Discrete Mathematics 311 (2011) 996–1005

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Ohba's conjecture for graphs with independence number five

Alexandr V. Kostochka [a,b], Michael Stiebitz [c], Douglas R. Woodall [d,*]

[a] Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA
[b] Sobolev Institute of Mathematics, Novosibirsk, Russia
[c] Institute of Mathematics, Technische Universität Ilmenau, D-98684 Ilmenau, Germany
[d] School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD, UK

ARTICLE INFO

Article history:
Received 27 July 2010
Received in revised form 31 January 2011
Accepted 22 February 2011
Available online 29 March 2011

Keywords:
Chromatic number
Vertex coloring
List coloring
List chromatic number
Choosability
Complete multipartite graph

ABSTRACT

Ohba has conjectured that if G is a k-chromatic graph with at most $2k+1$ vertices, then the list chromatic number or choosability $\mathrm{ch}(G)$ of G is equal to its chromatic number $\chi(G)$, which is k. It is known that this holds if G has independence number at most three. It is proved here that it holds if G has independence number at most five. In particular, and equivalently, it holds if G is a complete k-partite graph and each part has at most five vertices.

1. Introduction

Let G be a graph with vertex-set $V(G)$, chromatic number $\chi(G)$ and choosability (or list chromatic number) $\mathrm{ch}(G)$. Ohba [6] made the following conjecture.

Ohba's Conjecture. *If* $|V(G)| \leqslant 2\chi(G)+1$, *then* $\mathrm{ch}(G) = \chi(G)$.

Enomoto et al. [1] showed that the complete k-partite graph $K(4, 2, \ldots, 2)$ is not k-choosable if k is even, and so the upper bound on $|V(G)|$ in Ohba's conjecture would be sharp. The following weaker results are known.

Theorem A. *Let G be a graph. Then $\mathrm{ch}(G) = \chi(G)$ in the following cases:*

(i) $|V(G)| \leqslant \chi(G) + \sqrt{2\chi(G)}$ [6].
(ii) $|V(G)| \leqslant (2-\epsilon)\chi(G)$ $(0 < \epsilon < 1, |V(G)| \geqslant n_0(\epsilon))$ [8].
(iii) $|V(G)| \leqslant \frac{5}{3}\chi(G) - \frac{4}{3}$ [9].

Because every χ-chromatic graph is a subgraph of a complete χ-partite graph, Ohba's conjecture is true if and only if it is true for complete χ-partite graphs. It also suffices to prove it for graphs with the maximum number of vertices. It can thus be rephrased as follows.

* Corresponding author.
E-mail addresses: kostochk@math.uiuc.edu (A.V. Kostochka), michael.stiebitz@tu-ilmenau.de (M. Stiebitz), douglas.woodall@nottingham.ac.uk (D.R. Woodall).

0012-365X/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2011.02.026

Ohba's Conjecture Rephrased. *If G is a complete k-partite graph and $|V(G)| = 2k+1$, then $\mathrm{ch}(G) = \chi(G) = k$.*

In the following theorem the number of vertices is not necessarily equal to $2k+1$. Strings of the form $x, \ldots, x$ may be empty, provided that k is large enough (at least 1, unless otherwise stated), and $x * t$ denotes a string of t x's.

Theorem B. *Let G be any of the following complete k-partite graphs. Then* $\mathrm{ch}(G) = \chi(G) = k$.

(i) $K(2, \ldots, 2)$ [2].
(ii) $K(3, \ldots, 3, 2, \ldots, 2, 1, \ldots, 1)$ ($|V(G)| \leqslant 2k$) [7].
(iii) $K(3, 2, \ldots, 2, 1, \ldots, 1)$ [3].
(iv) $K(3, 3, 2, \ldots, 2, 1, \ldots, 1)$ ($k \geqslant 3$) [3].
(v) $K(4, 2, \ldots, 2)$ (k odd) [1].
(vi) $K(t+2, 2, \ldots, 2, 1*t)$ ($t \geqslant 0$) [6].
(vii) $K(r, 2, \ldots, 2, 1*t)$ ($t \geqslant 1, 2 \leqslant r \leqslant 2t+1$) [1].

Ohba's conjecture itself has been proved in the following cases.

Theorem C. *Let G be any of the following complete k-partite graphs of order $2k+1$. Then $\mathrm{ch}(G) = \chi(G) = k$.*

(i) $K(t+3, 2*s, 1*t)$ $(s \geqslant 0,\ t \geqslant 0,\ k = s+t+1)$.
(ii) $K(t+2, 3, 2*s, 1*t)$ $(s \geqslant 0,\ 0 \leqslant t \leqslant 4,\ k = s+t+2)$.
(iii) $K(3*(t+1), 2*s, 1*t)$ $(s \geqslant 0,\ t \geqslant 0,\ k = s+2t+1)$.
(iv) $K(4, 3, 3, 2*s, 1, 1, 1)$ $(s \geqslant 0,\ k = s+6)$.

Part (i) of Theorem C follows from Theorem B(iii) if $t = 0$, and it was proved by Enomoto et al. [1] for $t \geqslant 1$; for $t \geqslant 2$ it follows from Theorem B(vii). Part (ii) of Theorem C follows from part (i) if $t = 0$ and from Theorem B(iv) if $t = 1$; it was proved by Shen et al. [11] for $t = 2, 3$, and by Shen et al. [12] for $t = 4$. Part (iii) is the same as part (i) or part (ii) if $t = 0$ or 1, respectively; it was proved by He et al. [5] for $t = 2$, and by Shen et al. [10] in general. Part (iv) was proved by He et al. [5].

Theorem C(iii) implies that Ohba's conjecture holds for graphs with independence number at most three. The main result of this paper is the following improvement of this; it implies that Ohba's conjecture holds for graphs with independence number at most five.

Theorem 1. *Let $G = (V, E)$ be a complete k-partite graph such that $|V| \leqslant 2k+1$ and every part has at most five vertices. Then* $\mathrm{ch}(G) = \chi(G) = k$.

As we will explain in the next section, the method that we use to prove Theorem 1 is somewhat different from the methods used in most published proofs of parts of Theorem C. We were initially hopeful that our method could be used to prove the whole of Ohba's conjecture. However, we have not succeeded in finding a construction that will achieve this.

A *k-list-assignment* L to a graph G is an assignment of a list $L(v)$ of exactly k colors to each vertex v of G. An *L-coloring* of G is a proper coloring in which each vertex v is colored with a color from its own list $L(v)$. If G is L-colorable for every k-list-assignment L to G, then G is called *k-choosable*. The *choosability* $\mathrm{ch}(G)$ of G is the smallest k for which G is k-choosable.

The rest of this paper is devoted to a proof of Theorem 1.

2. Proof of Theorem 1

Let $G = (V, E)$ be a complete k-partite graph such that $|V| \leqslant 2k+1$ and every part has at most five vertices. Clearly $\mathrm{ch}(G) \geqslant \chi(G) = k$, and so it suffices to prove that $\mathrm{ch}(G) \leqslant k$, that is, G is k-choosable. Let m be a positive integer that is at least as large as the order of the largest part of G. We may assume that G has a part with at least three vertices, so that $m \geqslant 3$, since otherwise G is an induced subgraph of the complete k-partite graph with k parts of order 2, which is k-choosable by Theorem B(i). Let G have k_i parts of order i, for each $i \geqslant 1$, and let $k_0 := \sum_{i=2}^{m} k_i$, so that $k = k_0 + k_1$ and $|V| = \sum_{i=1}^{m} ik_i = k + \sum_{i=2}^{m}(i-1)k_i \leqslant 2k+1$. Let the parts of G be $U_1, \ldots, U_{k_1}$ of order 1 and $V_1, \ldots, V_{k_0}$ of order at least 2. Let L be a k-list-assignment to G. For a set $X \subseteq V$, let $L^{\cap}(X) := \bigcap_{v \in X} L(v)$. We wish to prove that G is L-colorable. By a simple inductive argument we may assume that

$$L^{\cap}(V_p) = \emptyset \quad \text{for each } p \in \{1, \ldots, k_0\}. \tag{1}$$

