Journal of Combinatorial Theory, Series B 99 (2009) 298–305

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Ore-type versions of Brooks' theorem

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ARTICLE INFO

Article history:
Received 1 September 2007
Available online 22 July 2008

Keywords:
Graph coloring
Brooks' theorem
Edge degree
Equitable coloring

ABSTRACT

The *Ore-degree* of an edge xy in a graph G is the sum $\theta(xy) = d(x) + d(y)$ of the degrees of its ends. In this paper we discuss colorings and equitable colorings of graphs with bounded *maximum Ore-degree*, $\theta(G) = \max_{xy \in E(G)} \theta(xy)$. We prove a Brooks-type bound on chromatic number of graphs G with $\theta(G) \geqslant 12$. We also discuss equitable and nearly equitable colorings of graphs with bounded maximum Ore-degree: we characterize r-colorable graphs with maximum Ore-degree at most $2r$ whose every r-coloring is equitable. Based on this characterization, we pose a conjecture on equitable r-colorings of graphs with maximum Ore-degree at most $2r$, which extends the Chen–Lih–Wu Conjecture and one of our earlier conjectures. We prove that our conjecture is true for $r = 3$.

1. Introduction

Two n-vertex graphs *pack*, if there exists an edge disjoint placement of these graphs into K_n. In other words, G_1 and G_2 *pack* if G_1 is isomorphic to a subgraph of the complement of G_2 (and vice versa).

A number of basic graph theoretic problems can be naturally expressed in the language of packing. For example, a proper (vertex) k-coloring of a graph G can be considered as a packing of G with a $|V(G)|$-vertex graph with k components each of which is a complete graph. In particular, an equitable k-coloring of a graph G can be viewed as a packing of G with the $|V(G)|$-vertex graph,

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[1] Supported in part by NSA grant MDA 904-03-1-0007.
[2] Supported in part by NSF grant DMS-0650784 and by grant 06-01-00694 of the Russian Foundation for Basic Research.

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2008.06.007

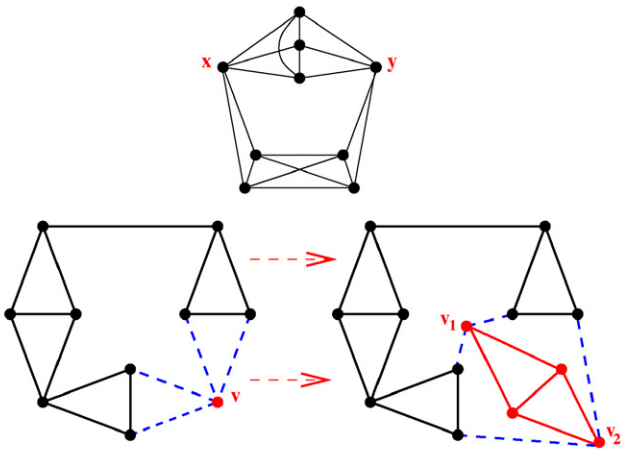

Fig. 1. Top: a graph with $\theta = 9$ and $\chi = 5$. Bottom: two graphs with $\theta = 7$ and $\chi = 4$.

whose components are cliques with either $\lfloor |V(G)|/k \rfloor$ or $\lceil |V(G)|/k \rceil$ vertices. Recall that an *equitable k-coloring* of a graph G is a proper k-coloring, for which any two color classes differ in size by at most one.

Another basic problem that can be viewed as a packing problem is the existence of hamiltonian cycles in a graph. The classical Dirac's Theorem [6] on the existence of hamiltonian cycles in each n-vertex graph with minimum degree at least $n/2$ can be stated in terms of packing as follows: *If $n \geqslant 3$ and G is an n-vertex graph with maximum degree at most $\frac{1}{2}n - 1$, then G packs with the cycle C_n of length n.*

Similarly, Ore's Theorem [18] on hamiltonian cycles is as follows: *If $n \geqslant 3$ and G is an n-vertex graph with $d(x) + d(y) \leqslant n - 2$ for each edge $xy \in E(G)$, then G packs with the cycle C_n.*

This statement motivates considering the notion of *Ore-degree* $\theta(xy)$ of an edge xy in a graph G as the sum, $d(x) + d(y)$, of the degrees of its ends in G. By definition, the Ore-degree of an edge xy is two greater than the degree of the vertex xy in the line graph of G, and coincides with the degree of xy in the total graph of G. We let *the Ore-degree* of a graph G be $\theta(G) = \max_{xy \in E(G)} \theta(xy)$. Thus, Ore's Theorem says that every n-vertex graph G with $n \geqslant 3$ and $\theta(G) \leqslant n - 2$ packs with the cycle C_n.

