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Large minors in graphs with given independence number

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ABSTRACT

A weakening of Hadwiger's conjecture states that every n-vertex graph with independence number α has a clique minor of size at least $\frac{n}{\alpha}$. Extending ideas of Fox (2010) [6], we prove that such a graph has a clique minor with at least $\frac{n}{(2-c)\alpha}$ vertices where $c > 1/19.2$.

1. Introduction

We use standard notation: for a graph G, $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively; $|G| := |V(G)|$ and $\|G\| := |E(G)|$. Also, $\Delta(G)$, $\alpha(G)$, $\omega(G)$, and $\eta(G)$ denote the maximum degree, the independence number, the clique number, and the order of a largest clique minor of G, respectively.

Hadwiger's conjecture [8] from 1943 (see [14] for a survey) states the following conjecture.

Conjecture 1.1. *For every k-chromatic graph G, K_k is a minor of G.*

Hadwiger's conjecture for $k = 4$ was proved by Dirac [3], the case $k = 5$ was shown equivalent to the Four Color Theorem by Wagner [15] and the case $k = 6$ was shown equivalent to the Four Color Theorem by Robertson et al. [12]. For $k \geq 7$, the conjecture remains open. Since $\alpha(G)\chi(G) \geq |V(G)|$ for every graph G, Hadwiger's conjecture implies the following conjecture.

Conjecture 1.2. *For every graph G, $\alpha(G)\eta(G) \geq |V(G)|$.*

Formally, this conjecture is weaker than Hadwiger's conjecture; however, Plummer et al. [11] showed that for graphs G with $\alpha(G) = 2$, the two conjectures are equivalent. In 1981, Duchet and Meyniel [4] showed that

$$(2\alpha(G) - 1)\eta(G) \geq |V(G)|. \quad (1)$$

In particular, this means that

$$\eta(G) \geq \frac{n}{3} \quad \text{for every } n\text{-vertex graph } G \text{ with } \alpha(G) = 2. \quad (2)$$

No significant improvement of (2) is known. Seymour suggested to prove that there exists an $\epsilon > 0$ such that if $\alpha(G) = 2$ and $|V(G)| = n$, then G has a complete minor of order $(1/3 + \epsilon)n$; but this also is not proved. For the case $\alpha(G) \geq 3$,

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several improvements of (1) were obtained recently (see, e.g. [9,10,16,1,6]). The best known bound for $\alpha(G) = 3$ is due to Kawarabayashi and Song [10]: they proved that $\eta(G) \geq \frac{n}{4}$ for every n-vertex graph G with $\alpha(G) = 3$. The best result for large $\alpha(G)$ is due to Fox [6]: he proved that $\eta(G) \geq |V(G)|/((2-c)\alpha(G))$, where $c \approx 1/57.5$ is a constant. Using the main tool of Fox [6], the notion of set potentials, together with additional ideas, we prove the following.

Theorem 1.1.

$$\eta(G) \geq \frac{|V(G)|}{(2-c)\alpha(G)},$$

where $c = (80 - \sqrt{5392})/126 > 1/19.2$.

In Section 2 we provide the key concepts and the outline of the proof of Theorem 1.1. In Section 3 we prove one of our key lemmas on the use of sets with large potential. In Section 4 we prove some properties of graphs with independence number 2, in Section 5 we describe three different ways to find sets with large potential. In Section 6 we have the final computation, and Section 7 contains a long proof of a lemma.

2. Preliminaries and outline of the proof

A *claw* in a graph G is an induced $K_{1,3}$-subgraph.

For a subset X of the vertex set of a graph G, $G[X]$ is the subgraph of G induced by X. Sometimes, we will identify X with $G[X]$. For example, by $\alpha(X)$ we denote $\alpha(G[X])$ and by $c(X)$ denote the number of components of $G[X]$. In these terms, for $X \subseteq V(G)$ Fox [6] defined the *potential* of X, $\phi(X) = \phi_G(X)$, as follows:

$$\phi(X) := 2\alpha(X) - |X| - c(X). \tag{3}$$

Also, for $X \subset V(G)$, $N(X)$ is the set of vertices in $V(G) - X$ that have neighbors in X.

A useful property of potentials is that if the vertex sets of the components of $G[X]$ are $X_1, \ldots, X_s$, then

$$\phi(X) := \sum_{i=1}^{s} \phi(X_i). \tag{4}$$

In view of (4), a component $G[X_i]$ of $G[X]$ will be called a *j-component* if $\phi(X_i) = j$. For $j = 1, 2, \ldots$, let $c_j(X)$ denote the number of j-components of $G[X]$, so that

$$c(X) = \sum_{j=1}^{s} c_j(X) \quad \text{and} \quad \phi(X) = \sum_{j=1}^{s} j c_j(X). \tag{5}$$

A graph G is *decomposable*, if there is a partition (V_1, V_2) of $V(G)$ into non-empty sets such that $\alpha(G[V_1]) + \alpha(G[V_2]) = \alpha(G)$, and *non-decomposable* otherwise. Fox [6] proved and used the fact that if a non-decomposable graph G contains an $X \subseteq V(G)$ with $\phi(X) = k$, then it contains a connected dominating set X' with $\phi(X') \geq 2k/7$. Extending his ideas we prove in Section 3 the following strengthening of this result.

Lemma 2.1. *Let G be a non-decomposable connected graph with independence number α. For every $X \subseteq V(G)$, G contains a connected dominating set $\widetilde{X}$ with $|\widetilde{X}| \leq 2\alpha - 2\phi(X)/3 - 1$.*[1]

Outline of the proof of Theorem 1.1: We assume that G is a minimal counter-example for our theorem. Let $n = |V(G)|$ and $\alpha = \alpha(G)$. If we find a connected dominating set $\widetilde{X}$ with at most $(2-c)\alpha$ vertices, then $\widetilde{X}$ can be contracted to be a vertex of a clique minor of G. Then it is sufficient to find in $G - \widetilde{X}$ a clique minor of size $\frac{|V(G)|}{(2-c)\alpha(G)} - 1$, which can be done by induction. To find such an $\widetilde{X}$, by Lemma 2.1, it is sufficient to find an $X \subset V(G)$ with $\phi(X) \geq 3c\alpha/2$. In Section 3 we prove Lemma 2.1. The rest of the proof of Theorem 1.1 tries to find either a subset of vertices with potential at least $3c\alpha/2$, or a clique minor of the required size.

We say that subsets $X_1, \ldots, X_k$ of $V(G)$ are *separated* if they are disjoint and there are no edges with ends in distinct X_i.

We follow the basic idea of Fox [6]: any graph G either has a large claw-free induced subgraph or has many vertex-disjoint claws. In the former case, a recent result of Fradkin [7] on minors in claw-free graphs can be used. In the latter case, either there are many separated claws forming a set with large potential, or by the induction assumption the subgraph of G induced by the vertex-disjoint claws has a large clique minor. However, we implement the idea in a different way, which together with Lemma 2.1 allows to improve the bound.

Let $\mathcal{A} = \{C_1, \ldots, C_m\}$ be a maximum family of separated claws in G, and let $A := \bigcup_{j=1}^{m} C_j$. Then $\phi(A) = m$. By the maximality of $\mathcal{A}$, the graph $F := G - A - N(A)$ is claw-free. The following theorem of Fradkin [7] gives an upper bound on the order of each component F' of F in terms of its independence number.

Theorem 2.2. *Let F' be a connected claw-free graph with $\alpha(F') \geq 3$. Then $\eta(F') \geq |F'|/\alpha(F')$.*

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1 We do not know whether our bound is best possible.

The known bounds for components with independence number 2 give weaker bounds than Theorem 2.2. To have a better control over the such components, we will use a corollary of the following result of Chudnovsky and Seymour [2].

Theorem 2.3. *Let F' be a graph with independence number 2, and $t := \lceil |F'|/2 \rceil$. If $|F'|$ is even and $\omega(F') \geq |F'|/4$, or $|F'|$ is odd and $\omega(F') \geq (|F'|+3)/4$, then F' has a clique minor of order at least t.*

So a large F has many components with independence number 2 and a small clique number.