The strategy of the proof is as follows. In contrast with the type of coloring argument used in most other proofs of similar results, we will construct directly a partition $\mathcal{Q} = (X_1, \ldots, X_q)$ of V such that each part of $\mathcal{Q}$ induces an independent subset of V, and we will then prove that G has an L-coloring in which two vertices have the same color if and only if they are in the same part X_i. This second stage is equivalent to proving that the family of sets $(L^{\cap}(X_1), \ldots, L^{\cap}(X_q))$ has a system of distinct representatives $(c_1, \ldots, c_q)$; we can then use color c_i on all vertices of X_i. The proof is divided into three parts: in Part 1 we define $\mathcal{Q}$, and in Part 2 we prove various lemmas, which we use in Part 3 to prove that the above family of sets satisfies Hall's condition and so has a system of distinct representatives. (A family of sets $\mathcal{S} = (S_1, \ldots, S_q)$ is said to satisfy *Hall's condition* if, for every subfamily $(S_{i_1}, \ldots, S_{i_r})$ of $\mathcal{S}$, $|S_{i_1} \cup \ldots \cup S_{i_r}| \geqslant r$. Hall [4] proved that this is a necessary and sufficient condition for there to exist a *system of distinct representatives* of $\mathcal{S}$, that is, a set of distinct elements $c_1, \ldots, c_q$ such that $c_i \in S_i$ for each i $(1 \leqslant i \leqslant q)$.)

We will not use the value of m, the upper limit on the order of a partite set, until after Claim 4 in Part 3, although to cope with larger values of m would require additional or more general lemmas in Part 2. However, we have not succeeded in proving the result even for $m = 6$, and we are doubtful whether this can be done with our present definition of $\mathcal{Q}$. Perhaps there is a variation of our construction that can be used to prove the result for all values of m, but we have not managed to find one.

Part 1. *Construction of the partition* $\mathcal{Q}$. For a partition $\mathcal{P}_t(p) = (X_1, \ldots, X_t)$ of a partite set V_p $(p \in \{1, \ldots, k_0\})$, let $f(\mathcal{P}_t(p)) := \sum_{i=1}^{t} |L^{\cap}(X_i)|$ and

$$g(\mathcal{P}_t(p)) := k + \sum_{i=1}^{t}(|L^{\cap}(X_i)| - k) = f(\mathcal{P}_t(p)) - (t-1)k. \tag{2}$$

Thus if $|V_p| = s$, so that $\mathcal{P}_s(p)$ is the partition of V_p into singletons, then $g(\mathcal{P}_s(p)) = k$; and if $\mathcal{P}_{s-1}(p)$ is a partition of V_p into $s-2$ singletons and a 2-set $X = \{x, y\}$, then $g(\mathcal{P}_{s-1}(p)) = |L^{\cap}(X)| = |L(x) \cap L(y)|$. For each $p \in \{1, \ldots, k_0\}$, let $s := |V_p|$ and define partitions $\mathcal{Q}_s(p), \ldots, \mathcal{Q}_2(p)$ recursively as follows. Let $\mathcal{Q}_s(p)$ be the partition of V_p into singletons. For $t = s-1, \ldots, 2$, let $\mathcal{Q}_t(p)$ be formed from $\mathcal{Q}_{t+1}(p)$ by merging two members so that $g(\mathcal{Q}_t(p))$ is as large as possible. In particular, $\mathcal{Q}_{s-1}(p)$ is obtained from $\mathcal{Q}_s(p)$ by merging two vertices having the maximum number of common colors in their lists.

Before defining the partition $\mathcal{Q}$, it will be helpful to explain our terminology. For $2 \leqslant t \leqslant s \leqslant m$, ${}^s\mathcal{A}_t$ will denote the set of parts V_p of G of order s that are divided into t parts in $\mathcal{Q}$, and $({}^s a_t) := |{}^s\mathcal{A}_t|$. (In fact, after specifying the sets ${}^s\mathcal{A}_t$, we will obtain $\mathcal{Q}$ from the natural partition $(U_1, \ldots, U_{k_1}, V_1, \ldots, V_{k_0})$ of V by replacing V_p with $\mathcal{Q}_t(p)$ for each set $V_p \in {}^s\mathcal{A}_t$; thus $\mathcal{Q}$ will be completely determined by the sets ${}^s\mathcal{A}_t$.) For $0 \leqslant q \leqslant m-2$, let $\mathcal{I}_q := \{(s,t) : 2 \leqslant t \leqslant m-q \text{ and } s-t = q\}$, $\mathcal{B}_q := \bigcup_{(s,t)\in\mathcal{I}_q} {}^s\mathcal{A}_t$ and $b_q := |\mathcal{B}_q| = \sum_{(s,t)\in\mathcal{I}_q} ({}^s a_t)$. For $0 \leqslant q \leqslant m-1$ and $2 \leqslant t \leqslant s \leqslant m$, let

$$\Theta_q := \sum_{i=q}^{m-2} b_i, \tag{3}$$

$${}^s\Omega_2^0 := {}^s\Omega_2^1 := \Theta_{s-1} + ({}^s a_2), \tag{4}$$

and, if $t \geqslant 3$,

$${}^s\Omega_t^0 := \Theta_{s-t} \quad \text{and} \quad {}^s\Omega_t^1 := \sum_{i=2}^{t} {}^s\Omega_i^0 = {}^s\Omega_2^0 + \sum_{j=s-t}^{s-3} \Theta_j. \tag{5}$$

We define ${}^s\Omega_1^1 := 0$, so that if $q = s-t$ then, by (4) and (5),

$${}^s\Omega_t^1 - {}^s\Omega_{t-1}^1 = {}^s\Omega_t^0 = \begin{cases} \Theta_{q+1} + ({}^s a_2) & \text{if } t = 2, \\ \Theta_q = \Theta_{q+1} + b_q & \text{if } t \geqslant 3. \end{cases} \tag{6}$$

Note that $\Theta_{m-1} = 0$ by (3), and so, by (4), (3) and the definition of b_q,

$${}^m\Omega_2^0 = {}^m\Omega_2^1 = ({}^m a_2) = b_{m-2} = \Theta_{m-2}. \tag{7}$$

Historical note. The main significance of ${}^s\Omega_t^0$ and ${}^s\Omega_t^1$ is that they are the threshold values of p and $g(\mathcal{Q}_t(p))$. respectively, between sets $V_p \in {}^s\mathcal{A}_t$ and sets $V_p \in {}^s\mathcal{A}_r$ for $r > t$; see Eqs. (13)–(15) below. In an earlier and simpler version of the proof, the case $t = 2$ in the construction of $\mathcal{Q}$ was treated in the same way as all other values of t. In that case there was no need to introduce ${}^s\Omega_t^0$ because it was always equal to Θ_q, where $q = s-t$ and

$$\Theta_q = b_{m-2} + b_{m-3} + \cdots + b_{q+1} + b_q \tag{8}$$

by (3); and ${}^s\Omega_t^1$ was always equal to $\sum_{j=q}^{s-2} \Theta_j$, which is

$$(t-1)(b_{m-2} + b_{m-3} + \cdots + b_{s-2}) + (t-2)b_{s-3} + (t-3)b_{s-4} + \cdots + 2b_{q+1} + b_q. \tag{9}$$

Unfortunately, this simpler proof did not work when $m = 5$. The construction of $\mathcal{Q}$ in the current proof treats the case $t = 2$ differently from larger values of t. Thus ${}^s\Omega_t^0$ is equal to (8) when $t > 2$, but when $t = 2$ it is necessary to replace the final term b_q by $({}^s a_2)$ if that is smaller. And ${}^s\Omega_t^1$ is obtained from (9) by subtracting $b_{s-2} - ({}^s a_2)$ (which is zero if $s = m$ and nonnegative otherwise).