In view of Dirac's and Ore's Theorems, we call upper bounds for properties of graphs in terms of maximum degree *Dirac-type bounds* and those in terms of the Ore-degree *Ore-type bounds*. The obvious (but sharp) Dirac-type bound on the chromatic number is

$$\chi(G) \leqslant \Delta(G) + 1, \tag{1}$$

where $\chi(G)$ is the chromatic number of G and $\Delta(G)$ is its maximum degree. Brooks' Theorem below characterizes the graphs for which (1) holds with equality.

Theorem 1 *(Brooks). If $\chi(G) = \Delta(G) + 1$, then either G contains the complete graph $K_{\Delta(G)+1}$ or $\Delta(G) = 2$ and G contains an odd cycle.*

The counterpart of (1) for $\theta(G)$ is

$$\chi(G) \leqslant \lfloor \theta(G)/2 \rfloor + 1. \tag{2}$$

The proof is also obvious and the bound is also attained at complete graphs. However for small odd θ there are more connected graphs for which (2) holds with equality.

Fig. 1 shows a connected graph G with $\theta(G) = 9$ and $\chi(G) = 5$ and two graphs with $\theta = 7$ and $\chi = 4$. The graph on the right is obtained from the one on the left by replacing a degree 4 vertex v with a copy of $K_4 - v_1 v_2$ so that $N(v) \subseteq N(v_1) \cup N(v_2)$ and the new graph H satisfies $\theta(H) = 7$. This

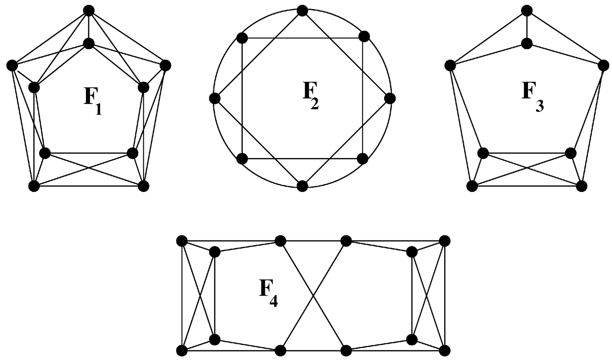

Fig. 2. One 5-equitable and three 4-equitable basic graphs.

creates two new vertices of degree 4 while maintaining $\chi(H) = 4$. Iterating this idea yields infinitely many such 2-connected graphs. Our first result shows that for graphs with Ore-degree at least 12 (i.e., with chromatic number at least 7), the only extremal connected graphs are complete graphs.

Theorem 2. *If* $7 \leqslant \chi(G) = \lfloor \theta(G)/2 \rfloor + 1$, *then G contains the complete graph* $K_{\chi(G)}$.

We believe that the result holds also for graphs G with $\chi(G) = 6$, but our method did not work in this case. Theorem 2 could also be stated as: *for* $k \geqslant 7$, K_k *is the only k-critical graph with maximum degree at most k whose vertices of degree k form an independent set.*

The analog of (1) for equitable coloring is the Hajnal–Szemerédi Theorem ([8], for a shorter proof see [10]).

Theorem 3. *For every positive integer r, each graph with* $\Delta(G) \leqslant r$ *has an equitable* $(r+1)$*-coloring.*

The theorem has interesting applications in extremal combinatorial and probabilistic problems, see e.g. [1,3,4,9,13,19,20]. It is easy to check that for odd r, $K_{r,r}$ has no equitable r-coloring. So there are new extremal examples for the case of equitable coloring. The problem of describing all extremal graphs is not resolved, but there are some conjectures. The first of them is due to Chen, Lih and Wu [5].

Conjecture 4. *Let G be a connected graph with* $\Delta(G) \leqslant r$*. Then G has no equitable r-coloring if and only if either*

(1) *G contains* K_{r+1}, *or*
(2) $r = 2$ *and G is an odd cycle, or*
(3) *r is odd and* $G = K_{r,r}$.

Some partial cases of Conjecture 4 were proved in [5,14,17,22,23]. In particular, Chen, Lih and Wu [5] proved that the conjecture holds for $r = 3$.

Unlike Brooks' Theorem, Conjecture 4 characterizes only connected extremal graphs for odd r. For example, for an odd $r \geqslant 3$, the graph consisting of two disjoint copies of $K_{r,r}$ has an equitable r-coloring, but the graph consisting of disjoint copies of $K_{r,r}$ and K_r does not. This construction can be generalized. We say that a graph H is *r-equitable* if $|H|$ is divisible by r, H is r-colorable and every r-coloring of H is equitable. If G contains $K_{r,r}$ and $G - K_{r,r}$ is r-equitable, then G does not have an equitable r-coloring. This motivates the study of equitable graphs.

