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DECOMPOSITIONS OF QUADRANGLE-FREE PLANAR GRAPHS

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Abstract

W. He *et al.* showed that a planar graph not containing 4-cycles can be decomposed into a forest and a graph with maximum degree at most 7. This degree restriction was improved to 6 by Borodin *et al.* We further lower this bound to 5 and show that it cannot be improved to 3.

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1. Introduction

He, Hou, Lih, Shao, Wang and Zhu [5] proved a number of results on decomposing a planar graph with specified girth conditions into a forest and another graph whose maximum degree is not too high. In the same paper, they also used these results to derive new upper bounds on the game chromatic number of graphs.

In particular, He et al [5] showed that a planar graph with girth g can be decomposed into a forest and a graph H such that the maximum degree, $\Delta(H)$, of H is at most 4, 2 or 1 if g is at least 5, 7 or 11, respectively. Kleitman [6] improved one of these results by showing that $\Delta(H) \leq 2$ if $g \geq 6$. Kleitman *et al.* [1] also proved that $\Delta(H) \leq 1$ when $g \geq 10$. This result was further improved by Borodin, Kostochka, Sheikh and Yu [3]: $\Delta(H) \leq 1$ if $g \geq 9$.

In addition to the results with girth conditions, He *et al.* [5] proved that a planar graph that does not contain 4-cycles (even though it may contain 3-cycles) can be decomposed into a forest and a graph H such that $\Delta(H) \leq 7$. This bound was improved to 6 by Borodin et al [4].

Our main result here is as follows.

Theorem 1. *Every planar graph without* 4*-cycles can be decomposed into a forest and a graph with maximum degree at most* 5.

The reason for forbidding 4-cycles is seen in the complete bipartite graph $K_{2,n}$, which is planar. Any forest F contained in $K_{2,n}$ has at most $n+1$ edges, and at least half of the remaining edges are incident with one of the two vertices of degree n in $K_{2,n}$. Thus, in any decomposition of $K_{2,n}$ into F and another graph, H, the maximum degree of H is at least $\frac{n-1}{2}$, i.e., is high if n is large. On the other hand, forbidding only 2-alternating 4-cycles (having two nonconsecutive vertices of degree 2) in a planar graph implies that this graph can be decomposed into a forest and an H with $\Delta(H) \leq 14$ (by Theorem 5 in [2] combined with Theorem 3.1 in [5]).

As explained in [5], Theorem 1 implies the following result.

Corollary 2. *The game chromatic number and the game coloring number of a planar graph without* 4*-cycles are at most* 9*.*

By an FH_k*-coloring* of a graph we mean a partition of its edges into a forest F and a graph H of maximum degree at most k, i.e., a coloring of the edge set of the graph with two colors, F and H, such that the set of edges colored F forms a forest and the set of edges colored H forms a graph of maximum degree at most k.

2. Proof of Main Result

Say that a vertex is *branching* if its degree is at least 3. Suppose that there exist counterexamples to Theorem 1. In the set of these counterexamples, consider the subset that contains graphs with the fewest branching vertices. Let G be a graph from this subset with the smallest total number of vertices and edges.

2.1. Basic structural properties of G

Claim 3. G is connected. ∎

Fix a planar embedding of G and let $F(G)$ denote the set of faces of G in this planar embedding. For $x \in V(G) \cup F(G)$, $d(x)$ denotes the degree of x, where the *degree* of a face is the length of a closed walk around the boundary of the face.

Claim 4. G has no vertex v with $d(v) = 1$.

Proof. Let $G' = G - v$. By the minimality of G, graph G' has an FH_5-coloring, and it suffices to color the edge incident with v with F to obtain an FH_5-coloring of G. ∎

Claim 5. Every edge in G is incident with a vertex of degree at least 7.