We will now construct $\mathcal{Q}$ by specifying the values of these parameters. We start with an *initialization step*: for all $s \geqslant 2$, set ${}^s\Omega_1^1 := \Theta_{m-1} := 0$. Then, for each $q = m-2, \ldots, 1$ in turn, we will carry out the following *construction procedure for* q, which specifies the values of $\mathcal{B}_q$, b_q, Θ_q, ${}^s\mathcal{A}_t$, $({}^s a_t)$, ${}^s\Omega_t^1$ and ${}^s\Omega_t^0$ for all $(s,t) \in \mathcal{I}_q$. For each value of q, we first set

$${}^s\Phi_t := {}^s\Omega_{t-1}^1 + \Theta_{q+1} \tag{10}$$

for each $(s,t) \in \mathcal{I}_q$. Note that the values of the two terms on the RHS of (10) have already been specified previously, the first in the initialization step if $t = 2$ and in the construction procedure for $q+1$ if $t > 2$ (which implies $q < m-2$), and the second in the initialization step if $q = m-2$ and in the construction procedure for $q+1$ if $q < m-2$. We then (see below) define $\mathcal{B}_q$, b_q, ${}^s\mathcal{A}_t$ and $({}^sa_t)$ for all $(s,t) \in \mathcal{I}_q$; and finally we set Θ_q by (3),

$$ {}^s\Omega_t^1 := {}^s\Phi_t + ({}^sb_t) \tag{11}$$

and

$$ {}^s\Omega_t^0 := \Theta_{q+1} + ({}^sb_t) \tag{12}$$

for each $(s,t) \in \mathcal{I}_q$, where $({}^sb_2) := ({}^sa_2)$ and $({}^sb_t) := b_{s-t} = b_q$ if $t \geqslant 3$. For example, if $m = 5$ then

$$\begin{array}{lll}
b_3 = ({}^5a_2), & b_2 = ({}^4a_2) + ({}^5a_3), & b_1 = ({}^3a_2) + ({}^4a_3) + ({}^5a_4), \\
\Theta_4 = 0, \quad \Theta_3 = b_3, & \Theta_2 = b_3 + b_2, & \Theta_1 = b_3 + b_2 + b_1, \\
{}^5\Phi_2 = 0, & {}^5\Omega_2^1 = ({}^5a_2) = b_3, & {}^5\Omega_2^0 = ({}^5a_2) = b_3, \\
{}^4\Phi_2 = b_3, & {}^4\Omega_2^1 = b_3 + ({}^4a_2), & {}^4\Omega_2^0 = b_3 + ({}^4a_2), \\
{}^5\Phi_3 = 2b_3, & {}^5\Omega_3^1 = 2b_3 + b_2, & {}^5\Omega_3^0 = b_3 + b_2, \\
{}^3\Phi_2 = b_3 + b_2, & {}^3\Omega_2^1 = b_3 + b_2 + ({}^3a_2), & {}^3\Omega_2^0 = b_3 + b_2 + ({}^3a_2), \\
{}^4\Phi_3 = 2b_3 + ({}^4a_2) + b_2, & {}^4\Omega_3^1 = 2b_3 + ({}^4a_2) + b_2 + b_1, & {}^4\Omega_3^0 = b_3 + b_2 + b_1, \\
{}^5\Phi_4 = 3b_3 + 2b_2, & {}^5\Omega_4^1 = 3b_3 + 2b_2 + b_1, & {}^5\Omega_4^0 = b_3 + b_2 + b_1.
\end{array}$$

For each q, after setting the values of ${}^s\Phi_t$ for all $(s,t) \in \mathcal{I}_q$ by (10), we define $\mathcal{B}_q$ iteratively in two stages as follows, starting with $\mathcal{B}_q := \emptyset$. Firstly, while there is a part $V_p \notin \bigcup_{i=q}^{m-2} \mathcal{B}_i$ such that $|V_p| = q+2$ and $g(\mathcal{Q}_2(p)) > {}^{q+2}\Phi_2 + |\mathcal{B}_q|$, choose such a part V_p for which $g(\mathcal{Q}_2(p))$ is as large as possible, and add V_p to $\mathcal{B}_q$. Secondly, when there is no such part V_p, but while there is a part $V_p \notin \bigcup_{i=q}^{m-2} \mathcal{B}_i$ such that $|V_p| > q+2$ and $g(\mathcal{Q}_{s-q}(p)) > {}^s\Phi_{s-q} + |\mathcal{B}_q|$, where $s := |V_p|$, choose such a part V_p for which $g(\mathcal{Q}_{s-q}(p)) - {}^s\Phi_{s-q}$ is as large as possible, and add V_p to $\mathcal{B}_q$. When there is no such part V_p, the construction of $\mathcal{B}_q$ terminates; we then set $b_q := |\mathcal{B}_q|$, ${}^s\mathcal{A}_t := \{V_p : |V_p| = s \text{ and } V_p \in \mathcal{B}_q\}$ and $({}^sa_t) := |{}^s\mathcal{A}_t|$ for each $(s,t) \in \mathcal{I}_q$, before setting the values of Θ_q, ${}^s\Omega_t^1$ and ${}^s\Omega_t^0$ by (3), (11) and (12). This completes the construction procedure for q.

Finally, after the sets $\mathcal{B}_{m-2}, \ldots, \mathcal{B}_1$ have been constructed, let $\mathcal{B}_0 := \{V_p : V_p \notin \bigcup_{i=1}^{m-2} \mathcal{B}_i\}$, $b_0 := |\mathcal{B}_0|$, and $\Theta_0 := k_0$, so that (3) holds when $q = 0$; and for $2 \leqslant s \leqslant m$ let ${}^s\mathcal{A}_s := \{V_p \in \mathcal{B}_0 : |V_p| = s\}$ and $({}^sa_s) := |{}^s\mathcal{A}_s|$. We now *reorder the parts* V_p so that they are numbered in the order in which they are assigned to a set $\mathcal{B}_q$; then $\mathcal{B}_q = \{V_p : \Theta_{q+1} < p \leqslant \Theta_q\}$ $(0 \leqslant q \leqslant m-2)$ and ${}^s\mathcal{A}_2 = \{V_p : \Theta_{q+1} < p \leqslant {}^s\Omega_2^0\}$ $(3 \leqslant s \leqslant m, q = s-2)$. But if $t \geqslant 3$ and $q = s-t$ then ${}^s\Omega_t^0 = \Theta_q$ by (5), and so, for all $t \geqslant 2$,

$$\text{if } V_p \in {}^s\mathcal{A}_t \quad \text{then } p \leqslant {}^s\Omega_t^0. \tag{13}$$

As already mentioned, we now define the partition $\mathcal{Q}$ by starting with the natural partition $(U_1, \ldots, U_{k_1}, V_1, \ldots, V_{k_0})$ of V, and for each s and t $(2 \leqslant t \leqslant s \leqslant m)$, and each $V_p \in {}^s\mathcal{A}_t$, replacing V_p by $\mathcal{Q}_t(p)$.

