag{14}$$

Case 1.3:  $U = \{u_1, u_2, u_3\}$  and  $G[U] = K_3$ . By the choice of  $U$ ,  $D(U) \geq 13$ . Some vertex in  $U$ , say  $u_1$ , has at least two neighbors in  $W_1$ . If  $vu_2, vu_3 \in E(G)$ , then  $G[v, u_2, u_3]$  is a triangle and  $G[W + u_1]$  contains either  $C_5^+$  or the  $T$ -graph. Hence we may assume that  $vu_3 \notin E(G)$  and, by (14),  $N(w) \cap U = \{u_1, u_2\}$ . Then by the above argument,  $d_{W_1}(u_3) \leq 1$ . So, to have  $D(U) \geq 13$ , we need  $N(u_2) \cap W_1 = N(u_1) \cap W_1 = W_1$  and  $d_{W_1}(u_3) = 1$ . In this case,  $G[U + v + w_3]$  contains the  $T$ -graph and  $G[W - w_3] = K_3$ .

Case 1.4:  $U = \{u_1, u_2, u_3, u_4\}$  and  $G[U] \supset K_4^-$  with  $u_1u_3$  as the only possible non-edge. By (14),  $G[U + v]$  contains either  $C_5^+$ , or the  $T$ -graph.

Case 1.5:  $U = \{u_1, u_2, u_3, u_4, u_5\}$  and  $G[U]$  contains cycle  $(u_1, u_2, u_3, u_4, u_5)$  and edge  $u_2u_5$ . By the choice of  $U$ ,  $D(U) \geq 21$ .

Subcase 1.5.1:  $d_U(v) = 5$ . Since  $D(U) \geq 21$ , some  $u \in U$  has at least two neighbors in  $W - w_1$ . Then  $G[U - u + v]$  contains  $C_5^+$  and  $G[W + u]$  contains either  $C_5^+$ , or the  $T$ -graph.

Subcase 1.5.2:  $d_U(v) = 4$ . Let  $u_i$  be the only non-neighbor of  $v$  in  $U$ . If  $d_U(u_i) < 4$ , then  $G[U - u_i + v]$  has more edges than  $G[U]$ , a contradiction to (\*). So,  $d_U(u_i) = 4$ . Hence for each  $j$ ,  $G[U - u_j + v]$  contains  $C_5^+$ . If follows that we are done if for some  $j$ ,  $u_j$  has at least two neighbors in  $W$ . Otherwise,  $D(U) \leq 3 \cdot 4 + 5 \cdot 1 = 17$ , a contradiction.

Subcase 1.5.3:  $d_U(v) = 3$ . Then  $e(W_1, U) \geq 21 - 9 = 12$ . If for some  $u \in U$ ,  $G[U - u + v]$  contains  $C_5^+$ , then  $u$  has at most one neighbor in  $W$ , because otherwise  $G[W + u]$  contains either  $C_5^+$  or the  $T$ -graph. Since at most 3 edges connecting  $W_1$  with  $U$  are missing, it yields:

(\*\*) There is at most one  $u \in U$  such that  $G[U - u + v]$  contains  $C_5^+$ .

If the non-neighbors of  $v$  in  $U$  are not consecutive on the cycle  $(u_1, \dots, u_5)$  (in which case they are  $u_{i-1}$  and  $u_{i+1}$  for some  $i \in \{1, 2, 3, 4, 5\}$ ), then both  $G[U - u_{i-1} + v]$  and  $G[U - u_{i+1} + v]$  contain  $C_5^+$ , a contradiction to (\*\*). So we assume below that the neighbors of  $v$  in  $U$  are  $u_{i-1}$ ,  $u_i$ , and  $u_{i+1}$ . Recall that  $u_2u_5 \in E(G)$ . Up to a symmetry, there are three possibilities:  $i = 3$ ,  $i = 2$  and  $i = 1$ . If  $i = 3$ , then either of  $G[U - u_1 + v]$  and  $G[U - u_3 + v]$  contains  $C_5^+$ , a contradiction to (\*\*), again. If  $i = 2$ , then either of  $G[U - u_1 + v]$  and  $G[U - u_4 + v]$  contains  $C_5^+$ . Thus, the last possibility is that  $i = 1$ . In this case,  $u_1$  has at most one neighbor in  $W$  and hence some  $u \in \{u_3, u_4\}$  is adjacent to all vertices in  $W_1$ . Then  $G[W + u]$  contains  $C_5^+$ , and  $G[U - u + v]$  contains the microphone graph. This contradicts Lemma 6.