Proof. Suppose u and v are two adjacent vertices of degree at most 6. Let $G' = G - uv$. Since the number of branching vertices in G' does not exceed that in G, and G' has fewer edges than G, G' has an FH_5-coloring. If both u and v have incident edges of color F, then we color uv with H; otherwise, color it with F. This gives an FH_5-coloring of G, a contradiction. ■

Claim 6. G has no vertices of degree 3.

Proof. Suppose that G has a vertex v of degree 3, with neighbors v_1, v_2, and v_3. Let G' be obtained from G by removing v, adding three new vertices x_1, x_2, and x_3, and the edges of the 6-cycle $v_1x_1v_2x_2v_3x_3$. By Claim 5, G' has fewer branching vertices. Thus by the minimality of G, G' has an FH_5-coloring c'. We use c' to construct an FH_5-coloring c for G: we let $c(vv_i) = F$ if both new edges at v_i in G' are colored F, and let $c(vv_i) = H$ otherwise. Then v has at most three incident edges colored H, the number of edges colored H in G incident with any vertex v_i does not exceed that in G', and no F-path in G going through v can appear. ■

For a vertex v, let $d^+(v)$ denote the total number of adjacent vertices of degree 2 and triangular faces incident with v.

Claim 7. G has no vertex v such that $d^+(v) > d(v)$.

Proof. We first note that by Claim 5, each closed walk $(w_1, \dots, w_t)$ encounters at least $\lceil \frac{t}{2} \rceil$ vertices of degree at least 7. Suppose that the neighbors of v in the clockwise direction are $w_1, \dots, w_t$. Let f_i be the face containing the walk (w_i, v, w_{i+1}), $i = 1, \dots, t$. Some vertices and/or faces with distinct labels can coincide. If a face f_i is a triangle, then by the first sentence of this proof, at least one of w_i and w_{i+1} is a vertex of degree at least 7. Call such a vertex x_i. Since G has no 4-cycles, all x_i are distinct. Thus the number of neighbors of degree at least 7 is at least the number of the incident triangular faces. Hence, the total number of adjacent 2-vertices and incident triangular faces is at most the number of adjacent vertices of degree 2 plus the number of adjacent vertices of degree at least 7, which is at most the degree of v. ■

2.2. Discharging and its consequences

Let the *initial charge* of every $x \in V(G) \cup F(G)$ be $\mu(x) = d(x) - 4$. By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we have

$$(1) \qquad \sum_{x\in V(G)\cup F(G)} (d(x)-4) = \sum_{x\in V(G)\cup F(G)} \mu(x) = -8 < 0.$$

A face f of degree at least 5 is *weak* if it is incident with $d(f)-3$ vertices of degree 2. Non-triangular faces that are not weak will be called *strong*. By Claim 5, every face f with $d(f) \geq 7$ is strong.

The vertices and faces of G discharge their initial charges by the following rules:

Rule 1. Every non-triangular face $f = uvw\ldots$ gives each incident vertex v of degree 2:

(a) 1, if either f is strong or f is a weak 6-face adjacent to the 3-face uvw;
(b) $\frac{1}{2}$, otherwise.

Rule 2. Every triangle gets $\frac{1}{2}$ from every incident vertex of degree at least 7.

Rule 3. Every vertex w of degree at least 7 gives $\frac{1}{2}$ to every adjacent 2-vertex v, unless v lies between non-triangular faces at least one of which is strong, in which case w gives only $\frac{1}{4}$ to v.

In the rest of the paper, we show that the final charge $\mu^*(x)$ is nonnegative for each $x \in V(G) \cup F(G)$, which is a contradiction to (1) since the total charge is preserved.