Note that if $q = s-t > 0$ then the construction of ${}^s\mathcal{A}_t$ formally terminates when there is no longer a part $V_p \notin \bigcup_{i=q}^{m-2} \mathcal{B}_i$ such that $|V_p| = q+2$ (if $t = 2$) or $|V_p| > q+2$ (otherwise) and $g(\mathcal{Q}_t(p)) > {}^s\Phi_t + |\mathcal{B}_q|$; and at this moment of formal termination, $|\mathcal{B}_q| = ({}^sa_t)$ if $t = 2$ and $|\mathcal{B}_q| = b_q$ if $t \geqslant 3$; i.e., $|\mathcal{B}_q| = ({}^sb_t)$. But ${}^s\Omega_t^1 = {}^s\Phi_t + ({}^sb_t)$ by (11), and so

$$g(\mathcal{Q}_t(p)) \geqslant {}^s\Omega_t^1 \quad \text{if } V_p \in {}^s\mathcal{A}_t \tag{14}$$

and

$$g(\mathcal{Q}_t(p)) \leqslant {}^s\Omega_t^1 \quad \text{if } V_p \in {}^s\mathcal{A}_r, r > t. \tag{15}$$

If $\mathcal{Q}_t(p) = (X_1, \ldots, X_t)$ then, by (2) and (14),

$$\sum_{i=1}^{t} |L^{\cap}(X_i)| \geqslant (t-1)k + {}^s\Omega_t^1 \quad \text{if } V_p \in {}^s\mathcal{A}_t. \tag{16}$$

Also, if $\mathcal{P}_t(p) = (X_1, \ldots, X_t)$ is any partition of V_p into t parts that is obtained by merging two parts of $\mathcal{Q}_{t+1}(p)$, then $g(\mathcal{P}_t(p)) \leqslant g(\mathcal{Q}_t(p))$ and so

$$g(\mathcal{P}_t(p)) \leqslant {}^s\Omega_t^1 \quad \text{if } V_p \in {}^s\mathcal{A}_r, r > t \tag{17}$$

by (15); thus

$$\sum_{i=1}^{t} |L^{\cap}(X_i)| \leqslant (t-1)k + {}^s\Omega_t^1 \quad \text{if } V_p \in {}^s\mathcal{A}_r, r > t; \tag{18}$$

in particular, if each part of $\mathcal{P}_t(p)$ is a singleton set except for X_t, then

$$|L^\cap(X_t)| \leqslant {}^s\Omega_t^1 \quad \text{if } V_p \in {}^s\mathcal{A}_r, r > t. \tag{19}$$

Part 2. *Lemmas*. To make it easier to apply Eqs. (17)–(19), in this part we will use t_p to denote the number of parts into which V_p is partitioned in $\mathcal{Q}$, so that if $V_p \in {}^s\mathcal{A}_t$ then $t_p = t$. Then to say that $V_p \in {}^s\mathcal{A}_r$ for some $r > t$ is the same as saying that $t_p > t$.

Lemma 1. *If* $2 \leqslant t < s \leqslant m$ *and* $V_p \in {}^s\mathcal{A}_t$, *then for every two members* X_1 *and* X_2 *of* $\mathcal{Q}_t(p)$,

$$|L^\cap(X_1) \cup L^\cap(X_2)| \geqslant k + {}^s\Omega_t^0.$$

Proof. If $t = 2$, then we see from (4) that ${}^s\Omega_2^1 = {}^s\Omega_2^0$, and it follows from (1) and (16) that

$$|L^\cap(X_1) \cup L^\cap(X_2)| = |L^\cap(X_1)| + |L^\cap(X_2)| \geqslant k + {}^s\Omega_2^1 = k + {}^s\Omega_2^0.$$

So assume that $t \geqslant 3$. Let $X' := X_1 \cup X_2$. Consider the partition $\mathcal{P}_{t-1}(p)$ obtained from $\mathcal{Q}_t(p)$ by merging X_1 and X_2 into X'. Since $V_p \in {}^s\mathcal{A}_t, g(\mathcal{Q}_t(p)) \geqslant {}^s\Omega_t^1$ by (14). Since $t_p = t > t-1, g(\mathcal{P}_{t-1}(p)) \leqslant {}^s\Omega_{t-1}^1$ by (17). Thus, using (2) in the fourth line,

$$\begin{aligned}
|L^\cap(X_1) \cup L^\cap(X_2)| &= |L^\cap(X_1)| + |L^\cap(X_2)| - |L^\cap(X_1) \cap L^\cap(X_2)| \\
&= |L^\cap(X_1)| + |L^\cap(X_2)| - |L^\cap(X')| \\
&= f(\mathcal{Q}_t(p)) - f(\mathcal{P}_{t-1}(p)) \\
&= k + g(\mathcal{Q}_t(p)) - g(\mathcal{P}_{t-1}(p)) \\
&\geqslant k + {}^s\Omega_t^1 - {}^s\Omega_{t-1}^1 \\
&= k + {}^s\Omega_t^0
\end{aligned}$$

since ${}^s\Omega_t^1 - {}^s\Omega_{t-1}^1 = {}^s\Omega_t^0$ by (6). This proves Lemma 1. □

Lemma 2. *If* $V_p \in {}^s\mathcal{A}_s \subseteq \mathcal{B}_0$ $(s \geqslant 3)$, *then for every pair of elements* $x, y \in V_p$,

$$|L(x) \cup L(y)| \geqslant 2k - {}^s\Omega_{s-1}^1 \geqslant 2k - {}^m\Omega_{m-1}^1.$$

Proof. Let $\mathcal{P}_{s-1}(p)$ be the partition of V_p consisting of $\{x, y\}$ and $s-2$ singleton sets. Since $t_p = s > s-1, |L(x) \cap L(y)| \leqslant {}^s\Omega_{s-1}^1$ by (19). Thus

$$|L(x) \cup L(y)| \geqslant |L(x)| + |L(y)| - {}^s\Omega_{s-1}^1 = 2k - {}^s\Omega_{s-1}^1.$$

Now, ${}^s\mathcal{A}_2 \subseteq \mathcal{B}_{s-2}$ and so $({}^s a_2) \leqslant b_{s-2}$. Also, $\Theta_{s-1} + b_{s-2} = \Theta_{s-2}$ by (3). Thus ${}^s\Omega_2^0 \leqslant \Theta_{s-2}$ by (4), with equality if $s = m$, by (7), and so (5) gives

$${}^s\Omega_{s-1}^1 = {}^s\Omega_2^0 + \sum_{j=1}^{s-3} \Theta_j \leqslant \sum_{j=1}^{s-2} \Theta_j \leqslant \sum_{j=1}^{m-2} \Theta_j = {}^m\Omega_{m-1}^1. \tag{20}$$

The result follows. □

Lemma 3. *Let* $V_p \in {}^s\mathcal{A}_t \subseteq \mathcal{B}_1$, *where* $s \geqslant 4$, *so that* $|V_p| = s = t+1$, *and let* $\mathcal{Q}_t(p) = (X_1, \ldots, X_t)$ *where* $X_i = \{x_i\}$ $(i = 1, \ldots, t-1)$ *and* $X_t = \{x_t, x_{t+1}\}$. *Then*

(a) $|L(x) \cup L(y)| \geqslant \frac{1}{2}(3k - {}^s\Omega_{s-2}^1)$ *for each* $x, y \in \{x_1, \ldots, x_{t-1}\}, x \neq y$;
(b) $|L(x) \cup L(y) \cup L^\cap(X_t)| \geqslant k + 2({}^s\Omega_{s-1}^1) - 3({}^s\Omega_{s-2}^1)$ *for each* $x, y \in \{x_1, \ldots, x_{t-1}\}, x \neq y$;
(c) *if* $s \geqslant 5$ *then* $|\bigcup_{i=1}^t L^\cap(X_i)| \geqslant \frac{1}{4}(7k - 3({}^s\Omega_{s-2}^1) - 2({}^s\Omega_{s-3}^1))$.