If an r-colorable graph G has a spanning subgraph whose components are all r-equitable, then G is also r-equitable. In Figs. 2 and 3 we show one 5-equitable graph F_1, three 4-equitable graphs

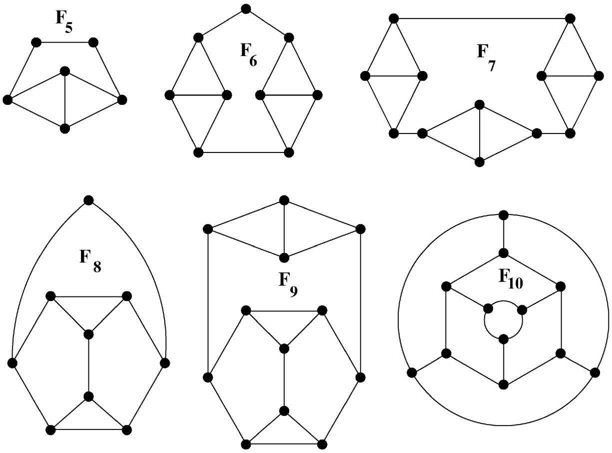

Fig. 3. Six 3-equitable basic graphs.

F_2, F_3, F_4 and six 3-equitable graphs $F_5, \dots, F_{10}$. Together with K_r, the r-equitable graphs from this list are the *r-basic* graphs. An *r-decomposition* of G is a partition of $V(G)$ into subsets $V_1, \dots, V_t$ such that each induced subgraph $G(V_i)$ is r-basic. We say that G is *r-decomposable* if it has an r-decomposition. So if r is odd and G is r-decomposable, then $G \cup K_{r,r}$ has no equitable r-coloring. In [12] we conjectured that this is the only obstacle that prevents an r-colorable graph with $\Delta(G) \leqslant r$ from having an equitable r-coloring.

Conjecture 5. *Suppose that $r \geqslant 3$ and G is an r-colorable graph with $\Delta(G) \leqslant r$. Then G has no equitable r-coloring if and only if r is odd and there exists $H \subseteq G$ such that $H = K_{r,r}$ and $G - H$ is r-decomposable.*

We proved that Conjecture 5 is equivalent to Conjecture 4 for each r, and thus that the conjecture holds for $r = 3$. The main tool was the following characterization of r-equitable graphs with maximum degree r. It uses the notion of *nearly equitable r-coloring*, i.e., a proper r-coloring of G that has exactly one color class of size $s - 1$, exactly one color class of size $s + 1$ and all other color classes of size s.

Theorem 6. *Let G be an r-colorable graph with $\Delta(G) \leqslant r$ and $|G|$ divisible by r. The following are equivalent*:

(A) *G is r-decomposable*;
(B) *G is r-equitable*;
(C) *G has an equitable r-coloring, but does not have a nearly equitable r-coloring.*

In [11] we proved the following analog of the Hajnal–Szemerédi Theorem in terms of Ore-degrees (also strengthening a conjecture in [15,16]).

Theorem 7. *For every $r \geqslant 3$, each graph G with $\theta(G) \leqslant 2r + 1$ has an equitable $(r + 1)$-coloring.*

The bound of this theorem is attained (apart from K_{r+1}) for every odd $0 < m \leqslant r$ at the graph $K_{m,2r-m}$. In the same paper, the following analog of the Chen–Lih–Wu Conjecture was proposed.

Conjecture 8. *Let G be a connected graph with $\theta(G) \leqslant 2r$ and $3 \leqslant r$. If G is distinct from K_{r+1} and from $K_{m,2r-m}$ for odd m, then G has an equitable r-coloring.*

Conjecture 8 was proved to hold for $r = 3$. We now think that probably the following common extension of Conjectures 5 and 8 holds true.

Conjecture 9. *Suppose that $r \geqslant 3$. An r-colorable, n-vertex graph G with $\theta(G) \leqslant 2r$ has no equitable r-coloring, if and only if n is divisible by r, and there exists $W \subseteq G$ such that $W = K_{m,2r-m}$ for some odd m and $G - W$ is r-decomposable.*

In this paper we prove that Conjecture 9 is equivalent to Conjecture 8 even for graphs with restricted number of vertices and restricted values of Ore-degree.

Theorem 10. *Assume that Conjecture* 8 *holds for all graphs with at most n vertices and Ore-degree at most* $2r$. *Let G be an r-colorable n-vertex graph with $\theta(G) \leqslant 2r$. Then G has no equitable r-coloring if and only if n is divisible by r and there exists $H \subseteq G$ such that $H = K_{m,2r-m}$ for some odd m and $G - H$ is r-decomposable.*

It follows that the conjecture holds for $r = 3$. Theorem 10 might help proving Conjecture 8 by induction, since we may prove the formally weaker statement, Conjecture 8, for some n-vertex graph using that the statement of Conjecture 9 is true for proper subgraphs of the graph.