If f is a 3-face, then $\mu^*(f) \geq 3-4+2\times\frac{1}{2} = 0$ by Rule 2, due to Claim 5. By Rule 1 combined with Claim 5, each 5-face ends with a charge of at least 0. For the same reasons, each 6-face f has $\mu^*(f) \geq 6-4-2\times 1 = 0$ if f is strong and $\mu^*(f) \geq 6-4-2\times\frac{1}{2}-1 = 0$ if f is weak. (Note that a weak 6-face cannot give 1 to more than one incident 2-vertex due to the absence of 4-cycles in G.) Finally, if $d(f) \geq 7$, then $\mu^*(f) \geq d(f)-4-\lfloor\frac{d(f)}{2}\rfloor\times 1 \geq 0$.

Now suppose $v \in V(G)$. If $d(v) = 2$, then $\mu(v) = -2$. Note that v cannot be incident with two triangle faces, since G has no multiple edges. If one of the faces, f_1, incident with v is a triangle, then the other, f_2, cannot be a weak 5-face, and hence $\mu^*(v) \geq 2-4+1+2\times\frac{1}{2} = 0$ by Rules 1 and 3. If both f_1 and f_2 are non-triangular, then $\mu^*(v) \geq 2-4+1+\frac{1}{2}+2\times\frac{1}{4} = 0$ provided that at least one of f_1, f_2 is strong; otherwise, $\mu^*(v) \geq 2-4+4\times\frac{1}{2} = 0$.

If $4 \leq d(v) \leq 6$, then $\mu^*(v) = \mu(v) = d(v)-4 \geq 0$. If $d(v) \geq 8$, then $\mu^*(v) \geq d(v)-4-d^+(v)\times\frac{1}{2} \geq d(v)-4-d(v)\times\frac{1}{2} = \frac{d(v)-8}{2} \geq 0$ by Claim 7. Vertices of degree 7 are considered in the next subsection.

2.3. Handling vertices of degree 7

Note that $d^+(v) \leq 6$ implies $\mu^*(v) \geq 7 - 4 - 6 \times \frac{1}{2} = 0$, so we can assume $d^+(v) = 7$ due to Claim 7. There are four configurations to consider.

Let the neighbors of v be labeled $v_1, \ldots, v_7$ in the clockwise order.

Configuration 1. Vertex v is adjacent to seven 2-vertices and not incident with any triangle.

Let the neighbor of v_i, $i \in \{1, \ldots, 7\}$, other than v be labeled w_i. Note that each w_i is of degree at least 7, and that they are all distinct, since G has no 4-cycles.

Let $G_{i,j}$ be the graph obtained from G by removing v and all 2-vertices adjacent to it, adding a new vertex x and the edges $w_i x$ and $w_j x$. Note that $G_{i,j}$ is still planar and has fewer branching vertices than G. If $G_{i,j}$ is C_4-free, then by the minimality of G, there exists an FH_5-coloring c' of $G_{i,j}$. We show how to extend c' to an FH_5-coloring c of G.

We let $c(w_l v_l) = c'(w_l x)$ for $l = i, j$ and $c(v_i v) = c(v_j v) = F$, while $c(v_l v) = H$ and $c(w_l v_l) = F$ for $l \neq i, j$. Note that if $c'(w_i x) = c'(w_j x) = F$, then there is no other F-path from w_i to w_j in $G_{i,j}$ and hence the F-path $w_i v_i v v_j w_j$ does not produce any F-colored cycle in coloring c of G.

Thus, we may assume that for each $i \neq j$, $i, j \in \{1, \ldots, 7\}$, graph $G_{i,j}$ has a 4-cycle, i.e., w_i has a common neighbor with w_j.

Claim 8. Let G' be obtained from G by deleting v and all its neighbors of degree 2. If every pair of vertices from the set $\{w_1, \ldots, w_7\}$ in G' has a common neighbor, then there exists a vertex distinct from these seven vertices that is adjacent to all of them.