Proof. Let $C_i := L^\cap(X_i)$ $(i = 1, \ldots, t)$. For $x, y \in \{x_1, \ldots, x_{t-1}\}$ $(x \neq y)$, let $d_{xy} := |L(x) \cap L(y)|$. Since we formed $\mathcal{Q}_t(p)$ by merging x_t with x_{t+1} to form X_t,

$$d_{xy} \leqslant |C_t|. \tag{21}$$

Since $V_p \in {}^s\mathcal{A}_{s-1}$, $(t-1)k + |C_t| \geqslant (t-1)k + {}^s\Omega_{s-1}^1$ by (16), and so

$$|C_t| \geqslant {}^s\Omega_{s-1}^1. \tag{22}$$

Since $t_p = s-1 > s-2$,

$$(s-4)k + d_{xy} + |C_t| \leqslant (s-3)k + {}^s\Omega_{s-2}^1$$

by (18), and similarly

$$(s-3)k + |L(x) \cap C_t| \leqslant (s-3)k + {}^s\Omega^1_{s-2},$$

so that

$$d_{xy} + |C_t| \leqslant k + {}^s\Omega^1_{s-2} \tag{23}$$

and

$$|L(x) \cap C_t| \leqslant {}^s\Omega^1_{s-2}. \tag{24}$$

By (21) and (23), $|L(x) \cap L(y)| = d_{xy} \leqslant \frac{1}{2}(k + {}^s\Omega^1_{s-2})$, so that

$$|L(x) \cup L(y)| = 2k - d_{xy} \geqslant \frac{1}{2}(3k - {}^s\Omega^1_{s-2}). \tag{25}$$

This proves (a). And by (23), (24), and the analog of (24) with y in place of x,

$$\begin{aligned} |L(x) \cup L(y) \cup C_t| &\geqslant |L(x)| + |L(y)| + |C_t| - |L(x) \cap L(y)| - |L(x) \cap C_t| - |L(y) \cap C_t| \\ &\geqslant k + k + 2|C_t| - (k + {}^s\Omega^1_{s-2}) - {}^s\Omega^1_{s-2} - {}^s\Omega^1_{s-2} \\ &\geqslant k + 2({}^s\Omega^1_{s-1}) - 3({}^s\Omega^1_{s-2}) \end{aligned}$$

by (22), which proves (b).

Now assume that $s \geqslant 5$, so that $t \geqslant 4$. Without loss of generality we may assume that $\mathcal{Q}_{t-1}(p)$ is formed from $\mathcal{Q}_t(p)$ by merging X_1 with either X_3 or X_t. Let $\mathcal{P}_{t-2}(p)$ be formed from $\mathcal{Q}_t(p)$ by merging all three of the sets X_1, X_3 and X_t; then $\mathcal{P}_{t-2}(p)$ can be formed from $\mathcal{Q}_{t-1}(p)$ by merging two parts, and every other part of $\mathcal{P}_{t-2}(p)$ is a singleton. Since $t_p = t > s-3$,

$$|C_1 \cap C_3 \cap C_t| \leqslant {}^s\Omega^1_{s-3} \tag{26}$$

by (19) applied to $\mathcal{P}_{t-2}(p)$. Let $x := x_2, y := x_3$ and $c_1 := |C_1 \setminus (C_2 \cup C_3)|$. In order to prove (c), it suffices to prove that

$$|C_1 \cup C_2 \cup C_3| + |C_1 \cup C_2 \cup C_t| \geqslant \frac{1}{2}(7k - 3({}^s\Omega^1_{s-2}) - 2({}^s\Omega^1_{s-3})), \tag{27}$$

since $\max\{a, b\} \geqslant \frac{1}{2}(a+b)$. By the definition of d_{xy},

$$|C_1 \cup C_2 \cup C_3| = c_1 + |C_2 \cup C_3| = c_1 + 2k - d_{xy}, \tag{28}$$

and

$$\begin{aligned} |(C_1 \setminus C_2) \cap C_t| &\leqslant |(C_1 \setminus (C_2 \cup C_3)) \cap C_t| + |C_1 \cap C_3 \cap C_t| \\ &\leqslant c_1 + {}^s\Omega^1_{s-3} \end{aligned} \tag{29}$$

by (26). Therefore, using (25), (21), (24) and (29) in the second line,

$$\begin{aligned} |C_1 \cup C_2 \cup C_t| &= |C_1 \cup C_2| + |C_t| - |C_2 \cap C_t| - |(C_1 \setminus C_2) \cap C_t| \\ &\geqslant \frac{1}{2}(3k - {}^s\Omega^1_{s-2}) + d_{xy} - {}^s\Omega^1_{s-2} - (c_1 + {}^s\Omega^1_{s-3}) \\ &= \frac{3}{2}(k - {}^s\Omega^1_{s-2}) - {}^s\Omega^1_{s-3} + d_{xy} - c_1, \end{aligned}$$

and so, adding in (28),

$$|C_1 \cup C_2 \cup C_3| + |C_1 \cup C_2 \cup C_t| \geqslant \frac{1}{2}(7k - 3({}^s\Omega^1_{s-2}) - 2({}^s\Omega^1_{s-3})).$$

This proves (27) and hence (c). □

Lemma 4. *Let $V_p \in {}^5\mathcal{A}_3 \subseteq \mathcal{B}_2$ and $\mathcal{Q}_3(p) = (X_1, X_2, X_3)$. Then*

$$|L^\cap(X_1) \cup L^\cap(X_2) \cup L^\cap(X_3)| \geqslant \frac{3}{2}k - {}^5\Omega^1_2.$$

Proof. As in Lemma 3, let $C_i := L^\cap(X_i)$ $(i = 1, 2, 3)$. We consider two cases.

Case 1. $X_1 = \{x_1\}, X_2 = \{x_2\}, X_3 = \{x_3, x_4, x_5\}$. Let $h := |C_1 \cap C_2| = |L(x_1) \cap L(x_2)|$. We may assume that $\mathcal{Q}_3(p)$ was obtained from $\mathcal{Q}_5(p)$ by first merging x_4 and x_5 to form $X_4 := \{x_4, x_5\}$, and then merging x_3 with X_4 to form X_3. Since x_1 was not merged with x_2 in either of these steps, it follows that $h \leqslant |L(x_4) \cap L(x_5)|$ and

$$2h + k \leqslant h + |L(x_3)| + |L(x_4) \cap L(x_5)| \leqslant |C_1| + |C_2| + |C_3| = 2k + |C_3|,$$

so that $|C_3| \geqslant 2h - k$. Since $t_p = 3 > 2$, $|C_1 \cap C_3| \leqslant {}^5\Omega_2^1$ by (19), and similarly $|C_2 \cap C_3| \leqslant {}^5\Omega_2^1$. Therefore

$$\begin{aligned} |C_3| - |(C_1 \cup C_2) \cap C_3| &\geqslant |C_3| - |C_1 \cap C_3| - |C_2 \cap C_3| \\ &\geqslant (2h - k) - {}^5\Omega_2^1 - {}^5\Omega_2^1 \\ &\geqslant h - k/2 - {}^5\Omega_2^1, \end{aligned}$$

since $(a \geqslant 0 \text{ and } a \geqslant b) \Longrightarrow a \geqslant \frac{1}{2}b$. Thus

$$\begin{aligned} |C_1 \cup C_2 \cup C_3| &= |C_1 \cup C_2| + |C_3| - |(C_1 \cup C_2) \cap C_3| \\ &\geqslant |C_1| + |C_2| - |C_1 \cap C_2| + h - k/2 - {}^5\Omega_2^1 \\ &= k + k - h + h - k/2 - {}^5\Omega_2^1 \\ &= 3k/2 - {}^5\Omega_2^1 \end{aligned}$$

as required.