Let I be a maximum independent set in $G[A \cup N(A)]$. Either I is large, and $A \cup I$ has potential at least $3c\alpha/2$, or I is small and we can apply the induction hypothesis to $G[A \cup N(A)]$. The worst case occurs when I is in the "middle range", and F has many components with independence number 2 and small clique number. In this case we find a new way to find subsets of vertices with potential larger than m. This last part of the proof is the most technical part of the paper.

Note that at the end of our proof, the case analysis could have been refined, but the improvement on c would have been relatively small, and the proof is rather technical. The approach cannot prove Conjecture 1.2 (which in our terms corresponds to $c = 1$): for example, if G contains $n/4$ vertex disjoint claws (covering $V(G)$), then the method yields only $c = 1/6$.

3. Finding small connected dominating sets

In this section, we prove Lemma 2.1. First, recall some known results.

Lemma 3.1 (*[1], Lemma 12*). *Let G be a connected graph with $\alpha(G) = k$. Let $v \in V(G)$. Then G contains a connected induced subgraph G' with $\alpha(G') = k$ such that $v \in V(G')$ and $|V(G')| \leq 2k - 1$.*

Claim 3.1 (*[6], Lemma 3*). *If X is a subset of the vertex set of a connected graph G, then there is a dominating set X' such that*

(i) *the potential of every component of $G[X']$ is positive;*
(ii) *$\phi(X') \geq \phi(X)$, $c(X') \leq c(X)$, and*
(iii) *each vertex in $V(G) - X'$ is adjacent to vertices in only one component of $G[X']$.*

Claim 3.2 (*[6], Corollary 1*). *If X is a dominating set in a non-decomposable graph G and $\alpha(X) = \alpha(G)$, then there is a connected dominating set X' containing X with $\phi(X') \geq \phi(X)$.*

We also need an easy observation.

Claim 3.3. *If X is a dominating set in a connected graph G, then there is a connected dominating set X' containing X with $|X'| \leq |X| + 2(c(X) - 1)$.*

Proof. If $c(X) = 1$, then $X' = X$ works. Proving the claim by induction on $c(X)$, suppose that the claim holds for all X with $c(X) < k$ for some $k \geq 2$. Let X be a dominating set in G such that the vertex sets of the components of $G[X]$ are $X_1, \ldots, X_k$. Let $V_i := X_i \cup N(X_i)$ for $i = 1, \ldots, k$. If $x \in V_i \cap V_j$ for some $i \neq j$, then let $X' := X \cup \{x\}$ and note that $c(X') \leq k - 1$, hence we are done by the induction hypothesis. In the case when the sets V_i form a partition of $V(G)$, since G is connected, there is some edge, say xy, that connects vertices from distinct V_is. Let $X_0 := X + x + y$. Then $c(X_0) \leq k - 1$ and by induction assumption, there is a connected X' containing X_0 with $|X'| \leq |X_0| + 2((k-1) - 1)$. This X' is what we need. □

Remark A. If $\widetilde{X}$ is a connected dominating set with $\alpha(\widetilde{X}) = \alpha$, then the inequality $|\widetilde{X}| \leq 2\alpha - 2k/3 - 1$ is equivalent to the inequality $\phi(\widetilde{X}) \geq 2k/3$. Thus by Claim 3.2, if we construct a dominating set X_0 with $\alpha(X_0) = \alpha$ and $\phi(X_0) \geq 2\phi(X)/3$, then the lemma will be proved.

Proof of Lemma 2.1. Let G and X satisfy the conditions of the lemma. By Claim 3.1, we may assume that X is dominating, each component of $G[X]$ has a positive potential and each vertex in $V(G) - X$ is adjacent to vertices in only one component of $G[X]$. Let $X_1, \ldots, X_s$ be the vertex sets of the components of $G[X]$, and for $i = 1, \ldots, s$, let $V_i := X_i \cup N(X_i)$. By the above, the sets $V_1, \ldots, V_s$ form a partition of $V(G)$. If for some i, $\alpha(V_i) = \alpha(X_i)$, then $\alpha(V_i) + \alpha(G - V_i) = \alpha(G)$, and hence G is decomposable. So

$$\alpha(V_i) > \alpha(X_i) \quad \text{for all } i \in \{1, \ldots, s\}. \tag{6}$$

By Claim 3.3, G has a connected dominating set X_0 with $|X_0| \leq |X| + 2(c(X) - 1)$. If this is at most $2\alpha - 2\phi(X)/3 - 1$ then the lemma is proved, otherwise

$$|X| + 2(c(X) - 1) > 2\alpha - 2\phi(X)/3 - 1,$$

and since $|X| + 2c(X) = 2\alpha(X) - \phi(X) + c(X)$, we have

$$2\alpha(X) - \phi(X) + c(X) - 2 > 2\alpha - 2\phi(X)/3 - 1. \tag{7}$$

Plugging the expressions from (5) into (7), we get

$$\frac{2c_1(X)}{3} + \frac{c_2(X)}{3} \geq c(X) - \frac{\phi(X)}{3} > 2(\alpha - \alpha(X)) + 1. \tag{8}$$

For each i and each $v \in V_i$, by Lemma 3.1, there exists a connected subset $Y_i(v) \subset V_i$ containing v with $\alpha(Y_i(v)) = \alpha(V_i) \geq 1 + \alpha(X_i)$ and $\phi(Y_i(v)) \geq 0$. Since G is connected, V_i contains a vertex x adjacent to some y in another $V_{i'}$. If some $x \in V_i$ has a neighbor y in some $V_{i'}$ for $i' \neq i$, then let $I(Y(x), y)$ be the set of indices ℓ such that X_ℓ has a neighbor in $Y_i(x) + y$. (Note that $I(Y(x), y) = \emptyset$ before the first change.) In this case, an (i, i', x, y)-*expansion* of X is the set X' obtained from X by replacing X_i with $Y_i(x) + y$. The components of $G[X']$ will be: (a) one component $X'_{i'}$ whose vertex set is obtained from merging $Y_i(x) + y$ with $V_{i'}$ and $\bigcup_{\ell \in I(Y(x),y)} X_\ell$, and (b) the components with vertex sets X_u, where $u \in \{1, \ldots, s\} - \{i, i'\} - I(Y(x), y)$. The component $X_{i'}$ containing y will be called the *attracting component* of X.

By construction, we have

(P1) $\alpha(X') \geq \alpha(X) - \alpha(X_i) + \alpha(Y_i(x)) \geq \alpha(X) + 1$.

Since $\phi(X_{i'}) \geq 1$ and $\phi(Y_i(x)) \geq 0$, connecting them together via y and possible merging with other components keeps the potential of the resulting set positive, i.e.,

(P2) $\phi(X'_{i'}) \geq \phi(X_{i'}) \geq 1$.

By construction and (4),

(P3) $\phi(X') \geq \phi(X) - \phi(X_i)$.

Since each X_j dominates V_j and $Y_i(x)$ dominates V_i,

(P4) X' is dominating.

An (i, i')-*expansion* of X is an (i, i', x, y)-*expansion* of X for some $x \in V_i$ and $y \in V_{i'}$. An *expansion* of X is any (i, i')-*expansion* of X.