Proof. *Case* 1. First suppose that there is a chord of type $w_i w_{i+3}$ for some i in G', say, $w_1 w_4$. There are two possibilities: (a) neither edge $w_1 w_3$ nor edge $w_2 w_4$ is present, and (b) exactly one of them exists (both cannot be present by planarity). If (a) holds, then for w_2 to have a common neighbor with w_5, the edges $w_1 w_2$ and $w_1 w_5$ must be present, and similarly in order for w_3 to have a common neighbor with w_6 the edges $w_3 w_4$ and $w_4 w_6$ must be present. But by planarity, edges $w_1 w_5$ and $w_4 w_6$ cannot be present at the same time.

Suppose now that (b) holds; say, $w_1 w_3 \in E(G)$. In order for w_2 to have a common neighbor with any of the vertices in the set $\{w_5, w_6, w_7\}$, the edge $w_1 w_2$ must exist. If both edges $w_2 w_3$ and $w_3 w_4$ exist, then the 4-cycle

$w_1w_2w_3w_4w_1$ arises. Thus, one of these two edges does not exist. If the edge w_2w_3 is not there, then the face containing w_2, w_3 and v in the original graph G is strong. Indeed, it either contains w_1, in which case it is a strong 6-face, or is a face with at least 7 edges (if there is a w_2 to w_3 path, it must have at least 3 edges, since 2 edges would produce a 4-cycle with the edges w_1w_2 and w_1w_3). Note that the presence of a strong face at v ensures that $\mu^*(v) \geq 0$. Similarly, if the edge w_3w_4 is absent, then the face incident with w_3, w_4 and v in G is strong. Thus, there is no chord of type w_iw_{i+3} if v has a negative final charge.

Case 2. Now suppose there is no chord of the kind w_iw_{i+3} but there is a chord of the kind w_iw_{i+2}, say, w_1w_3. Since $w_3w_6 \notin E(G)$, to have a path of length 2 from w_6 to w_2, there should be edges w_1w_6 and w_1w_2. Similarly, to have a path of length 2 from w_2 to w_5, the edges w_2w_3 and w_3w_5 should be present. Now, because of planarity, there cannot be a path of length 2 between w_4 and w_7, a contradiction.

Case 3. Finally, suppose there are no chords in this set of vertices. To produce a path of length 2 from w_1 to w_4, there should be a common neighbor w of w_1 and w_4. Also, there must be a common neighbor w' of w_2 and w_6. By planarity, w' must be the same vertex as w. Similarly, w_5 must also be adjacent to w, and in fact all the other vertices must be adjacent to w to get all the paths of length 2. ■

By Claim 8, we can assume that there is a vertex w adjacent to each w_i. Since G has no 4-cycles, either $w_1w_2 \notin E(G)$ or $w_2w_3 \notin E(G)$. Suppose that $w_1w_2 \notin E(G)$. Then the face incident with v, v_1 and v_2 is strong. Hence, $\mu^*(v) \geq 0$.

Configuration 2. Vertex v is adjacent to six 2-vertices and incident with one triangle.

Let v_7 have degree larger than 2, and let the triangle incident with v contain v_7 and v_6. Let the neighbors of v_i, $i = 1, ..., 6$, be labelled w_i; in particular, $v_7 = w_6$. Like in Configuration 1 above, consider $G_{i,j}$ for $1 \leq i < j \leq 6$ (we do not delete v_7). If some $G_{i,j}$ is C_4-free, then by the minimality of G as a counterexample, there exists an FH_5-coloring c' of $G_{i,j}$. We will show that one can extend c' to an FH_5-coloring c of G.

Let $c(v_7v) = F$, $c(w_lv_l) = c'(w_lx)$ for $l \in \{i, j\}$, while $c(w_lv_l) = F$ and $c(v_lv) = H$ for $l \in \{1, \ldots, 6\} - \{i, j\}$. It remains to color v_iv and v_jv.

If at least one of w_iv_i, w_jv_j, say the first, is colored with H, then we let $c(v_iv) = F$, $c(v_jv) = H$. Suppose both w_iv_i and w_jv_j are colored with F. This means that w_i and w_j are not connected by an F-path in $G_{i,j} - x$, which implies that v_7 is not connected by an F-path in $G_{i,j} - x$ either with w_i or with w_j. Suppose with the first; then we let $c(v_iv) = F$ and $c(v_jv) = H$.