Case 2. $X_1 = \{x_1\}, X_2 = \{x_2, x_3\}, X_3 = \{x_4, x_5\}$. We may assume that $|C_2| \geqslant |C_3|$, so that $\mathcal{Q}_3(p)$ was formed from $\mathcal{Q}_5(p)$ by first merging x_2 and x_3 into X_2, and then merging x_4 and x_5 into X_3. Since in the second step we did not merge x_1 and X_2,

$$|C_1 \cap C_2| + k + k \leqslant k + |C_2| + |C_3|. \tag{30}$$

Since $t_p = 3 > 2$, $|C_1 \cap C_3| + |C_2| \leqslant k + {}^5\Omega_2^1$ by (18), which with (30) gives

$$|C_1 \cap C_2| + |C_1 \cap C_3| \leqslant |C_3| + {}^5\Omega_2^1. \tag{31}$$

Similarly, $|C_1| + |C_2 \cap C_3| \leqslant k + {}^5\Omega_2^1$, so that $|C_2 \cap C_3| \leqslant {}^5\Omega_2^1$. And since $V_p \in {}^5\mathcal{A}_3$,

$$|C_1| + |C_2| + |C_3| \geqslant 2k + {}^5\Omega_3^1$$

by (16), so that

$$2|C_2| \geqslant |C_2| + |C_3| \geqslant k + {}^5\Omega_3^1. \tag{32}$$

Therefore, using (31) in the second line,

$$\begin{aligned} |C_1 \cup C_2 \cup C_3| &\geqslant |C_1| + |C_2| + |C_3| - (|C_1 \cap C_2| + |C_1 \cap C_3|) - |C_2 \cap C_3| \\ &\geqslant k + |C_2| + |C_3| - (|C_3| + {}^5\Omega_2^1) - {}^5\Omega_2^1 \\ &= k + |C_2| - 2({}^5\Omega_2^1) \\ &\geqslant \frac{1}{2}(3k + {}^5\Omega_3^1) - 2({}^5\Omega_2^1) \end{aligned}$$

by (32). But, by (4), ${}^5\Omega_2^1 = \Theta_4 + ({}^5a_2) \leqslant \Theta_4 + b_3 = \Theta_3 \leqslant \Theta_2$, and so ${}^5\Omega_3^1 = {}^5\Omega_2^1 + \Theta_2 \geqslant 2({}^5\Omega_2^1)$ by (6); thus the lemma is proved. □

Part 3. *Completion of the proof of* Theorem 1. We must prove that the family $(L^\cap(X) : X \in \mathcal{Q})$ has a system of distinct representatives, which we can then use to form an L-coloring of G as described near the beginning of the proof. Suppose that there is no such system of distinct representatives. Then, by Hall's Theorem, there exists $\mathcal{R} \subseteq \mathcal{Q}$ such that $|\bigcup_{X\in\mathcal{R}} L^\cap(X)| < |\mathcal{R}|$. Let $C_\mathcal{R} := \bigcup_{X\in\mathcal{R}} L^\cap(X)$, so that $|C_\mathcal{R}| < |\mathcal{R}|$.

Note that, by (20) and (3),

$${}^m\Omega_{m-1}^1 = \sum_{q=1}^{m-2} \Theta_q = \sum_{q=1}^{m-2}\sum_{i=q}^{m-2} b_i = \sum_{i=1}^{m-2} ib_i = \sum_{q=1}^{m-2} qb_q. \tag{33}$$

For $2 \leqslant t \leqslant m$, let $\mathcal{A}_t := \bigcup_{s=t}^m {}^s\mathcal{A}_t$ and $a_t := |\mathcal{A}_t|$. Then $\sum_{s=t}^m ({}^sa_t) = a_t$, $\sum_{t=2}^s ({}^sa_t) = k_s$, and $\sum_{t=2}^{m-q} ({}^{t+q}a_t) = b_q$. So

$$|\mathcal{Q}| - k = \sum_{t=2}^m (t-1)a_t = \sum_{2\leqslant t\leqslant s\leqslant m} (t-1)({}^sa_t), \tag{34}$$

$$|V| - k = \sum_{s=2}^m (s-1)k_s = \sum_{2\leqslant t\leqslant s\leqslant m} (s-1)({}^sa_t), \tag{35}$$

and, by (33),

$$ {}^{m}\Omega^{1}_{m-1} = \sum_{q=1}^{m-2} q b_q = \sum_{2\leqslant t\leqslant s\leqslant m} (s-t)({}^{s}a_t). \tag{36} $$

Hence $|\mathcal{Q}| + {}^{m}\Omega^{1}_{m-1} = |V|$. Since $|V| \leqslant 2k+1$, it follows that

$$ |\mathcal{Q}| \leqslant 2k + 1 - {}^{m}\Omega^{1}_{m-1}. \tag{37} $$

Claim 1. *Suppose $\mathcal{R}$ contains more than one subset of V_p for some part $V_p \in {}^{\sigma}\mathcal{A}_\sigma \subseteq \mathcal{B}_0$. Then $\sigma \geqslant 3$, and for any three distinct elements $x, y, z \in V_p$ the following hold: $|L(x) \cap L(y)| \geqslant \frac{1}{2}k$, and*

$$ |L(x) \cup L(y)| = |(L(x) \cap L(y)) \cup L(z)| = |C_{\mathcal{Q}}| = |\mathcal{Q}| - 1. \tag{38} $$

Proof. Suppose first that $\mathcal{R}$ contains both (singleton) subsets of some part $V_p \in {}^{2}\mathcal{A}_2$. Then, by (1) and (37), $|C_{\mathcal{R}}| \geqslant 2k \geqslant 2k - {}^{m}\Omega^{1}_{m-1} \geqslant |\mathcal{Q}| - 1 \geqslant |\mathcal{R}| - 1$. Thus equality holds throughout, so that ${}^{m}\Omega^{1}_{m-1} = 0$ and $\mathcal{R} = \mathcal{Q}$. By (36), $b_q = 0$ for all $q \geqslant 1$. By the construction of $\mathcal{Q}$, this means that, for each part V_p, $L(x) \cap L(y) = \emptyset$ for each two vertices $x, y \in V_p$. Since we are assuming that there is a part V_p such that $|V_p| \geqslant 3$, this implies that $|C_{\mathcal{R}}| \geqslant 3k$. This contradiction shows that $\sigma \geqslant 3$.

Suppose now that $\mathcal{R}$ contains two (singleton) subsets of some part $V_p \in {}^{\sigma}\mathcal{A}_\sigma$, where $\sigma \geqslant 3$. Then, by Lemma 2 and (37), $|C_{\mathcal{R}}| \geqslant 2k - {}^{m}\Omega^{1}_{m-1} \geqslant |\mathcal{Q}| - 1 \geqslant |\mathcal{R}| - 1$, and so equality holds throughout, including in Lemma 2, and $\mathcal{R} = \mathcal{Q}$ and $|C_{\mathcal{Q}}| = 2k - {}^{m}\Omega^{1}_{m-1} = |\mathcal{Q}| - 1$. Thus $\mathcal{R}$ contains all σ (singleton) subsets of V_p, and $L(x) \cup L(y) = C_{\mathcal{Q}}$ for each pair of distinct elements $x, y \in V_p$. By (1), this means that each color in $C_{\mathcal{Q}}$ is in the list of exactly $\sigma - 1$ vertices of V_p, so that $|C_{\mathcal{Q}}| = \sigma k/(\sigma - 1)$, and for each vertex $x \in V_p$, $L(x)$ omits a different set of $k/(\sigma - 1)$ colors from $C_{\mathcal{Q}}$. Thus, for each two distinct vertices $x, y \in V_p$,

$$ |L(x) \cap L(y)| = (\sigma - 2)k/(\sigma - 1) \geqslant k/2 $$

since $\sigma \geqslant 3$, and (38) holds because every color not in $L(x)$ or $L(y)$ is in $L(z)$. □

We now divide the proof into two cases, which we will deal with simultaneously.

Case 1. $\mathcal{R}$ does not contain more than one subset of any part $V_p \in \mathcal{B}_0$. In this case we define $\mathcal{Q}' := \mathcal{Q}$ and $\mathcal{R}' := \mathcal{R}$.

Case 2. $\mathcal{R}$ contains more than one subset of V_p for some part $V_p \in {}^{\sigma}\mathcal{A}_\sigma \subseteq \mathcal{B}_0$. We choose one such part $V_{p'}$ and form $\mathcal{Q}'$ from $\mathcal{Q}$ by merging two vertices x', y' of $V_{p'}$ into a single set $X' = \{x', y'\}$; that is, we replace $\mathcal{Q}_\sigma(p')$ by $\mathcal{Q}_{\sigma-1}(p')$. If $|C_{\mathcal{R}'}| \geqslant |\mathcal{R}'|$ for every subset $\mathcal{R}' \subseteq \mathcal{Q}'$, where $C_{\mathcal{R}'} := \bigcup_{X\in\mathcal{R}'} L^{\cap}(X)$, then the family $(L^{\cap}(X) : X \in \mathcal{Q}')$ has a system of distinct representatives, which we can use to form an L-coloring of G. So we may assume that there is a set $\mathcal{R}' \subseteq \mathcal{Q}'$ such that $|C_{\mathcal{R}'}| < |\mathcal{R}'|$.

