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Hadwiger numbers and over-dominating colourings

Thomas Böhme [a], Alexandr Kostochka [b,c], Andrew Thomason [d,*]

[a] Institut für Mathematik, Technische Universität Ilmenau, Ilmenau, Germany
[b] Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA
[c] Sobolev Institute of Mathematics, Novosibirsk-90, 630090, Russia
[d] DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, UK

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To Carsten Thomassen on his 60th birthday

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ABSTRACT

Motivated by Hadwiger's conjecture, we say that a colouring of a graph is *over-dominating* if every vertex is joined to a vertex of each other colour and if, for each pair of colour classes C_1 and C_2, either C_1 has a vertex adjacent to all vertices in C_2 or C_2 has a vertex adjacent to all vertices in C_1.

We show that a graph that has an over-dominating colouring with k colours has a complete minor of order at least $2k/3$ and that this bound is essentially best possible.

1. Introduction

The *Hadwiger number* $h(G)$ of the graph G is the order of the largest complete minor of G. Hadwiger's conjecture notoriously asserts that $h(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G. It is open even for graphs with independence number two and for uniquely k-colourable graphs.

A *dominating set* in a graph G is a set of vertices $X \subseteq V(G)$ such that every vertex not in X has a neighbour in X. We say that a proper vertex colouring f of a graph G is *dominating* if every colour class is a dominating set: that is, every vertex has a neighbour in each colour class apart from its own. Connected dominating sets are helpful when seeking complete minors, since if X is a connected dominating set then it can be seen, by contracting X to a single vertex, that $h(G) \geq 1 + h(G - X)$. Duchet and Meyniel [2] showed that a graph G of independence number α has a connected dominating set of order at most $2\alpha - 1$, and hence $h(G) \geq |G|/(2\alpha - 1)$, where $|G|$ is the number of vertices of G.

The case of Hadwiger's conjecture when G has independence number two has received particular attention recently. Every colouring of such a graph in which each colour class has size two is dominating. Moreover, if such a k-colouring f is a unique k-colouring of G, then for each pair of colour classes C_1 and C_2 of f, C_1 has a vertex adjacent to both vertices in C_2 and vice versa. This motivated the following definition. A proper vertex colouring f of a graph G is *over-dominating* if

(a) f is dominating: that is, every vertex has a neighbour in each colour class apart from its own, and
(b) for each pair of colour classes C_1 and C_2 of f, either C_1 has a vertex adjacent to all vertices in C_2 or C_2 has a vertex adjacent to all vertices in C_1.

The goal of this note is to understand how large the Hadwiger number of a graph G must be if it has an over-dominating k-colouring. However it is of interest at the same time to see what other implications there are on the graph structure. In particular, the chromatic number can be much less than k, though not arbitrarily small. Clearly $\chi(G) \leq k$ and equality can be attained; the largest possible value of k is given by the next theorem.

* Corresponding author.
E-mail addresses: thomas.boehme@theoinf.tu-ilmenau.de (T. Böhme), kostochk@math.uiuc.edu (A. Kostochka), a.g.thomason@dpmms.cam.ac.uk (A. Thomason).

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Theorem 1. *If $\chi(G) = c$ and G has an over-dominating k-colouring then $k = c$ if $c \leq 2$ and $c \leq k \leq 2^c - c - 1$ otherwise. Moreover this inequality is best possible.*

Proof. Let $g : V(G) \to [c]$ be a proper c-colouring of G with colour set $[c] = \{1, \dots, c\}$, and let f be an over-dominating k-colouring. Associate with each colour class C of f the subset $g(C) \subseteq [c]$ of those colours assigned by g to the vertices in C. By definition, $g(C) \neq \emptyset$. Furthermore, condition (b) implies that for any two colour classes C_1 and C_2, $g(C_1) \neq g(C_2)$ holds, since if say $x \in C_2$ is joined to all of C_1 then $g(x) \notin g(C_1)$. Finally, condition (a) implies that if $|g(C_1)| = 1$, say $g(C_1) = \{i\}$, then $i \notin g(C_2)$, since a vertex in C_2 of colour i could not be joined to anything in C_1.

Let there be t classes C with $|g(C)| = 1$, using say colours $c-t+1, \dots, c$. Then the remaining $k-t$ classes are associated with distinct subsets of $[c-t]$ having size at least two. Thus $k \leq t + 2^{c-t} - (c-t) - 1 = 2^{c-t} + 2t - c - 1$. It can be checked that the maximum of $2^{c-t} + 2t - c - 1$ over $0 \leq t \leq c$ equals c when $c \leq 2$ and equals $2^c - c - 1$ for $c \geq 3$.