It remains to assume that every pair of vertices in the set $\{w_1, \ldots, w_6\}$ has a common neighbor. Then we make use of the following claim.

Claim 9. Let G' be obtained from G by deleting v and all its neighbors of degree 2. If in G', every pair of vertices from the set $\{w_1, \ldots, w_6\}$ has a common neighbor, then there exists a vertex x distinct from these six vertices such that x is adjacent to all of them.

Proof. The proof is similar to that of Claim 8. In particular, Case 3 is identical and is omitted from the proof below. The only remark on Case 1 is that $w_6w_3 \notin E(G)$, which implies that a strong face at v is not adjacent to the triangle at v and therefore saves $\frac{1}{2}$ for v. Case 2 is a little bit different and given in full detail below.

Case 2. There is no chord of the kind w_iw_{i+3}, but there is a chord w_iw_{i+2}. Since w_{i+1} has a common neighbor with a w_{i+4}, we can assume by symmetry that $w_iw_{i+4} \in E(G)$. Note that w_{i+3} can reach w_{i+5} only through w_{i+4} (which means that $w_{i+3}w_{i+4} \in E(G)$) and w_{i+1} only through w_{i+2} (which means that $w_{i+3}w_{i+2} \in E(G)$). This yields a 4-cycle $w_iw_{i+2}w_{i+3}w_{i+4}$, a contradiction. ∎

By the same reasoning as in the proof of Configuration 1, at least one of the edges w_2w_3 or w_3w_4, say the first, is not present in G. Then the face incident with v, v_2 and v_3 is strong. This face is not adjacent to the triangle at v and has at least two vertices of degree 2, which implies that v gives $\frac{1}{4}$ to each of them.

Configuration 3. Vertex v is adjacent to five 2-vertices and incident with two triangles, where each triangle contains one of these five 2-vertices.

W.l.o.g., we can assume that $d(v_1) > 2$ and there is a triangle vv_1v_2 at v. There are two cases to consider as shown in Figure 1.

(A) $d(v_6) > 2$ and there is a triangle vv_5v_6, and

(B) $d(v_7) > 2$ and there is a triangle vv_6v_7.

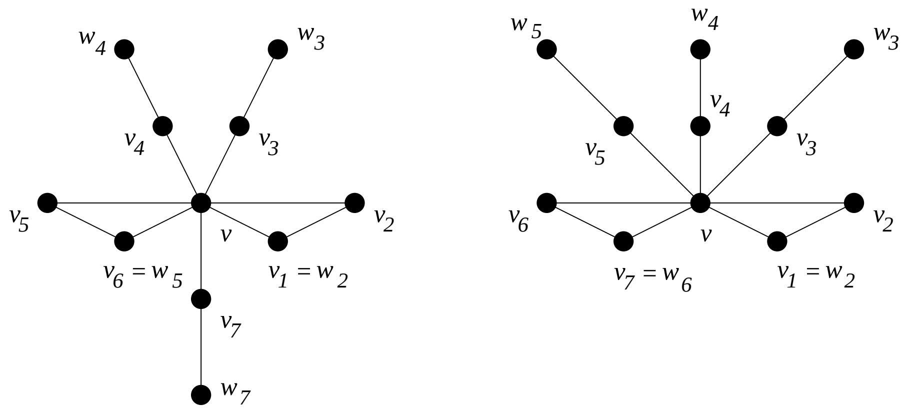

Figure 1. Two cases of Configuration 3.