Claim 2. *$\mathcal{R}'$ contains at most one subset of each part $V_p \in \mathcal{B}_0$.*

Proof. This holds by hypothesis in Case 1, and so it suffices to prove it in Case 2. In this case, if $\mathcal{R}'$ contains two singleton subsets $\{x\}, \{y\}$ of some part $V_p \in \mathcal{B}_0$, then $\{x\}, \{y\} \in \mathcal{Q} = \mathcal{R}$ and so

$$ |C_{\mathcal{R}'}| \geqslant |L(x) \cup L(y)| = |\mathcal{Q}| - 1 = |\mathcal{Q}'| \geqslant |\mathcal{R}'| $$

by (38), which is a contradiction. And if $\mathcal{R}'$ contains the set $X' = \{x', y'\}$ and another subset $\{z\}$ of $V_{p'}$, then

$$ |C_{\mathcal{R}'}| \geqslant |(L(x') \cap L(y')) \cup L(z)| = |\mathcal{Q}| - 1 = |\mathcal{Q}'| \geqslant |\mathcal{R}'| $$

by (38). In every case we have a contradiction. □

Claim 3. $k_0 - k_2 \leqslant \frac{1}{2}(k+1)$.

Proof. Recall that $k = k_0 + k_1$. If $k_0 - k_2 \geqslant k_1 + 2$ then

$$ |V(G)| = \sum_{i=1}^{m} i k_i \geqslant 3(k_0 - k_2) + 2k_2 + k_1 \geqslant 2(k_0 - k_2) + 2k_2 + 2k_1 + 2 = 2k + 2, $$

a contradiction. Thus $k_0 - k_2 \leqslant k_1 + 1$, and so $2(k_0 - k_2) \leqslant k_0 - k_2 + k_1 + 1 \leqslant k + 1$, as required. □

Claim 4. *$\mathcal{R}'$ contains at least three subsets of some part V_p.*

Table 1
Coefficients for use in the case $m = 5$.

	$(^5a_2)$	$(^4a_2)$	$(^5a_3)$	$(^3a_2)$	$(^4a_3)$	$(^5a_4)$	$(^2a_2)$	$(^3a_3)$	$(^4a_4)$	$(^5a_5)$
$\lvert\mathcal{Q}\rvert - k$ (34)	1	1	2	1	2	3	1	2	3	4
$\lvert V\rvert - k$ (35)	4	3	4	2	3	4	1	2	3	4
$^5\Omega_4^1$ (36)	3	2	2	1	1	1	0	0	0	0
$^5\Omega_3^1$	2	1	1	0	0	0	0	0	0	0
$^5\Omega_2^1$	1	0	0	0	0	0	0	0	0	0
$^4\Omega_3^1$	2	2	1	1	1	1	0	0	0	0
$^4\Omega_2^1$	1	1	0	0	0	0	0	0	0	0
(40)	1	1	2	1	2	3	0	0	0	0
$\frac{3}{4}(\lvert V\rvert - k) - \frac{3}{4}(^5\Omega_3^1) - \frac{1}{2}(^5\Omega_2^1)$ (41)	1	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{3}{2}$	$\frac{9}{4}$	3	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{9}{4}$	3
(42)	1	1	2	1	2	2	0	0	0	0
$2(^4\Omega_3^1) - 3(^4\Omega_2^1)$ (43)	1	1	2	2	2	2	0	0	0	0
(44)	1	1	2	1	1	2	0	0	0	0
$\frac{1}{2}(\lvert V\rvert - k) - {^5\Omega_2^1}$ (45)	1	$\frac{3}{2}$	2	1	$\frac{3}{2}$	2	$\frac{1}{2}$	1	$\frac{3}{2}$	2
(46)	1	1	1	1	1	2	0	0	0	0
$\frac{1}{2}(\lvert V\rvert - k - {^5\Omega_3^1})$ (47)	1	1	$\frac{3}{2}$	1	$\frac{3}{2}$	2	$\frac{1}{2}$	1	$\frac{3}{2}$	2

Proof. By Claim 2, $\mathcal{R}'$ contains at most one subset of each part $V_p \in \mathcal{B}_0$. Suppose it contains at most two subsets of each other part V_p. We consider two cases.

Case 4.1. $|\mathcal{R}'| \geqslant k+1$. Since $|\mathcal{R}'| > k$, $\mathcal{R}'$ must contain exactly two subsets of some $V_p \notin \mathcal{B}_0$. Choose the maximum such p, so that $|\mathcal{R}'| \leqslant k+p$, and let $V_p \in {^s\mathcal{A}_t}$, so that $p \leqslant {^s\Omega_t^0}$ by (13). By Lemma 1, $|C_{\mathcal{R}'}| \geqslant k + {^s\Omega_t^0} \geqslant k+p \geqslant |\mathcal{R}'|$, a contradiction.

Case 4.2. $|\mathcal{R}'| \leqslant k$. Then $|C_{\mathcal{R}'}| \leqslant k-1$, and so $\mathcal{R}'$ contains no singleton set. But for each part $V_p = \{x, y\}$ of order 2, $L(x) \cap L(y) = \emptyset$ by (1), and so $\{x\}$ and $\{y\}$ are singleton sets in $\mathcal{Q}'$. Thus $\mathcal{R}'$ contains no subset of any part V_p such that $|V_p| \leqslant 2$. By Lemma 1, $\mathcal{R}'$ contains at most one subset of any part V_p. Thus $|\mathcal{R}'| \leqslant k_0 - k_2 \leqslant \frac{1}{2}(k+1)$, by Claim 3.

If $\mathcal{R}'$ contains the set $X' = \{x', y'\}$ in Case 2, then $|C_{\mathcal{R}'}| \geqslant |L(x') \cap L(y')| \geqslant \frac{1}{2}k$ by Claim 1; thus $|\mathcal{R}'| \geqslant \frac{1}{2}k + 1 > \frac{1}{2}(k+1)$, which contradicts the previous paragraph. Since $\mathcal{R}'$ contains no singleton set, it follows that $\mathcal{R}'$ contains no subset of any part $V_p \in \mathcal{B}_0$.

Let p be the maximum index such that some subset X of V_p belongs to $\mathcal{R}'$, so that $|\mathcal{R}'| \leqslant p$. Assume that $V_p \in {^s\mathcal{A}_t}$, so that $p \leqslant {^s\Omega_t^0}$ by (13). Then $g(\mathcal{Q}_t(p)) \geqslant {^s\Omega_t^1} \geqslant {^s\Omega_t^0}$ by (14) and (5). Since $|L^\cap(X_i)| \leqslant k$ for every set $X_i \in \mathcal{Q}_t(p)$, it follows from (2) that

$$|L^\cap(X)| = g(\mathcal{Q}_t(p)) - \sum(|L^\cap(X_i)| - k) \geqslant g(\mathcal{Q}_t(p)),$$

where the sum is taken over the $t-1$ sets $X_i \in \mathcal{Q}_t(p)$ such that $X_i \neq X$; thus $|C_{\mathcal{R}'}| \geqslant |L^\cap(X)| \geqslant {^s\Omega_t^0} \geqslant p \geqslant |\mathcal{R}'|$, which is a contradiction. This completes the proof of Claim 4. □

We now consider the relevant values of m. We need only prove the result for $m = 5$, since it then holds for all smaller values of m; but it is now so quick to finish the proof for $m = 4$ that we do it anyway. Suppose $m = 4$. By Claim 2, $\mathcal{R}'$ contains at most one subset of each part $V_p \in \mathcal{B}_0$. Thus, by a slight modification of (34),

$$|\mathcal{R}'| \leqslant k + \sum_{2 \leqslant t < s \leqslant m} (t-1)(^s a_t) = k + (^4a_2) + (^3a_2) + 2(^4a_3). \tag{39}$$

Note that, by the definition of b_q and (3)–(5) with $m = 4$, $b_2 = (^4a_2)$, $b_1 = (^3a_2) + (^4a_3)$, $^4\Omega_2^1 = \Theta_3 + (^4a_2) = 0 + b_2$, and $^4\Omega_3^1 = {^4\Omega_2^1} + \Theta_1 = 2b_2 + b_1$. By Claim 4, $\mathcal{R}'$ contains at least three subsets of some part V_p. The only possibility is that $V_p \in {^4\mathcal{A}_3}$, so that, by Lemma 3(b),

$$\begin{aligned}|C_{\mathcal{R}'}| &\geqslant k + 2(^4\Omega_3^1) - 3(^4\Omega_2^1) = k + 2(2b_2 + b_1) - 3b_2 \\ &= k + b_2 + 2b_1 = k + (^4a_2) + 2(^3a_2) + 2(^4a_3) \geqslant |\mathcal{R}'|\end{aligned}$$

by (39). But we are assuming that $|C_{\mathcal{R}'}| < |\mathcal{R}'|$, and this contradiction completes the proof when $m = 4$.