If all the components of $G[X]$ are 1-components, then choose one of them and call it *senior*. Otherwise, *senior* components are all j-components for all $j \geq 2$. Since G is connected, if $c_1(X) \geq 1$ and X is not connected, then there exist i and i' such that X_i is a 1-component, $X_{i'}$ is a senior component, and a vertex $x \in X_i$ is adjacent to a vertex $y \in X_{i'}$. Take any such pair (i, i') and perform an (i, i')-expansion of X. The component obtained by merging $X_{i'}$ with $Y_i(x) + y$ (and maybe some others) is considered senior, again. Repeat such merges until either the resulting set is connected, or the resulting set has an independent subset of size α, or the resulting set does not have 1-components. Let Z be the resulting set and suppose that we made exactly z expansions. By (P1) and (8),

$$z \leq \alpha - \alpha(X) < \frac{c_1(X)}{3} + \frac{c_2(X)}{6} - \frac{1}{2}. \tag{9}$$

By (P3),

$$\phi(Z) \geq \phi(X) - z. \tag{10}$$

By (P4), Z is dominating. Since $\phi(X) \geq c(X)$, by (9) and (10),

$$\phi(Z) > \phi(X) - \frac{1}{3}c(X) + \frac{1}{2} > \frac{2}{3}\phi(X). \tag{11}$$

Case 1: $c(Z) = 1$. By (11), $2\alpha(Z) - |Z| - 1 > \frac{2}{3}\phi(X)$, i.e.,

$$|Z| < 2\alpha(Z) - 1 - \frac{2}{3}\phi(X) \leq 2\alpha - 1 - \frac{2}{3}\phi(X).$$

Case 2: $\alpha(Z) = \alpha$. By (11) and Remark A we are done.

Case 3: $c(Z) \geq 2$, $\alpha(Z) < \alpha$, and $c_1(Z) = 0$. Note that in this case there was at least one non-1-component in X, so the senior components in X were j-components for $j \geq 2$. This implies that at any expansion, no new 2-component arises, and in particular, $c_2(X) \geq c_2(Z)$. Our strategy and the computations will be similar to above, but we will eliminate all 2-components. Note that after each expansion, each component of the new X' either was a component of the set X before expansion, or is the result of merging of a senior component with some other components, and hence the potential of the new component is not less than it was before expansion. It follows that for every expansion from X' to X'', $c(X') - c(X'') \geq c_1(X') - c_1(X'')$. This yields that

$$c(X) - c(Z) \geq c_1(X) - c_1(Z) = c_1(X). \tag{12}$$

Also, if before an expansion from X' to X'', the attracting component $X'_{i'}$ was a 2-component and after it is a j-component for some $j \geq 3$, then $\phi(X'') \geq \phi(X')$. So, (10) can be strengthened as follows:

$$\phi(Z) \geq \phi(X) - z + c_2(X) - c_2(Z). \tag{13}$$

By Claim 3.3, G has a connected dominating set X_0 with $|X_0| \leq |Z| + 2(c(Z) - 1)$. Also as above, if this X_0 does not satisfy the lemma, i.e., $|X_0| > 2\alpha - 2\phi(X)/3 - 1$, then similarly to (7), and using the definition of the potential function $\phi(Z)$, we have

$$2\alpha(Z) - \phi(Z) + c(Z) - 2 > 2\alpha - 2\phi(X)/3 - 1. \tag{14}$$

By (12), (10) and (5), this gives

$$2(\alpha - \alpha(Z)) < z - 1 - \phi(X)/3 + c(X) - c_1(X) \leq z - 1 - \frac{c_1(X)}{3} + \frac{c_2(X)}{3}. \tag{15}$$

In this case, we continue extensions. If all the components of $G[Z]$ are 2-components, then choose any one of them and call it *senior*. Otherwise, *senior* components are all j-components for all $j \geq 3$. Since G is connected, if $c_2(Z) \geq 1$ and $c(Z) \geq 2$, then there exist i and i' such that Z_i is a 2-component, $Z_{i'}$ is a senior component, and vertex $x \in Z_i$ is adjacent to a vertex $y \in Z_{i'}$. Take any such a pair (i, i') and perform an (i, i')-expansion of Z. The component obtained by merging $Z_{i'}$ with $Y_i(x)+y$ (and maybe some others) is considered senior, again. Repeat such merges until either the resulting set is connected, or the resulting set has an independent subset of size α, or the resulting set does not have 2-components. Let U be the resulting set and suppose that we made exactly u expansions after Z was obtained. By (P1) and (15),

$$u \leq \alpha - \alpha(Z) < \frac{1}{2}\left(z - 1 - \frac{c_1(X)}{3} + \frac{c_2(X)}{3}\right). \tag{16}$$

By (P3),

$$\phi(U) \geq \phi(Z) - 2u. \tag{17}$$

By (P4), U is dominating. By (16), (17) and (10),

$$\phi(U) > \phi(Z) - \left(z - 1 - \frac{c_1(X)}{3} + \frac{c_2(X)}{3}\right) > \phi(X) - 2z + 1 + \frac{c_1(X)}{3} - \frac{c_2(X)}{3}.$$

So, by (9) and (5),

$$\phi(U) > \phi(X) - \frac{c_1(X)}{3} - \frac{2c_2(X)}{3} + 2 \geq \frac{2\phi(X)}{3} + 2. \tag{18}$$

Subcase 3.1: $c(U) = 1$ or $\alpha(U) = \alpha$. Similarly to Cases 1 and 2, we are done by (18).

Subcase 3.2: $c(U) \geq 2, \alpha(U) \leq \alpha - 1$ and $c_1(U) = c_2(U) = 0$. As it was observed above, at every expansion, no component that was not a 2-component before the expansion, becomes such a component after it. In particular, this implies that if all components of $G[Z]$ were 2-components, then at the end, only a senior component survives, i.e. we have Case 3.1. Another implication is that

$$c(Z) - c(U) \geq c_2(Z) - c_2(U) = c_2(Z). \tag{19}$$

By Claim 3.3, G has a connected dominating set X_0 with $|X_0| \leq |U| + 2(c(U) - 1)$. If this X_0 is larger than what we want to achieve in the proof, then $|X_0| > 2\alpha - 2\phi(X)/3 - 1$ and we have

$$2\alpha(U) - \phi(U) + c(U) - 2 > 2\alpha - 2\phi(X)/3 - 1.$$

By (19), (17) and (13), this yields

$$\begin{aligned} 2(\alpha - \alpha(U)) &< \frac{2\phi(X)}{3} - 1 - \phi(Z) + 2u + c(Z) - c_2(Z) \\ &\leq \frac{2\phi(X)}{3} - 1 - \phi(X) + z - c_2(X) + c_2(Z) + 2u + c(Z) - c_2(Z). \end{aligned}$$

So, by (12),

$$2(\alpha - \alpha(U)) < -\frac{\phi(X)}{3} - 1 + z + 2u + c(X) - c_1(X) - c_2(X).$$

By (16), (5) and (9), the left-hand side is at most

$$\begin{aligned} &-1 - \frac{\phi(X)}{3} + 2z - 1 - \frac{c_1(X)}{3} + \frac{c_2(X)}{3} + \sum_{j=3}^{\infty} c_j(X) \\ &\leq -2 - \frac{1}{3}\sum_{j=1}^{\infty} jc_j(X) + \frac{2c_1(X)}{3} + \frac{c_2(X)}{3} - 1 - \frac{c_1(X)}{3} + \frac{c_2(X)}{3} + \sum_{j=3}^{\infty} c_j(X) \leq -3. \end{aligned}$$

Since $\alpha - \alpha(U) \geq 1$, this is a contradiction. □

4. Graphs with independence number 2

A graph F is *good* if $\eta(F)\alpha(F) \geq |V(F)|$, and *bad* otherwise. In particular, by Theorem 2.3, if $\alpha(F) = 2$ and $\omega(F) \geq \frac{|F|+1}{4}$ then F is good (if $|F|$ is odd, this follows by the integrality of $\omega(F)$). Hence,

$$\text{if } F \text{ is a bad graph with } \alpha(F) = 2, \text{ then } 4\omega(F) - 1 \leq |F|. \tag{20}$$

Theorem 4.1. *Let F be an n-vertex graph with independence number 2, and*

$$w = \omega(F) \leq (2+n)/4. \tag{21}$$

Then $\eta(F) \geq \frac{n+2w-2}{3}$.