Our argument below makes no distinction between (A) and (B), except in notation. For $i \in \{2, \ldots, 7\}$, if $d(v_i) = 2$, then denote the vertex adjacent to v_i other than v by w_i. In particular, $w_2 = v_1$ and either $w_5 = v_6$ (Case (A)) or $w_6 = v_7$ (Case (B)). We delete all 2-vertices adjacent to v and add one of the following 2-paths to obtain a graph G': either $v_1x_1w_4$ or $w_3x_3v_6$ in (A), and either $v_1x_1w_4$ or $w_3x_3v_7$ in (B). Note that vertex x_1 or x_3, as the case may be, is a newly-added vertex.

Note that G' has fewer branching vertices than G. First suppose G' is C_4-free; then let c' be an FH_5-coloring of G'. We use c' to design an FH_5-coloring c of G. To fix notation, we give the argument for the case when G' contains 2-path $v_1x_1w_4$.

We define an intermediate coloring c_0 of G and then adjust it to obtain a desired coloring c. Let e^* be the edge other than vv_1 joining v with a vertex of degree greater than 2, i.e., $e^* = vv_6$ in (A) and $e^* = vv_7$ in (B). Let $c_0(vv_1) = c'(vv_1)$ and $c_0(e^*) = c'(e^*)$. Also for each i such that $d(v_i) = 2$, we let $c_0(vv_i) = H$ and $c_0(v_iw_i) = F$.

If $c_0(vv_1) = c_0(e^*) = F$ then c_0 is already the desired FH_5-coloring c of G. We are also easily done if $c_0(vv_1) = c_0(e^*) = H$ by letting $c(vv_1) = c(vv_2) = F$ and $c(v_1v_2) = H$.

Suppose $c_0(vv_1) = F$ and $c_0(e^*) = H$. At this point, c_0 is not an FH_5-coloring as there are six edges incident to v that are colored H. We will recolor some edges to obtain FH_5-coloring c. In the case that $c'(v_1x_1) = H$, c is obtained by recoloring vv_2 with F and v_1v_2 with H. Now if $c'(x_1w_4) = H$, we let $c(v_4w_4) = H$ and $c(vv_4) = F$. So we can assume that $c'(v_1x_1) = c'(x_1w_4) = F$; now it suffices to let $c(vv_4) = F$.

Finally, suppose that $c_0(vv_1) = H$ and $c_0(e^*) = F$. The case $c'(x_1w_4) = H$ is resolved as in the previous paragraph. If $c'(v_1x_1) = H$, we can let $c(v_1v_2) = H$ and $c(vv_2) = F$. So we can assume that $c'(v_1x_1) = c'(x_1w_4) = F$. This implies that there is no F-path from v_1 to w_4. Now the end of e^* other than v is not joined by an F-path either with v_1 or with w_4, and it suffices to let $c(vv_1) = F$ or $c(vv_4) = F$, respectively.

Thus, our initial supposition that G' is C_4-free has been ruled out. So we have that G has a 2-path joining v_1 with w_4 (in order for there to be a C_4 in G'). The same argument, with a slight difference in notation, shows that G has a 2-path joining w_3 with v_6 if (A) happens and with v_7 otherwise. This means that G has a vertex w adjacent to v_1, w_3, w_4, and either v_6 (if (A) is the case) or v_7 (otherwise).

If the face $f_3 = v_3vv_4\ldots$ is strong, then v gives only $\frac{1}{4}$ to each of v_1 and v_2, and $\mu^*(v) \geq 7 - 4 - 5 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$. If, on the other hand, face f_3 is weak, then, since $d(w) > 2$ and there is a path w_3ww_4, it follows that f_3 must be a 5-face incident with edge qw_3w_4. In that case, the faces $f_2 = v_2vv_3\ldots$ and $f_4 = v_4vv_5\ldots$ are strong since edges w_2w_3 and w_4w_5 cannot exist, and again $\mu^*(v) \geq 0$.

Configuration 4. Vertex v is adjacent to four 2-vertices and incident with three triangles, each containing one of these 2-vertices.