Assume now that $m = 5$. For convenience, in Table 1 we have tabulated the coefficients of the terms $(^s a_t)$ occurring in various expressions. The analog of (39) is

$$|\mathcal{R}'| \leqslant k + \sum_{2 \leqslant t < s \leqslant m} (t-1)(^s a_t) = k + (^5a_2) + (^4a_2) + 2(^5a_3) + (^3a_2) + 2(^4a_3) + 3(^5a_4). \tag{40}$$

By Lemma 3(c), and since $|V| \leqslant 2k+1$, if $\mathcal{R}'$ contains all four subsets of some part $V_p \in {^5\mathcal{A}_4}$ then

$$|C_{\mathcal{R}'}| \geqslant \frac{1}{4}(7k - 3(^5\Omega_3^1) - 2(^5\Omega_2^1)) \geqslant k + \frac{3}{4}(|V| - k - 1) - \frac{3}{4}(^5\Omega_3^1) - \frac{1}{2}(^5\Omega_2^1). \tag{41}$$

It follows from (40) and (41) (comparing the coefficients in Table 1) that $|C_{\mathcal{R}'}| \geqslant |\mathcal{R}'| - \frac{3}{4}$, which contradicts the supposition that $|C_{\mathcal{R}'}| \leqslant |\mathcal{R}'| - 1$. Thus we deduce that $\mathcal{R}'$ contains at most three subsets of each part $V_p \in {}^5\mathcal{A}_4$, so that

$$|\mathcal{R}'| \leqslant k + ({}^5a_2) + ({}^4a_2) + 2({}^5a_3) + ({}^3a_2) + 2({}^4a_3) + 2({}^5a_4). \qquad (42)$$

By Lemma 3(b), if $\mathcal{R}'$ contains all three subsets of some part $V_p \in {}^4\mathcal{A}_3$, then

$$|C_{\mathcal{R}'}| \geqslant k + 2({}^4\Omega_3^1) - 3({}^4\Omega_2^1). \qquad (43)$$

It follows from (42) and (43) (comparing the coefficients in Table 1) that $|C_{\mathcal{R}'}| \geqslant |\mathcal{R}'|$, which contradicts the supposition that $|C_{\mathcal{R}'}| \leqslant |\mathcal{R}'| - 1$. Thus we deduce that $\mathcal{R}'$ contains at most two subsets of each part $V_p \in {}^4\mathcal{A}_3$, so that

$$|\mathcal{R}'| \leqslant k + ({}^5a_2) + ({}^4a_2) + 2({}^5a_3) + ({}^3a_2) + ({}^4a_3) + 2({}^5a_4). \qquad (44)$$

By Lemma 4, if $\mathcal{R}'$ contains all three subsets of some part $V_p \in {}^5\mathcal{A}_3$, then

$$|C_{\mathcal{R}'}| \geqslant \frac{3}{2}k - {}^5\Omega_2^1 \geqslant k + \frac{1}{2}(|V| - k - 1) - {}^5\Omega_2^1. \qquad (45)$$

It follows from (44) and (45) that $|C_{\mathcal{R}'}| \geqslant |\mathcal{R}'| - \frac{1}{2}$, which contradicts the supposition that $|C_{\mathcal{R}'}| \leqslant |\mathcal{R}'| - 1$. We deduce that $\mathcal{R}'$ contains at most two subsets of each part $V_p \in {}^5\mathcal{A}_3$, so that

$$|\mathcal{R}'| \leqslant k + ({}^5a_2) + ({}^4a_2) + ({}^5a_3) + ({}^3a_2) + ({}^4a_3) + 2({}^5a_4). \qquad (46)$$

By Claim 4, $\mathcal{R}'$ contains at least three subsets of some part V_p. Since we have ruled out all other possibilities, $\mathcal{R}'$ must contain exactly three subsets, and hence two or more singleton subsets, of some part $V_p \in {}^5\mathcal{A}_4$. Thus, by Lemma 3(a),

$$|C_{\mathcal{R}'}| \geqslant \frac{1}{2}(3k - {}^5\Omega_3^1) \geqslant k + \frac{1}{2}(|V| - k - 1 - {}^5\Omega_3^1). \qquad (47)$$

It follows from (46) and (47) that $|C_{\mathcal{R}'}| \geqslant |\mathcal{R}'| - \frac{1}{2}$, which contradicts the supposition that $|C_{\mathcal{R}'}| \leqslant |\mathcal{R}'| - 1$. This completes the proof of Theorem 1.

Acknowledgements

The first author's research was supported in part by NSF grant DMS-0965587 and by grant 08-01-00673 of the Russian Foundation for Basic Research. We are grateful to the referees, whose thoughtful suggestions helped to improve the presentation of the paper.

References

[1] H. Enomoto, K. Ohba, K. Ota, J. Sakamoto, Choice number of some complete multi-partite graphs, Discrete Math. 244 (2002) 55–66.
[2] P. Erdős, A.L. Rubin, H. Taylor, Choosability in graphs. Proc. west coast conf. on combinatorics, graph theory and computing, Arcata, 1979, Congr. Numer. 26 (1980) 125–157.
[3] S. Gravier, K. Maffray, Graphs whose choice number is equal to their chromatic number, J. Graph Theory 27 (1998) 87–97.
[4] P. Hall, On representatives of subsets, J. London Math. Soc. 10 (1935) 26–30.
[5] W. He, L. Zhang, D.W. Cranston, Y. Shen, G. Zheng, Choice number of complete multipartite graphs $K_{3*3,2*(k-5),1*2}$ and $K_{4,3*2,2*(k-6),1*3}$, Discrete Math. 308 (2008) 5871–5877.
[6] K. Ohba, On chromatic-choosable graphs, J. Graph Theory 40 (2002) 130–135.
[7] K. Ohba, Choice number of complete multipartite graphs with part size at most three, Ars Combin. 72 (2004) 133–139.
[8] B. Reed, B. Sudakov, List colouring of graphs with at most $(2 - o(1))\chi$ vertices, Beijing, 2002, in: Proceedings of the International Congress of Mathematicians, vol. III, Higher Ed. Press, Beijing, 2002, pp. 587–603.
[9] B. Reed, B. Sudakov, List colouring when the chromatic number is close to the order of the graph, Combinatorica 25 (2005) 117–123.
[10] Y. Shen, W. He, G. Zheng, Y. Li, Ohba's conjecture is true for graphs with independence number at most three, Appl. Math. Lett. 22 (2009) 938–942.
[11] Y. Shen, W. He, G. Zheng, Y. Wang, L. Zhang, On choosability of some complete multipartite graphs and Ohba's conjecture, Discrete Math. 308 (2008) 136–143.
[12] Y.-F. Shen, G.-P. Zheng, W.-J. He, Chromatic choosability of a class of complete multipartite graphs, J. Math. Res. Exposition 27 (2007) 264–272.