Proof. If $w = 1$ then $n \leq 2$, and the claim holds: $\eta(G) = 1 > (2+2-2)/3$. So let $w \geq 2$. Let W be the vertex set of a clique of size w in F. Let $P_1, \ldots, P_t$ be a maximum set of vertex-disjoint induced paths of length 2 in $F - W$. Consider $F_0 := F - W - P_1 - \cdots - P_t$. By the maximality of t, each component of F_0 is a clique, and hence $|F_0| \leq 2w$. So, $3t + w + 2w \geq n$, i.e.,

$$t \geq \left\lceil \frac{n-3w}{3} \right\rceil. \tag{22}$$

Let $t' := \left\lceil \frac{n-4w+2}{3} \right\rceil$. Since $w \geq 2$, $t' \leq t$. Let $F_1 := G - P_1 - \cdots - P_{t'}$. Then

$$|F_1| = n - 3t' \leq n - (n - 4w + 2) = 4w - 2.$$

So by (20), $\eta(F_1) \geq \left\lceil \frac{|F_1|}{2} \right\rceil$. Since each of $P_1, \ldots, P_{t'}$ forms a connected dominating set in F,

$$\eta(F) \geq t' + \eta(F_1) \geq t' + \frac{n-3t'}{2} = \frac{n-t'}{2} \geq \frac{n-(n-4w+4)/3}{2} = \frac{n+2w-2}{3},$$

as claimed. □

The known values of Ramsey numbers ($R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$, $R(3,6) = 18$, $R(3,7) = 23$) together with (20) yield:

Corollary 4.1. *If F is a bad graph with $\alpha(F) = 2$, then $\omega(F) \geq 7$.*

The next fact is a corollary of (20).

Lemma 4.2. *Let G_0 be a bad graph with $\alpha(G_0) = 2$. Then for every two cliques Q_1 and Q_2 in G_0 with $Q_1 \neq \emptyset$, there are vertices $q \in Q_1$ and $p_1, p_2 \in N(q) - Q_1 - Q_2$ such that $p_1p_2 \notin E(G_0)$.*

Proof. Let $w = \omega(G_0)$. Let $q \in Q_1$, $C = N(q) - Q_1 - Q_2$, $B = V(G_0) - N(q) - q$ and $B' = V(G_0) - Q_1 - Q_2 - C = B - Q_1 - Q_2$. Since $\alpha(G_0) = 2$, B is a clique in G_0. If C is not a clique, then the lemma holds, so assume that C is a clique. Then $|C| \leq w - 1$, $|B'| \leq |B| \leq w$, $|Q_1| \leq w$ and $|Q_2| \leq w$. Moreover, if q is adjacent to all vertices in Q_2, then $|Q_2| \leq w - 1$, otherwise $B' \neq B$ and so $|B'| \leq w - 1$. In any case, $|G_0| = |C| + |B'| + |Q_1| + |Q_2| \leq 4w - 2$, a contradiction to (20). □

We will apply this lemma in the following form.

Lemma 4.3. *Let G be a graph and $G_0 = (V_0, E_0)$ be a bad induced subgraph of G with $\alpha(G_0) = 2$. Let $W \subset V(G) - V_0$ be such that*

(a) *for every $w \in W$, $N(w) \cap V_0$ is a clique (maybe empty);*
(b) *there are $w_1, w_2 \in W$ such that $\{w_1\} \subseteq N(V_0) \cap W \subseteq \{w_1, w_2\}$.*

Then $\phi(W') \geq 1 + \phi(W)$ for some $W' \subseteq W \cup V_0$.

Proof. For $j = 1, 2$, let $Q_j = N(w_j) \cap V_0$. By (a), Q_1 and Q_2 are cliques in G_0. So by Lemma 4.2, there is a $q \in Q_1$ and $p_1, p_2 \in N(q) - Q_1 - Q_2$ such that $p_1p_2 \notin E(G_0)$. Let $W' := W \cup \{q, p_1, p_2\}$. By (b), $\alpha(W') = \alpha(W) + 2$. Since the number of components of $G(W)$ and $G(W')$ is the same, we have $\phi(W') - \phi(W) \geq 2 \cdot 2 - 3 = 1$. □

We will also use the following extensions of Lemmas 4.2 and 4.3.

Lemma 4.4. *Let G_0 be an n-vertex bad graph with $\alpha(G_0) = 2$ and $w := \omega(G) \geq 6$. Let $j \geq 3$ and $Q_1, \ldots, Q_j$ be cliques in G_0 not all empty. Then either there are vertices $q \in \bigcup_{i=1}^{j} Q_i$ and $p_1, p_2 \in N(q) - \bigcup_{i=1}^{j} Q_i$ such that $p_1p_2 \notin E(G_0)$, or $|G_0| \leq (j+2)(w-1)$.*

Proof. Suppose that G_0 is a counter-example to the lemma. Let Q_1 be a non-empty clique in our family, and $q \in Q_1$. Similarly to the proof of Lemma 4.2, let $A = N(q) - \bigcup_{i=1}^{j} Q_i$, $B = V(G_0) - N(q) - q$ and $B' = V(G_0) - A - \bigcup_{i=1}^{j} Q_i = B - \bigcup_{i=1}^{j} Q_i$. Since $\alpha(G_0) = 2$, B is a clique in G_0. If A is not a clique, then the first claim of the lemma holds, so we can assume that A is a clique. Also, since G_0 has no cliques of size $w + 1$, $|Q_i - B - q| \leq w - 1$ for all $i = 2, \ldots, j$. It follows that

$$|G_0| = |A \cup B \cup \bigcup_{i=1}^{j} (Q_i - B - q)| \leq |A| + |B| + |Q_1| + \sum_{i=2}^{j} |Q_i - B - q| \tag{23}$$

$$\leq (w-1) + w + w + (j-1)(w-1) = (j+2)(w-1) + 2. \tag{24}$$

By (23) and (24), in order the second statement of the lemma to fail we need all the conditions below to be satisfied:
(a) $|Q_1| \geq w-1$ and $|Q_i - B - q| \geq w-2$ for $i = 2, \ldots, j$ (in particular, each Q_i is non-empty);
(b) since we can choose Q_1 ourselves, by (a), $|Q_i| \geq w-1$ for all $i = 1, \ldots, j$;
(c) for all $i, i' \in \{1, \ldots, j\}$ with $i \neq i'$,

$$(w - |Q_i|) + |Q_i \cap Q_{i'}| \leq 1 \tag{25}$$

(otherwise, choose Q_i as Q_1, then choose $q \in Q_1$ so that either $|(Q_1 \cap Q_{i'}) - q| \geq 2$ or $|Q_1| = w-1$ and $|(Q_1 \cap Q_{i'}) - q| = 1$, and then apply (23));

(d) no vertex belongs to more than two Q_is: if $v \in Q_1 \cap Q_2 \cap Q_3$, then choose $q \in Q_1 - v$, and v will be counted 3 times (in Q_1, in $Q_2 - B$ and in $Q_3 - B$).

Now that (a)–(d) hold, we may assume that $|Q_1| \leq |Q_2| \leq \cdots \leq |Q_j|$. Let $H = G[Q_1 \cup Q_2 \cup Q_3]$. If some $v \in V(H)$ has degree at least 4 in the complement, $\overline{H}$, then taking its clique as Q_1 and v as q, (23) yields $|G_0| \leq (j+2)(w-1)$. So,

$$\Delta(\overline{H}) \leq 3 \text{ and } \overline{H} \text{ is triangle-free.} \tag{26}$$

It was proved in [5,13] that if F is a triangle-free graph with maximum degree at most 3 then $\alpha(F) \geq 5|F|/14$. Applying this to $\bar{H}$ we obtain that $\omega(H) \geq 5|H|/14$.

Because of (b), we have two cases.

Case 1: $|Q_1| = w-1$. By (c), $Q_1 \cap (Q_2 \cup Q_3) = \emptyset$. Suppose first that $|Q_2| = w-1$. Since $\omega(G) = w < 2w-2$, there are $q \in Q_1$ and $q' \in Q_2$ with $qq' \notin E(G)$. So in (23), $|Q_2 - B| \leq w-2$ and the lemma follows. Thus $|Q_2| = |Q_3| = w$. If there exists $v \in Q_2 \cap Q_3$, then by (a), there exists $q \in Q_1 \cap N(v)$; so that in (23), vertex v will be counted twice. Thus $Q_2 \cap Q_3 = \emptyset$. Therefore, $|V(H)| = 3w-1$. Then by (26),

$$w = \omega(H) = \alpha(\overline{H}) \geq 5|V(H)|/14 = (15w-5)/14,$$

hence $w \leq 5$, a contradiction to Corollary 4.1.