Let v be adjacent to 2-vertices v_2, v_4, v_6 and v_7, and the three triangles are vv_1v_2, vv_3v_4 and vv_5v_6. Furthermore, let w_7 be the neighbor of v_7 other than v.

Let G' be the graph obtained by removing v and 2-vertices adjacent to v, adding vertices z_1, z_3, and z_5, and adding the edges of the 6-cycle $C^+ = v_1z_1v_3z_3v_5z_5$. Note that G' is C_4-free since the common neighbor, v, of v_1, v_3, v_5 in G has been removed.

Since G' has fewer branching vertices than G, it is FH_5-colorable. Let c' be an FH_5-coloring of G'. We use c' to design an FH_5-coloring c of G as follows.

Again, we first define an intermediate coloring c_0 and then we adjust it to obtain a desired coloring c. Let $c_0(vv_{i+1}) = c_0(vv_7) = H$, and $c_0(v_iv_{i+1}) = c_0(v_7w_7) = F$ whenever $i \in \{1, 3, 5\}$. Also for $i \in \{1, 3, 5\}$, we let $c_0(vv_i) = H$ if v_i is incident with at least one edge of C^+ colored H (in coloring c'); otherwise, we let $c_0(vv_i) = F$.

Note that at least one of vv_1, vv_3 and vv_5 is colored H since C^+ cannot be totally colored with F in coloring c'. Thus, we have three cases to consider.

Case 1. Exactly one of the edges vv_1, vv_3 and vv_5 is colored H. Without loss of generality, suppose $c_0(vv_1) = H$, $c_0(vv_3) = c_0(vv_5) = F$. Here, c_0 is already a desired FH_5-coloring c of G since $G - v$ has no F-path from v_3 to v_5 due to the absence of F-cycles in G' going through v_3z_3 and z_3v_5.

Case 2. Exactly two of the edges vv_1, vv_3 and vv_5 are colored H. Thus, c_0 is not an FH_5-coloring (as there are six edges incident to v that are colored H in c_0). We will recolor some edges to yield an FH_5-coloring c.

Without loss of generality, suppose $c_0(vv_1) = c_0(vv_3) = H$, $c_0(vv_5) = F$. By our construction of c_0 from c', this implies that $c'(z_3v_5) = c'(v_5z_5) = F$ in G', and at least one edge of C^+ incident to v_1 is colored H in c' and similarly at least one edge of C^+ incident to v_3 is colored H in c'. This yields three subcases to consider.

Subcase 2.1. Both the edges of C^+ incident to one of v_1 or v_3 are colored H: say, $c'(v_1z_1) = c'(v_1z_5) = H$. To obtain c, swap the colors on the edges vv_2 and v_1v_2 (similarly, swap the colors on edges vv_4 and v_3v_4, if it had been v_3 whose both edges in C^+ were colored H).

So, now we are left with two subcases where each of v_1 and v_3 has exactly one of their C^+ edges colored F and one colored H.

Subcase 2.2. One of the edges of C^+ at v_1 or v_3 colored F is incident with one of the two edges of C^+ at v_5 (both of are which colored F in the case under consideration): suppose, without loss of generality, that $c'(v_1z_5) = F$. We recolor vv_2 with F to obtain a desired FH_5-coloring c of G (since there is no F-cycle in G' going through v_1z_5 and z_5v_5, there is no F-cycle containing the F-path $v_1v_2vv_5$ in G).

Subcase 2.3. $c'(v_1z_5) = c'(v_3z_3) = H$, which means that $c'(v_1z_1) = c'(v_3z_1) = F$. This implies the absence of an F-path from v_1 to v_3 in c_0. As a consequence, either there is no F-path from v_5 to v_1 or there is no F-path from v_3 to v_5. If the former is true, we recolor vv_2 with F to obtain c; otherwise, we recolor vv_4 with F.

Case 3. All three of the edges vv_1, vv_3, and vv_5 are colored H in c_0. In c_0, all the seven edges of v are colored H. To obtain c, it suffices to recolor edges vv_1 and vv_2 with F, and recolor v_1v_2 with H.