The preceding argument establishes the inequality of the theorem and also indicates how equality can be attained when $c \geq 3$. Let G have vertex set $\{(\sigma, i) : \sigma \subseteq [c], |\sigma| \geq 2, i \in \sigma\}$. Join (σ, i) to (τ, j) if $\sigma \neq \tau$ and $i \neq j$. The map $(\sigma, i) \mapsto i$ is a c-colouring of G. Let $f((\sigma, i)) = \sigma$. Then f is a k-colouring where $k = 2^c - c - 1$. Let $C_1 = f^{-1}(\sigma)$ and $C_2 = f^{-1}(\tau)$ be two colour classes of f. If $(\sigma, i) \in C_1$ then, since $|\tau| \geq 2$, there exists $j \in \tau$ with $j \neq i$, so (σ, i) is joined to (τ, j) in C_2. Thus condition (a) is satisfied. Moreover, since $\sigma \neq \tau$, there is an element $\ell \in \sigma \triangle \tau$. Suppose, say, that $\ell \notin \sigma$ and $\ell \in \tau$. Then the vertex $(\tau, \ell) \in C_2$ is joined to every vertex in C_1, which is to say that condition (b) is satisfied. Thus f is an over-dominating k-colouring. □

We remark that the first part of the proof of Theorem 1 shows that the graph G defined in the last part of the proof is, in some sense, a "universal" graph for c-colourable graphs having an over-dominating k-colouring, or at least for those in which no colour class of the k-colouring is also a colour class of the c-colouring, insofar as any such graph must be a subgraph of G or of a blowup of it (meaning that we allow more than one vertex labelled (σ, i)).

We return now to our main aim of determining the minimum possible Hadwiger number amongst graphs with an over-dominating k-colouring. The above-mentioned work of Duchet and Meyniel [2] implies that graphs of order $2k$ with independence number two have a complete minor on at least $\lceil 2k/3 \rceil$ vertices. Our first result gives the same lower bound for all graphs having an over-dominating k-colouring. Our second result shows that this bound is asymptotically best possible for these graphs.

2. Lower bound

Theorem 2. *Let G be a graph that has an over-dominating k-colouring. Then $h(G) \geq 1 + \lfloor 2k/3 \rfloor$.*

Proof. We use induction on k. For $k = 1, 2$, the statement is evident. It is also true for $k = 3$ because in this case G must contain a triangle. Suppose that the theorem holds for each $k' < k$. Suppose that a graph G has an over-dominating k-colouring f with colour classes $C_1, \dots, C_k$.

For each pair $\{i, j\} \subseteq [k]$, condition (b) of the definition of an over-dominating k-colouring means that there is a vertex in C_i joined to all of C_j or a vertex of C_j joined to all of C_i. Choose exactly one such vertex and attach the *tag* $\{i, j\}$ to it. Thus exactly $\binom{k}{2}$ tags are attached altogether. A vertex with one or more tags is said to be *tagged*.

Suppose first that there is some $i \in [k]$ such that C_i contains more than one tagged vertex. So C_i contains a vertex v_1 tagged with $\{i, j_1\}$ and another vertex v_2 tagged with $\{i, j_2\}$, where $j_1 \neq j_2$. Let $W_1 = C_{j_1} \cup \{v_1\}$, $W_2 = C_{j_2} \cup \{v_2\}$, and $G' = G - C_i - C_{j_1} - C_{j_2}$. By the definition of a tag, W_1 and W_2 induce connected subgraphs of G. Since f is dominating, each of C_1 and C_2 is a dominating set in G, so W_1 and W_2 are connected dominating sets. By contracting W_1 and W_2 to single vertices we see that $h(G) \geq 2 + h(G')$. Now the graph G' has an over-dominating $(k-3)$-colouring $f|_{G'}$. By the induction hypothesis, $h(G') \geq 1 + \lfloor 2(k-3)/3 \rfloor$, and so $h(G) \geq 1 + \lfloor 2k/3 \rfloor$.

Thus we may assume that for each $i \in [k]$, C_i contains at most one tagged vertex; define x_i to be this vertex if it exists, and otherwise let x_i be any vertex of C_i. For each pair $\{i, j\} \subseteq [k]$, either x_i or x_j has the tag $\{i, j\}$, and either way $x_i x_j \in E(G)$. Therefore the set $X = \{x_1, \dots, x_k\}$ induces a complete subgraph of G, and so $h(G) \geq k$. This proves the theorem. □

We remark that the proof still works if condition (b), on pairs of colour classes, is dropped for every pair containing (say) colour 1.

3. Upper bound

Theorem 3. *There is a sequence $G_1, G_2, G_3, \dots$ of graphs, where for each $k \geq 1$ the graph G_k has order $2k$ and has an over-dominating k-colouring with each colour class having size two, such that $h(G_k) = 2k/3 + o(k)$.*

To prove the theorem we consider random graphs G_k of order $2k$ generated as follows. The vertex set of G_k is $\{a_i, b_i : 1 \leq i \leq k\}$. None of $a_i b_i$, $1 \leq i \leq k$ is an edge. Between each pair $\{a_i, b_i\}$ and $\{a_j, b_j\}$ with $i \neq j$, exactly one edge is missing and the other three are present; the choice of missing edge is made at random, and the choices for different pairs $\{i, j\}$ are independent. Thus there are $4^{\binom{k}{2}}$ equiprobable choices for G_k.