Subcase 1.5.4:  $d_U(v) = 2$ . Then  $e(W - w_1, U) \geq 21 - 6 = 15$ . It follows that all edges connecting  $U$  with  $W - w_1$  are present in  $G$ . We may assume that the neighbors of  $v$  in  $U$  are  $u_i$  and either  $u_{i+1}$  or  $u_{i+2}$ . Then either of  $G[u_i, u_{i+1}, u_{i+2}, v, w_3]$  and  $G[W - w_3 + u_{i-1} + u_{i-2}]$  contains  $C_5^+$ .

CASE 2.  $N(v) \cap W = \{w_2, w_4\}$ . Then  $v$  and  $W$  form the  $T$ -graph, and we are done by Lemma 7.

CASE 3.  $N(v) \cap W = \{w_4\}$ . For every  $Y \subseteq V(G)$ , let  $D(Y) = 3d_Y(v) + d_Y(w_1) + d_Y(w_2) + d_Y(w_3)$ . Since  $D(V(G)) \geq 3\sigma_2$ , we have  $D(V(G) - W - v) \geq 3\sigma_2 - (3 + 3 + 3 + 3) \geq (4n - 3) - 12 > 4(n - 5)$ , and hence there exists a component of  $H'_0$  mapped to a set  $U \subset V(G)$  with  $D(U) > 4|U|$ . Let  $W_4 = W - w_4$ .

Proofs of the Cases 3.1 (when  $|U| = 1$ ), 3.2 (when  $|U| = 2$ ) and 3.4 (when  $|U| = 4$ ) are exact repetitions of proofs of the Cases 1.1, 1.2, and 1.4, respectively. Also, (14) holds if  $|U| \geq 3$  for the same reasons as in Case 1.

Case 3.3:  $U = \{u_1, u_2, u_3\}$  and  $G[U] = K_3$ . By the choice of  $U$ ,  $D(U) \geq 13$ . If some  $w \in \{w_1, w_3\}$  has at least two neighbors in  $U$ , then  $G[W - w] = K_3$  and, by (14),  $G[U + v + w]$  contains either  $C_5^+$  or the  $T$ -graph. Otherwise,  $e(W_4, U) \leq 5$  and to have  $D(U) \geq 13$ , we need  $d_U(v) = 3$ . In this case, we still have  $e(W_4, U) \geq 4$  and hence some  $w \in \{w_1, w_3\}$  has a neighbor in  $U$ . Then  $G[U + v + w]$  contains the microphone graph and again  $G[W - w] = K_3$ .

Case 3.5:  $U = \{u_1, u_2, u_3, u_4, u_5\}$  and  $G[U]$  contains cycle  $(u_1, u_2, u_3, u_4, u_5)$  and edge  $u_2u_5$ . By the choice of  $U$ ,  $D(U) \geq 21$ . The proofs of the subcases when  $d_U(v)$  equals 5, 4, and 2 word-by-word repeat the proofs of the subcases 1.5.1, 1.5.2, and 1.5.4, respectively. So, we need to handle only the case  $d_U(v) = 3$ .

Since  $D(U) \geq 21$ , we have  $e(W_4, U) \geq 12$ . Then some  $x \in \{w_1, w_3\}$  has at least four neighbors in  $U$ , and some  $x' \in W_4 - x$  (also having at least four neighbors in  $U$ ) has at least two common neighbors (say  $u$  and  $u'$ ) with  $v$  in  $U$ . Since  $vw_4 \in E(G)$ ,  $G[W - x + v + u]$  (and  $G[W - x + v + u']$ ) contains either  $C_5^+$  or the  $T$ -graph. Hence we are done if  $G[U - u + x]$  or  $G[U - u' + x]$  contains  $C_5^+$ . If neither of  $G[U - u + x]$  and  $G[U - u' + x]$  contains  $C_5^+$ , then  $d_U(x) = 4$  and  $d_U(x') = 4$ . Furthermore, if  $u_i$  is the non-neighbor of  $x$  in  $U$ , then the common neighbors of  $v$  and  $x'$  in  $U$  are only  $u_{i-1}$  and  $u_{i+1}$ . It follows that  $N_U(w_j) = U - u_i$  for  $j = 1, 2, 3$  and  $N_U(v) = \{u_{i-1}, u_i, u_{i+1}\}$ . Then either of  $G[u_{i-1}, u_i, u_{i+1}, v, w_1]$  and  $G[W - w_1 + u_{i-2} + u_{i+2}]$  contains  $C_5^+$ .

So, all cases are considered and the theorem is proved.

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