So, we have proved that the final charge of each 7-vertex is non-negative. This completes the proof of Theorem 1. ■

3. Remarks

The proof above shows that a planar C_4-free graph has one of the configurations that cannot exist in a minimal counterexample. Since these configurations can be identified in polynomial time (in terms of the number of vertices), our proof yields a polynomial time algorithm for decomposing a planar C_4-free graph into a forest and a graph of maximum degree at most 5.

In the rest of this section, we construct a planar C_4-free graph that cannot be decomposed into a forest and a graph of maximum degree 3. This leaves open the question whether every planar C_4-free graph can be decomposed into a forest and a graph of maximum degree at most 4.

It is well-known that there are infinitely many cubic planar graphs with girth 5, and the dodecahedron (having 20 vertices) is the smallest of them. Let G be an n-vertex cubic planar graph with girth 5, having m edges and f faces. We construct a planar graph G' from G as follows:

(1) for every face f, add a vertex v_f and connect v_f with each vertex on the boundary of f with an edge and then subdivide such edge with a new vertex;

(2) for every edge e of the original graph G, add a vertex and join it by edges with both ends of e.

Denote the numbers of vertices, edges, faces, and vertices of degree 2 in G' by n', m', f' and n'_2, respectively. By construction,

$$n' = n + (f + 2m) + m = n + 3 \times \frac{3}{2}n + \frac{1}{2}n + 2 = 6n + 2,$$

$$m' = 3m + 4m = 7m = \frac{21}{2}n, \qquad \text{and} \qquad n'_2 = m + 2m = \frac{9}{2}n.$$

Suppose $E(G')$ has a partition into a spanning tree F and a subgraph H such that $\Delta(H) \leq 3$. Since F is a spanning tree we have:

(a) for every 2-vertex v in G', $\deg_H(v) \leq 1$;

(b) for each face f of G, at least one 2-vertex in G' adjacent to v_f has two neighbors in F and hence is isolated in H.

Therefore, $\sum_{v\in V(G')} \deg_H(v) \leq 3(n'-n'_2)+n'_2-f = 3(6n+2)-2\times\frac{9}{2}n-\frac{1}{2}n-2 = \frac{17}{2}n+4$.

Hence, $m' = |E(F)|+|E(H)| \leq n'-1+\frac{1}{2}(\frac{17}{2}n+4) = \frac{41}{4}n+3 < \frac{21}{2}n = m'$, a contradiction for all $n \geq 14$.

References

[1] A. Bassa, J. Burns, J. Campbell, A. Deshpande, J. Farley, M. Halsey, S. Michalakis, P.-O. Persson, P. Pylyavskyy, L. Rademacher, A. Riehl, M. Rios, J. Samuel, B. Tenner, A. Vijayasaraty, L. Zhao and D. J. Kleitman, *Partitioning a planar graph of girth ten into a forest and a matching*, manuscript (2004).

[2] O.V. Borodin, *Consistent colorings of graphs on the plane*, Diskret. Analiz **45** (1987) 21–27 (in Russian).

[3] O. Borodin, A. Kostochka, N. Sheikh and G. Yu, *Decomposing a planar graph with girth nine into a forest and a matching*, European Journal of Combinatorics **29** (2008) 1235–1241.

[4] O. Borodin, A. Kostochka, N. Sheikh and G. Yu, *M-degrees of quadrangle-free planar graphs*, J. Graph Theory **60** (2009) 80–85.

[5] W. He, X. Hou, K.W. Lih, J. Shao, W. Wang and X. Zhu, *Edge-partitions of planar graphs and their game coloring numbers*, J. Graph Theory **41** (2002) 307–317.

[6] D.J. Kleitman, *Partitioning the edges of a girth* 6 *planar graph into those of a forest and those of a set of disjoint paths and cycles*, manuscript.

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