Case 2: $|Q_1| = |Q_2| = |Q_3| = w$. If $|V(H)| \geq 3w-1$, then we repeat the end of the previous paragraph. So, $|V(H)| \leq 3w-2$. By (c), (d) and the symmetry between Q_1, Q_2, and Q_3, we may assume that there are distinct $q_2, q_3 \in Q_1$ such that for $i = 2, 3$, $Q_1 \cap Q_i = \{q_i\}$. Then for $q \in Q_1 - q_2 - q_3$, vertices q_2 and q_3 are counted twice in (23), and the lemma follows. □

Lemma 4.5. *Let G be a graph and F be a bad induced subgraph of G with $\alpha(G_0) = 2$. Let $k \geq 3$ and*

$$\eta(F) < \frac{k+4}{3(k+2)}|F|. \tag{27}$$

Let $W \subset V(G) - V(F)$ be such that

(a) *for every $w \in W$, $N(w) \cap F$ is a clique (maybe empty);*
(b) *there are $w_1, \ldots, w_k \in W$ such that $\emptyset \neq N(V(F)) \cap W \subseteq \{w_1, \ldots, w_k\}$.*

Then $\phi(W') \geq 1 + \phi(W)$ for some $W' \subseteq W \cup V(F)$.

Proof. Let $n_0 = |F|$ and $w = \omega(F)$. For $j = 1, \ldots, k$, let $Q_j = N(w_j) \cap V(F)$. By (a), $Q_1, \ldots, Q_k$ are cliques in F. Since F is bad, we have $w \leq (n_0+1)/4$, hence by Theorem 4.1, $\eta(F) \geq \frac{n_0+2(w-1)}{3}$. Together with (27), this yields $n_0 > (w-1)(k+2)$. So by Lemma 4.4, there is $q \in \bigcup_{i=1}^{k} Q_i$ and $p_1, p_2 \in N(q) - \bigcup_{i=1}^{k} Q_i$ such that $p_1p_2 \notin E(G_0)$. Let $W' := W \cup \{q, p_1, p_2\}$. By (b), $\alpha(W') = \alpha(W) + 2$. Since the number of components of $G(W)$ and $G(W')$ is the same, we have $\phi(W') - \phi(W) \geq 2 \cdot 2 - 3 = 1$. □

5. Finding sets with large potential

Let $\mathcal{A} = \{C_1, \ldots, C_m\}$ be a maximum collection of separated claws in G, and let $A = \bigcup_{i=1}^{m} V(C_i)$. Then $\alpha(A) = 3m$, $|A| = 4m$ and $\phi(A) = 2\alpha(A) - |A| - c(A) = m$. Fox [6] used A as a set with large potential. Since G is a minimum counter-example, it does not have sets of potential at least $3c\alpha/2$. We will try to find sets with larger potential in three different ways, and if each of the new sets will have potential less than $3c\alpha/2$, then we get a system of inequalities that leads to a contradiction.

Given A, we let $G' = G - A - N(A)$ and $\alpha' := \alpha(G')$. By the maximality of A, G' is claw-free, and by the definition of G' we also have $\alpha' \leq \alpha - 3m$. Let I be a maximum independent set in $A \cup N(A)$.

If a component D of $G[A \cup I]$ has potential greater than (respectively, smaller than and equal to) the number of claws in A contained in it, then we call D a *positive* (respectively, *negative* and *neutral*) component. Let $\mathcal{D}^+$ (respectively, $\mathcal{D}^-$ and $\mathcal{D}^0$) denote the set of positive (respectively, negative and neutral) components of $G[A \cup I]$. Also $\mathcal{D}$ denotes the set of all components of $G[A \cup I]$. Similarly $\mathcal{D}_j^+$ (respectively, $\mathcal{D}_j^-$, $\mathcal{D}_j^0$, and $\mathcal{D}_j$) is the set of components in $\mathcal{D}^+$ (respectively, $\mathcal{D}^-$, $\mathcal{D}^0$, and $\mathcal{D}$) containing exactly j claws.

5.1. First attempt

Our first set R_1 is obtained from A by replacing the claws contained in components of $\mathcal{D}^+$ with these components themselves. Let

$$\widetilde{f} := \sum_{j=1}^{\infty} \sum_{D \in \mathcal{D}_j^+} (\phi(D) - j).$$

By construction, $\phi(R_1) = m + \widetilde{f}$, hence

$$m + \widetilde{f} < 3c\alpha/2. \tag{28}$$

5.2. Second attempt

Let $G_0 := G[A \cup N(A)]$ and G_3 be the subgraph of G' induced by the good components. Let $\alpha' = \alpha(G')$ and for $i = 0, 3$, let $\alpha_i = \alpha(G_i)$. Write $n = |G|$, $n_0 = |G_0|$, $n' = |G'|$. Then

$$(2 - c)\alpha_0\eta \geq n_0 \quad \text{and} \quad n_0 + n' = n > (2 - c)\alpha\eta. \tag{29}$$

A family $\mathcal{B}$ of bad components of G' is *bearable*, if $\sum_{B \in \mathcal{B}} |B| \leq \frac{7}{3}|\mathcal{B}|\eta$. Since each bad component in G' has independence number 2, this is equivalent to

$$\sum_{B \in \mathcal{B}} |B| \leq \frac{7}{6}\alpha(G[\mathcal{B}])\eta.$$

Recall that $|F| \leq \eta\alpha(F)$ for each good component F of G'. Let α_4 be the size of a maximum independent set of a bearable family $\mathcal{B}$ of bad components of G'. Then

$$|n'| \leq \alpha_3\eta + \frac{7}{6}\alpha_4\eta + \frac{3}{2}(\alpha' - \alpha_3 - \alpha_4)\eta = \left(\frac{3}{2}\alpha' - \frac{1}{2}\alpha_3 - \frac{1}{3}\alpha_4\right)\eta.$$

Together with (29) and the fact that G is a counter-example, we obtain

$$(2 - c)\alpha_0 + \frac{3}{2}\alpha' - \frac{1}{2}\alpha_3 - \frac{1}{3}\alpha_4 \geq (2 - c)\alpha. \tag{30}$$

Let $y := \alpha - 3m - \alpha'$. Then (30) can be rewritten as

$$(2 - c)\alpha_0 \geq \frac{1 - 2c}{2}\alpha + 4.5m + 1.5y + 0.5\alpha_3 + \frac{1}{3}\alpha_4. \tag{31}$$

Let M be a largest matching between I and a maximum independent set I'in $G' - G_3$ (since all components of $G' - G_3$ are bad, I' contains two vertices in each such component). By König–Egerváry Theorem, $|M| \geq \alpha_0 + \alpha' - \alpha - \alpha_3$. By (30) and the definition of y, we infer that

$$|M| \geq \frac{(1 - 2c)\alpha'}{2(2 - c)} - \frac{(3 - 2c)\alpha_3}{2(2 - c)} + \frac{\alpha_4}{3(2 - c)} = \frac{(1 - 2c)(\alpha - 3m - y)}{2(2 - c)} - \frac{(3 - 2c)\alpha_3}{2(2 - c)} + \frac{\alpha_4}{3(2 - c)}. \tag{32}$$

Let H be the auxiliary bipartite (multi)graph such that one partite set of H is I, the vertices of the other partite set, call it T, are the bad components of G', and the edges of H are defined as follows: if $v \in I$ is adjacent in G to two non-adjacent vertices in a component $W \in T$, then in H we draw two edges connecting v with W, and if $N_G(v) \cap W$ is a non-empty clique in G, then in H we draw one edge connecting v with W.