We shall prove shortly that G_k almost surely satisfies the statement of Theorem 3. Before doing so, however, in view of Theorem 1 we point out that the clique number and the chromatic number of G_k can be estimated very well by using standard facts about random graphs (though we do not subsequently make use of these estimates). Observe that the subgraph of G_k spanned by $\{a_1, \ldots, a_k\}$ is just an ordinary random graph $G(k, 3/4)$, as is the subgraph spanned by $\{b_1, \ldots, b_k\}$. Almost surely, the clique numbers of these two subgraphs are $(2 + o(1)) \log_{4/3} k$ and their chromatic numbers are $(1/2 + o(1))k/\log_4 k$ (see [1]). Hence, almost surely, the clique number of G_k lies between $(2 + o(1)) \log_{4/3} k$ and $(4 + o(1)) \log_{4/3} k$, and the chromatic number of G_k lies between $(1/2 + o(1))k/\log_4 k$ and $(1 + o(1))k/\log_4 k$.

Rather than proving just that $h(G_k) = 2k/3 + o(k)$, we shall in fact prove a similar statement for a more general random graph $G_k(\ell)$ of order $(2 + \ell)k$, where $\ell \geq 0$. The vertex set of $G_k(\ell)$ is the disjoint union of the sets $\{a_i, b_i, d_{i,1}, \ldots, d_{i,\ell}\}$, $1 \leq i \leq k$. Each of these sets is an independent set in $G_k(\ell)$. For each *ordered* pair (i, j) with $i \neq j$, we select one vertex of $\{a_i, b_i\}$ at random, and join it to all vertices in $\{a_j, b_j, d_{j,1}, \ldots, d_{j,\ell}\}$. Notice that there are $4^{\binom{k}{2}}$ equiprobable choices for $G_k(\ell)$, and that $G_k = G_k(0)$; indeed, the subgraph of $G_k(\ell)$ induced by the vertex set $\{a_i, b_i : 1 \leq i \leq k\}$ is exactly G_k. Since none of the $d_{i,j}$'s are joined to each other, the clique and chromatic numbers of $G_k(\ell)$ exceed those of G_k by at most one, and hence the remarks above about these parameters for G_k apply equally to $G_k(\ell)$.

Of course, by construction, $G_k(\ell)$ has an over-dominating k-colouring; indeed, in this colouring, every two colour classes span a subgraph which is the union of two stars. So Theorem 2 tells us that $h(G_k(\ell)) \geq h(G_k) \geq 2k/3$. Our reason for introducing $G_k(\ell)$ is to give a examples of larger and sparser graphs than G_k that have an over-dominating k-colouring and still have small Hadwiger number. Large graphs, even sparse ones, typically have large Hadwiger numbers simply because they offer many opportunities to construct dominating sets using a small proportion of the vertices. Indeed, this fate befalls $G_k(\ell)$ once $\ell = \Omega(\log k)$, as we indicate at the end of the paper. In light of this, the next theorem is best possible. As previously discussed, Theorem 3 is a consequence of the case $\ell = 0$ of this theorem. The case $\ell = 0$ is in fact easier to prove, as we shall point out, but we include the general case for the reasons stated.

Theorem 4. *If $\ell = o(\log k)$ then $h(G_k(\ell)) \leq 2k/3 + o(k)$ almost surely.*

Proof. Consider a subcontraction of $G_k(\ell)$ to K_h, where $h = h(G_k(\ell))$. It consists of disjoint connected subgraphs F_u of $G_k(\ell)$, $u \in V(K_h)$, such that, between each pair of subgraphs F_u, F_v with $u \neq v$, there is an edge of $G_k(\ell)$. Let $D_i = \{d_{i,1}, \ldots, d_{i,\ell}\}$, $1 \leq i \leq k$. Observe that, for each u, we may assume that $|V(F_u) \cap D_i| \leq 1$ for $1 \leq i \leq k$, since the vertices in D_i have the same neighbours. Moreover if $V(F_u) \cap \{a_i, b_i\} \neq \emptyset$ we may assume $V(F_u) \cap D_i = \emptyset$, since any neighbour of a vertex in D_i is a neighbour of both a_i and b_i.