Let F be a maximum matching in H. Since each $W \in T$ was incident with at most two edges of M, by (32) we have

$$|F| \geq \frac{|M|}{2} \geq \frac{(1 - 2c)(\alpha - 3m - y) - (3 - 2c)\alpha_3 + 2\alpha_4/3}{4(2 - c)}. \tag{33}$$

Consider the following procedure. Let $H_0 := H$.

Step h, $h \geq 1$: If $d_{H_{h-1}}(v) \leq 1$ for each $v \in I \cap V(H_{h-1})$, then stop and let $b := h - 1$. Otherwise,

(a) choose some $v \in I \cap V(H_{h-1})$ with $d_{H_{h-1}}(v) \geq 2$ and call it v_h;
(b) let $H_h := H_{h-1} - N_{H_{h-1}}(v_h)$;
(c) go to Step $h + 1$.

Let $\widetilde{G}$ (respectively, $\widetilde{G'}$) be the graph obtained from G (respectively, G') by deleting all the components of G' in $\bigcup_{h=1}^{b} N_H(v_h)$. By the construction,

$$\text{for each } w \in I,\ N(w) \cap V(\widetilde{G'}) \text{ is a clique.} \tag{34}$$

Let $\widetilde{F} = F \cap E(H_b)$ and $x := |\bigcup_{h=1}^{b} N_H(v_h)| - b$. Since $|N_{H_{h-1}}(v_h)| \geq 2$ for each h,

$$x \geq b. \tag{35}$$

Then

$$|\widetilde{F}| \geq |F| - b - x \geq |F| - 2x. \tag{36}$$

Since $d_{H_b}(v) \leq 1$ for every $v \in I$, H_b is the union of stars with centers in T. Thus, we can construct $\widetilde{F}$ by choosing any edge at each vertex in $T \cap V(H_b)$. So we will choose $\widetilde{F}$

with the fewest edges incident with vertices of I in neutral components. (37)

Our second construction of a set with a large potential is as follows. We start from the set $P_0 := A$ of m claws and for $h = 1, \ldots, b$, let P_h be obtained from P_{h-1} by adding the vertex v_h (from the definition of H_h) and a maximum independent set in $G[N_G(v_h) \cap \bigcup_{C \in N_{H_{h-1}}(v_h)} C]$. The last set P_b is our second set R_2. At each step h, we

(a) add $1 + d_{H_{h-1}}(v_h)$ vertices,
(b) do not increase the number of components of the induced subgraph, and
(c) increase the maximum independent set by $d_{H_{h-1}}(v_h)$.

Hence

$$\phi(R_2) - \phi(A) = \phi(P_b) - \phi(P_0) = \sum_{h=1}^{b}(2d_{H_{h-1}}(v_h) - (1 + d_{H_{h-1}}(v_h))) = \sum_{h=1}^{b}(d_{H_{h-1}}(v_h) - 1) = x.$$

It follows that

$$m + x = \phi(A) + (\phi(R_2) - \phi(A)) = \phi(R_2) < 3c\alpha/2. \tag{38}$$

5.3. Third attempt

Lemma 5.1. *Let D be the vertex set of a component of $G[A \cup I]$ that contains exactly h claws. Then $I \cap D$ is incident with at most h edges in $\widetilde{F}$.*

Proof. Suppose that $I \cap D$ is incident with $h + 1$ edges $w_1W_1, \ldots, w_{h+1}W_{h+1}$ in H_b. By Lemma 4.2, for $j = 1, \ldots, h + 1$, there are $u_j, u_j', u_j'' \in W_j$ such that $G[\{w_j, u_j, u_j', u_j''\}]$ is a claw with center u_j. By (34), for all $j \neq j'$, w_j does not have neighbors in $W_{j'}$. Hence replacing in A the h claws of D with the new $h + 1$ claws we get a contradiction to the maximality of A. □

An immediate consequence of Lemma 5.1 is

Corollary 5.1. $|\widetilde{F}| \leq m$. □

A neutral component D of $G[A \cup I]$ is *h-weak*, if

(i) D contains exactly h claws in A;
(ii) D is incident with exactly h edges in $\widetilde{F}$;
(iii) if $B_1, \ldots, B_h$ are the bad components of G' connected by edges in $\widetilde{F}$ with D, then $D \cup \bigcup_{j=1}^{h} B_j$ does not contain a set of potential $h + 1$.

We call a component *weak* if it is h-weak for some $h \geq 2$.

Lemma 5.2. *Let D be the vertex set of an h-weak component of $\widetilde{G}[A \cup I]$, and $B_1, \ldots, B_h$ be the bad components of $\widetilde{G}$ connected by edges in $\widetilde{F}$ with D. Then the family $\mathcal{B} := \{B_1, \ldots, B_h\}$ is bearable.*

Proof. Let $I_D := I \cap D$. Since D is neutral, $|I_D| \leq 5h + 1$. Let $C_1, \ldots, C_h$ be the claws of A contained in D. Since we are in $\widetilde{G}$,

each $v \in I_D$ has neighbors (necessarily forming a clique) in at most one B_j. (39)

Since $h \geq 2$, there is a vertex $v_D \in I_D$ adjacent to at least two distinct claws, say to C_1 and C_2. We claim that

$$v_D \text{ has no neighbors in } \bigcup_{j=1}^{h} B_j. \tag{40}$$

Indeed, if v_D is adjacent to $w \in \bigcup_{j=1}^{h} B_j$, then the set $(A \cap D) \cup \{v_D, w\}$ has $4h + 2$ vertices, at most $h - 1$ components and independence number $3h + 1$; so it has potential at least $h + 1$, a contradiction to (iii) from the definition of h-weak components.

Suppose that for $j = 1, \ldots, h$, component B_j has b_j neighbors in D. By (39) and (40),

$$\sum_{j=1}^{h} b_j \leq 5h. \tag{41}$$

By Lemma 4.3, $b_j \geq 3$ for all j. So, by Lemma 4.5, for $j = 1, \dots, h$, $|B_j| \leq \frac{3(b_j+2)}{b_j+4}\eta$. It follows, using (41), that

$$\sum_{j=1}^{h} |B_j| \leq \eta \left(3h - 6\sum_{j=1}^{h} \frac{1}{b_j+4}\right) \leq \eta \left(3h - 6\sum_{j=1}^{h} \frac{1}{5+4}\right) = \frac{7h}{3}\eta.$$

This proves the lemma. □

Suppose that exactly x' edges of $\widetilde{F}$ are incident with weak components of $G[A \cup I]$. The immediate consequence of Lemma 5.2 is

Corollary 5.2. $\alpha_4 \geq 2x'$.

Our third attempt to construct a set of large potential starts from $A \cup I$ and we compare the potential of the construction with $|I \cap D| - 5h$. The procedure is that we replace the negative components in $G[A \cup I]$ with the original claws, and then modify 1-weak and neutral non-weak components by deleting some vertices from them and/or adding some vertices from the bad components of G' adjacent to them via edges in $\widetilde{F}$ in order to increase their potential. Since distinct edges in $\widetilde{F}$ connect I to different components in G', there will be no conflict. The resulting set is our third set R_3.

Observe first that if $G[A \cup I]$ has exactly s components, then

$$\begin{aligned}\phi(A \cup I) &= 2|I| - |A \cup I| - s = |I| - |A| + |A \cap I| - s \\ &= |I| - 4m + |A \cap I| - s = \alpha_0 - 5m + (m - s) + |A \cap I|.\end{aligned} \tag{42}$$

We view $\alpha_0 - 5m$ as $\sum_{h=1}^{\infty} \sum_{D \in \mathcal{D}_h} (|I \cap D| - 5h)$, and will count, how large in comparison with $|I \cap D| - 5h$ can we make the potential of a component $D \in \mathcal{D}_h$.

Lemma 5.3. *Let $h \geq 1$ and $D \in \mathcal{D}_h$. Let $B_1, \dots, B_{t(D)}$ be the bad components in G' adjacent via edges in $\widetilde{F}$ to D. Then $\bigcup_{j=1}^{t(D)} B_j \cup D$ contains a vertex set X_D of potential at least*

(a) $|I \cap D| - 5h + (t(D) - 1)$ *if D is positive or weak;*
(b) $|I \cap D| - 5h + t(D)$ *if D is negative or neutral but not weak.*

Proof. Since $D \in \mathcal{D}_h$,

$$\phi(D) = 2|I \cap D| - |D| - 1 \geq |I \cap D| - 4h - 1 = (|I \cap D| - 5h) + (h - 1). \tag{43}$$

By Lemma 5.1, $t(D) \leq h$. This already implies (a).

Suppose D is negative. By (43), $\phi(D) \geq (|I \cap D| - 5h) + (h - 1)$. Then by the definition of negative components, $\phi(D - I) \geq \phi(D) + 1 \geq (|I \cap D| - 5h) + h$. Since $t(D) \leq h$, we obtain (b) for negative components.

Finally, suppose that D is neutral but not weak. If $t(D) \leq h-1$, then the lemma holds by (43). Otherwise, by the definition of weak components, there exists a set D' of potential $h + 1$ contained in $D \cup \bigcup_{j=1}^{t(D)} B_j$, where $B_1, \dots, B_{t(D)}$ are the bad components of G' connected by edges in $\widetilde{F}$ with D. In this case, we replace D with this D'. □

The following lemma has a long proof which is deferred to the final section.

Lemma 5.4. *Let D be a component in $A \cup I$ with $\phi(D) = 1$, $h(D) = 1$, $A \cap D \cap I = \emptyset$ such that D is incident with an edge in $\widetilde{F}$. Let B be the bad component incident with this edge. Then there is a $D' \subset D \cup B$ with $\phi(D') \geq 2$.*

Recall that x' was defined as the number of edges in $\widetilde{F}$ incident with weak components. Let x^+ denote the number of edges in $\widetilde{F}$ incident with positive components, and let $x^- = |\widetilde{F}| - x' - x^+$ denote the number of edges in $\widetilde{F}$ incident with negative or neutral non-weak or 1-weak components. Let $\mathcal{D}^w$ denote the family of weak components and $\mathcal{D}_0^+$ denote the set of positive components that are incident with at least one edge in $\widetilde{F}$. The last two lemmas imply the following.

Lemma 5.5. *There exists a set R_3 of potential at least $X := \alpha_0 - 5m + x^- + 0.5x' + x^+ - |\mathcal{D}_0^+|$.*

Proof. Let R_3 be the union of sets guaranteed by the last two lemmas for components of $G[A \cup I]$. By Lemmas 5.3(b) and 5.4, negative and neutral non-weak and 1-weak components contribute to $X - (\alpha_0 - 5m)$ at least x^-. Since each weak component is h-weak for some $h \geq 2$, by Lemma 5.3(a), the weak components contribute to $X - (\alpha_0 - 5m)$ at least $x' - |\mathcal{D}^w| \geq x'/2$. Finally, again by Lemma 5.3(a), the positive components contribute to $X - (\alpha_0 - 5m)$ at least $x^+ - |\mathcal{D}_0^+|$. This proves the lemma. □

Since $x^+ - |\mathcal{D}_0^+| \geq 0$, by Lemma 5.5 we have

$$\frac{3}{2}c\alpha - (\alpha_0 - 5m) \geq \frac{2}{3}(x^- + (x^+ - |\mathcal{D}_0^+|)) + \frac{1}{2}x' = \frac{2}{3}(x^- + x^+ - |\mathcal{D}_0^+| + x') - \frac{1}{6}x'.$$

Since $x^- + x' + x^+ = |\widetilde{F}|$, we conclude that $\frac{3}{2}c\alpha - (\alpha_0 - 5m) \geq \frac{2}{3}(|\widetilde{F}| - |\mathcal{D}_0^+|) - \frac{1}{6}x'$. Since every positive component contributes at least 1 to $\widetilde{f}$, we have $\widetilde{f} \geq |\mathcal{D}^+|$. Thus,

$$\alpha_0 - 5m + \frac{2}{3}(|\widetilde{F}| - \widetilde{f}) - \frac{1}{6}x' < \frac{3}{2}c\alpha. \tag{44}$$

6. Final computation

We start from (44). Plugging in the bound for α_0 from (31) and using Corollary 5.2 to exclude α_4, we have

$$\frac{(1-2c)\alpha+9m+3y+\alpha_3+\frac{4}{3}x'}{2(2-c)}+\frac{2}{3}|\widetilde{F}|<\frac{2}{3}\widetilde{f}+x'/6+\frac{3c\alpha}{2}+5m.$$

Using (36) and (33),

$$\frac{(1-2c)\alpha+9m+3y+\alpha_3+\frac{4}{3}x'}{2(2-c)}+\frac{(1-2c)(\alpha-3m-y)-(3-2c)\alpha_3+4x'/3}{6(2-c)}-\frac{4x}{3}$$
$$<\frac{2}{3}\widetilde{f}+x'/6+\frac{3c\alpha}{2}+5m.$$

Simplifying and moving m and x to the right, we have

$$\frac{4(1-2c)\alpha+(8+2c)y+2c\alpha_3+(10/3+c)x'}{6(2-c)}<\frac{4x}{3}+\frac{2}{3}\widetilde{f}+\frac{3c\alpha}{2}+5m+m\frac{-27+3(1-2c)}{6(2-c)}.$$

By (38) and (28), the RHS is at most

$$\frac{3c\alpha}{2}+\left(\frac{4x}{3}+\frac{2}{3}\widetilde{f}+2m\right)+m\frac{12-24c}{6(2-c)}\le\frac{9}{2}c\alpha+\frac{(12-24c)(1.5c\alpha-x)}{6(2-c)},$$

so moving everything to the left hand side and multiplying by $6(2-c)$ we get

$$\left((4-8c)-\frac{9c}{2}6(2-c)-\frac{3c}{2}(12-24c)\right)\alpha+(8+2c)y+2c\alpha_3+\left(\frac{10}{3}+c\right)x'+(12-24c)x<0. \tag{45}$$

The coefficient at α is $4-80c+63c^2$. Since the coefficients at y, x', α_3 and x are positive, for (45) to hold, the coefficient at α must be negative. In other words, $4-80c+63c^2<0$. But this inequality does not hold for $c=(80-\sqrt{5392})/126>1/19.2$.

7. Proof of Lemma 5.4

First we will characterize the components D satisfying the conditions of the lemma. Let D be a component containing only one claw of A and $I_D=I\cap D-A$. Label the vertices of D as follows: v is the root of A, u_1,u_2,u_3 are the leaves of A and $w_1,w_2,\dots$ are the vertices in I_D. First we show that $|I_D|=\alpha(D)=6$:

If $\alpha(D)\le 5$ then $v(D)=2\alpha(D)-\phi(D)-c(D)=2(\alpha(D)-1)\le\alpha(D)+3$ so $|I_D\cap A|>0$, which contradicts our assumption that $I_D\cap A=\emptyset$.

If $\alpha(D)\ge 7$ then $v(D)=2\alpha(D)-\phi(D)-c(D)=2(\alpha(D)-1)>\alpha(D)+4$ so $|I_D|>\alpha(D)$, which is not possible.

Thus, $\alpha(D)=6$. Then $1=\phi(D)=2\cdot 6-4-|I_D|-1$, so $|I_D|=6$. Observe the following:

(i) *For every i, $|N(u_i)\cap I_D|\ge 2$. Similarly $|N(v)\cap I_D|\ge 2$.* Otherwise if w_j is the unique neighbor of u_i or of v, then $\alpha(D-w_j)=6$, so $\phi(D-w_j)\ge 2$, a contradiction.
(ii) *For every i there is a j such that $N(w_j)\cap D=\{u_i\}$.* Otherwise $D-u_i$ is connected, so its potential is at least 2.
(iii) *$D-v$ is not connected, otherwise* $\phi(D-v)\ge 2$.
(iv) *For every i, $|N(u_i)\cap I_D|\le 3$*, otherwise $\phi(N(u_i)\cup\{u_i\})\ge 2$.
(v) *For every j there is an i such that $w_ju_i\in E(G)$.* Otherwise $\phi(\{v,u_1,u_2,u_3,w_j\})=2$.