Let $L = \sqrt{\ell \log k}$; thus $\ell = o(L)$ and $L = o(\log k)$. Let $X = \{u \in V(K_h) : |F_u| \geq L\}$. Then $|X| \leq (2 + \ell)k/L = o(k)$. Let $S = \{a_i, b_i : 1 \leq i \leq k\}$. Let $U = \{u \in V(K_h) - X : |F_u \cap S| = 1\}$ and $W = \{w \in V(K_h) - X : |F_w \cap S| = 2\}$. To prove the theorem it suffices to show that $|U| = o(k)$ and $|W| = o(k)$ almost surely, because $h \leq |X| + |U| + |W| + |S|/3$ and $|S| = 2k$. (Observe that if $\ell = 0$ then U spans a clique, and so the stated estimates on the clique number of G_k give $|U| = o(k)$ immediately.)

We call any subset $Y \subseteq V(G_k(\ell))$ such that $|Y| \leq L$, $|Y \cap S| \leq 2$ and $|Y \cap (\{a_i, b_i\} \cup D_i)| \leq 1$ for $1 \leq i \leq k$, a *potential component*. By the remarks above, the vertex set of each component F_u, $u \in U \cup W$ is a potential component; the only point to observe here is that, if $u \in W$, the connectivity of F_u means it is impossible that $V(F_u) \cap S = \{a_i, b_i\}$, because every edge of F_u meets a_i or b_i and these vertices have no common neighbours outside S.

Let Y and Z be two disjoint potential components. They are said to be *unrelated* if there is no i, $1 \leq i \leq k$, for which $\{a_i, b_i\} \subseteq Y \cup Z$. Since by definition neither Y nor Z can contain both a_i and b_i, this condition simply precludes a_i being in one of the sets and b_i being in the other. In the random generation of $G_k(\ell)$, for each vertex $a \in Y \cap S$, and for each (except at most one) vertex $z \in Z$, with probability $1/2a$ is joined to z. Thus the probability that there is no edge in $G_k(\ell)$ between unrelated Y and Z is at least 2^{-4L}.

If the potential components $Y_1, \ldots, Y_t \subseteq V(G_k(\ell))$ are pairwise unrelated then the $\binom{t}{2}$ events that there is an edge between Y_i and Y_j, $1 \leq i < j \leq t$, are independent. Thus the probability that there are edges between each pair of sets is at most $(1 - 2^{-4L})^{\binom{t}{2}}$. There are at most $|G_k(\ell)|^{Lt}$ ways to choose t sets of vertices in $G_k(\ell)$, each of size at most L. So the probability $P(t)$ that there are t pairwise unrelated potential components with an edge between each pair satisfies $P(t) \leq ((2 + \ell)k)^{tL}(1 - 2^{-4L})^{\binom{t}{2}}$. Since $2 + \ell < k$, and $1 - 2^{-4L} = 1 - 2^{-o(\log k)} < 1 - 1/\sqrt{k} < e^{-\sqrt{k}}$ for large k, we have $P(t) < k^{2tL}e^{-\sqrt{k}t^2/4}$. For fixed $\epsilon > 0$ this means that $P(\epsilon k) < \exp\{2\epsilon kL \log k - \epsilon^2 k^{3/2}/4\} = o(1)$. Thus if $G_k(\ell)$ contains t such components then almost surely $t = o(k)$.

Now each $u \in U$ satisfies $|V(F_u) \cap S| = 1$, so we may find at least $|U|/2$ vertices $u \in U$ for which the sets $V(F_u)$ are pairwise unrelated potential components. Thus $|U| = o(k)$ as desired.

Consider now the multigraph with vertex set $\{1, \ldots, k\}$, such that for each $u \in W$ we insert an edge between i and j if $V(F_u) \cap \{a_i, b_i\} \neq \emptyset$ and $V(F_u) \cap \{a_j, b_j\} \neq \emptyset$. This graph has $|W|$ edges and maximum degree at most two, so it is a union of paths and cycles, and hence we may find a set of at least $|W|/3$ disjoint edges in it. The sets $V(F_u)$ corresponding to these disjoint edges are pairwise unrelated potential components. It follows that $|W| = o(k)$, so completing the proof. □

The bound on ℓ in Theorem 4 is best possible in the order of magnitude. We just sketch a crude argument to show that if $\ell = 20 \log k$ then $h(G_k(\ell)) = 2k$ almost surely. Take k random injections $f_i : D_i \to S$ and, for $a \in S$, let U_a be all those vertices mapping to a. Then $|U_a|$ is $\mathrm{Bin}(k, \ell/2k)$ and the event $|U_a| < 8 \log k$ has probability $o(k^{-1})$, so $|U_a| \geq 8 \log k$ for all $a \in S$. Let W_a be the neighbours of a in U_a; then similarly $|W_a| \geq 3 \log k$ for all $a \in S$. The probability that $b \in S$ has no neighbour in W_a is then at most $o(k^{-2})$, so there is an edge between $a \cup W_a$ and $b \cup W_a$ for all $a, b \in S$.

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