By (i)–(v), every u_i has 2 or 3 neighbors in $D-A$. We may assume w.l.o.g. that $u_iw_i\in E(G)$ for every $i=1,2,3$, and w_i is not adjacent to u_j for $i\ne j$ (by (ii)). We consider four cases:

Case 1: $|N(u_1)\cap I_D|=|N(u_2)\cap I_D|=|N(u_3)\cap I_D|=2$. In this case, there are two possibilities for D, see Fig. 1.

Case 2: $|N(u_1)\cap I_D|=3$, $|N(u_2)\cap I_D|=|N(u_3)\cap I_D|=2$. W.l.o.g., we may assume that $u_1w_4,u_1w_5,u_3w_6\in E(G)$.

Assume first that $u_2w_6\in E(G)$. At least one of vw_4, vw_5 is an edge, but not both; otherwise $\phi(v,w_4,w_5,u_2,u_3)=2$. By (i), vw_6 is an edge, hence up to symmetry the only possible D in this case is on the right-hand side of Fig. 2.

Assume now that $u_2w_5\in E(G)$. Then by (i), v has at least two neighbors among w_4,w_5,w_6, see the left-hand side of Fig. 2.

Case 3: $|N(u_1)\cap I_D|=|N(u_2)\cap I_D|=3$, $|N(u_3)\cap I_D|=2$. W.l.o.g., we may assume that $u_1w_4,u_1w_5,u_3w_6\in E(G)$. If $u_2w_6\in E(G)$ then (iii) is violated, so $u_2w_4,u_2w_5\in E(G)$. W.l.o.g. $vw_4\in E(G)$. Additionally, v has at least one neighbor in $\{w_5,w_6\}$, see Fig. 3.

Case 4: $|N(u_1)\cap I_D|=|N(u_2)\cap I_D|=|N(u_3)\cap I_D|=3$. This is not possible, as (iii) is violated.

Observation 1. Let B be a bad component of G'. Assume that $vw_i\in E(G)$, $N(w_i)\cap\{u_1,u_2,u_3\}=\{u_\ell\}$ for some ℓ and $N(w_i)\cap B\ne\emptyset$. Then by Lemma 4.2 there exist $q,p_1,p_2\in B$ such that $w_iq,qp_1,qp_2\in E(G)$, and $p_1p_2\notin E(G)$. Therefore $\{u_1,u_2,u_3\}-\{u_\ell\}$ together with $\{v,w_i,x,z_1,z_2\}$ induces a graph with potential 2. So we shall assume that this does not happen.

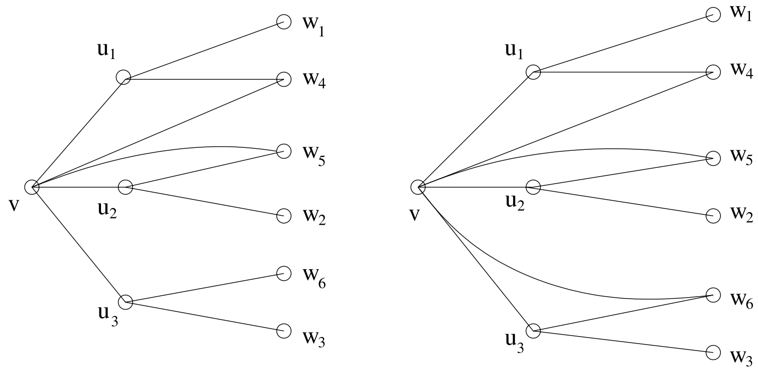

Fig. 1. $\alpha(D) = 6$, degree sequence 2, 2, 2.

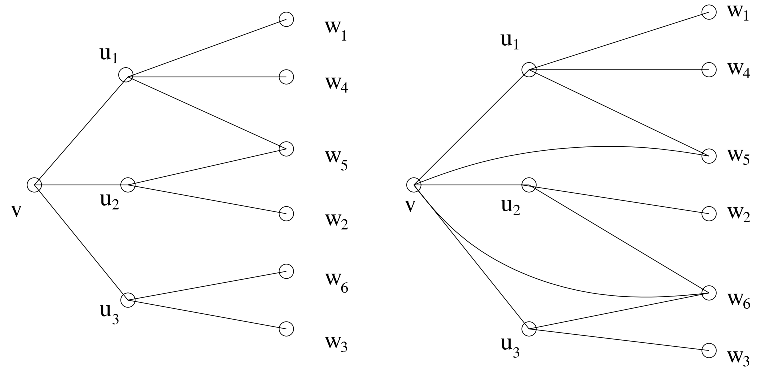

Fig. 2. $\alpha(D) = 6$, degree sequence 3, 2, 2. On the left-hand side, v is adjacent to at least two of w_4, w_5, w_6.

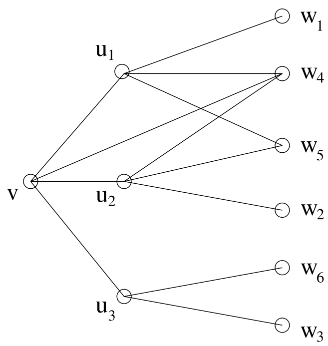

Fig. 3. $\alpha(D) = 6$, degree sequence 3, 3, 2; v is adjacent to at least one of w_5 and w_6.

In the rest of the proof we consider all the graphs listed in the figures. Our strategy will be to check if Lemmas 4.2 and 4.3 could be applied, i.e. we try to find a $W \subset D$ containing at most two w_i's with edges to B. If such a W was found, the proof could be completed.

Right-hand side of Fig. 1: By Observation 1, we have to check only $i \in \{1, 2, 3\}$, and by symmetry we may assume that $i = 1$. Then $W = \{v, u_1, u_2, u_3, w_1, w_4\}$ works.

Left-hand side of Fig. 1: By Observation 1, we have to check only $i \in \{1, 2, 3, 6\}$. The proof is exactly the same as in the previous case.

Right-hand side of Fig. 2: By Observation 1, we have $i \neq 5$. If $i = 1$ or $i = 4$, then $W = \{u_1, w_1, w_4\}$ works. If any of w_3 or w_6 has an edge to B, and none of w_1, w_4, w_5 does, then $W = \{v, u_1, u_3, w_1, w_4, w_5, w_3, w_6\}$ works.

Left-hand side of Fig. 2: Here we have four graphs to consider, and more or less the same argument works for all. If $i = 3$ or $i = 6$ then we take the set $W = \{v, u_1, u_2, u_3, w_3, w_6\}$. Now assume that neither w_3, nor w_6 is adjacent to G', so we have to check the case that at least three of the other four w_j are. If any of w_1, w_4, w_5 is not adjacent to B, then $W = \{u_1, w_1, w_4, w_5\}$ works. If each of w_1, w_4, w_5 is adjacent to B and $vw_4 \in E(G)$, then by Observation 1 we are done. If $vw_4 \notin E(G)$ then $vw_4, vw_5 \in E(G)$. In this final case we set $W = \{v, u_1, w_1, w_4\}$.

Fig. 3: Here we have three graphs to check, the same argument works for all. If $i = 3$ or $i = 6$ then we take the set $W = \{v, u_1, u_2, u_3, w_3, w_6\}$. If at least one and at most two of w_1, w_4, w_5 are adjacent to B, then $D' = \{u_1, w_1, w_4, w_5\}$ works for us. Similar statement holds for w_2, w_4, w_5. The remaining case is that each of w_1, w_2, w_4, w_5 is adjacent to B. If $vw_5 \in E(G)$ then $W = \{v, u_3, w_4, w_5\}$ works, and if $vw_5 \notin E(G)$ then we can choose $W = \{v, u_2, w_2, w_5\}$. □

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