In the next section we discuss ordinary colorings and prove Theorem 2. In the last section we discuss equitable colorings and Conjecture 9.

2. Ordinary coloring

The aim of this section is to prove Theorem 2. For convenience, we restate it here in a slightly different, but equivalent, form.

Theorem 11. *If $r \geqslant 6$, $\theta(G) \leqslant 2r + 1$, and $\chi(G) = r + 1$, then G contains K_{r+1}.*

Assume that $\theta(G) = 2r + 1$, and $\chi(G) = r + 1$. Then G contains an $(r + 1)$-critical subgraph G'. Since G' is $(r + 1)$-critical, $\delta(G') \geqslant r$. This together with $\theta(G) \leqslant 2r + 1$ yields

$$\Delta(G') \leqslant r + 1 \text{ and the set } B \text{ of vertices of degree } r + 1 \text{ is independent.} \tag{3}$$

Therefore, Theorem 2 can be restated in yet another form.

Theorem 12. *If $r \geqslant 6$, then the only $(r + 1)$-critical graph satisfying* (3) *is the complete graph K_{r+1}.*

Suppose for a contradiction that G is a minimal $(r + 1)$-critical graph that is distinct from K_{r+1} and satisfies (3). Let B (from “big”) be the set of vertices of degree $r + 1$ and S (from “small”) be the (remaining) set of vertices of degree r in G. We prove Theorem 12 in a series of lemmas. The first is a special case of a well-known theorem of Gallai [7, Theorem E.1].

Lemma 13. *Every block in $G(S)$ is either a complete graph or an odd cycle. Moreover, if G' is obtained from G by deleting every K_r-component of $G(S)$, then in every r-coloring of G', the neighborhood in B of each K_r-component of $G(S)$ is monochromatic.*

This lemma has the following immediate consequence.

Lemma 14. *If $C \notin \{K_1, K_r\}$ is a component of $G(S)$ then $2(r - 1) \leqslant |E(B, C)|$.*

The next lemma is a special case of a theorem of Stiebitz [21, Theorem 4.2] settling a conjecture by Gallai.

Lemma 15. *The number of components in $G(S)$ is at least $|B|$.*

We can add more detail on the structure of G.

Lemma 16. *G does not contain $K_{r+1} - e$.*

Proof. Suppose that $F := K_{r+1} - e \subseteq G$. By criticality, $G - F$ has an r-coloring f. Using the Ore-degree of G, $|E(F, G - F)| \leqslant 4 < r$. Thus we can extend f to an r-coloring of G by permuting the color classes of an r-coloring of F. This is a contradiction. □

Lemma 17. *Every $b \in B$ has at most one neighbor in any K_r-component of $G(S)$.*

Proof. Suppose that $b \in B$ has at least two neighbors in a K_r-component C of $G(S)$. Since G does not contain K_{r+1}, B contains a neighbor b' of C distinct from b. Let $G' := G - C + bb'$. Since $d_{G-C}(b) \leqslant r - 1$, graph G' satisfies (3). By Lemma 16, G' does not contain K_{r+1}. So by criticality, G' has an r-coloring f with $f(b) \neq f(b')$, a contradiction to Lemma 13. □

Lemma 17 yields that the neighborhood $N(C)$ of every K_r-component C of $G(S)$ is a set of cardinality r. Consider the r-uniform hypergraph H with the vertex set B whose edges are neighborhoods of K_r-components of $G(S)$.

Lemma 18. *The hypergraph H defined above has no cycles. In particular, no two edges of H share more than one vertex.*

Proof. Suppose that $(b_1, \dots, b_k)$ is a shortest cycle in H. This means that there are K_r-components $C_1, \dots, C_k$ of $G(S)$ such that $b_i \in N(C_i) \cap N(C_{i+1})$ for $i = 1, \dots, k$ (we consider $k + 1 \equiv 1$). Since we have chosen a shortest cycle in H, each C_i has a neighbor $b'_i \notin \{b_1, \dots, b_k\}$ (vertices b'_i can coincide for distinct i). Let G' be obtained from $G - C_1 - \cdots - C_k$ by adding the k edges $b_1b'_1, \dots, b_kb'_k$. Since the degree of each b_i in $G - C_1 - \cdots - C_k$ is at most $r - 1$, the graph G' satisfies (3).

If $Q := K_{r+1} \subseteq G'$ then Q contains some edge $b_ib'_i$. Since B is independent, Q contains at most one such edge. Thus G contains $K_{r+1} - e$, contradicting Lemma 16. So G' does not contain K_{r+1}. By the minimality of G, G' (and also $G - C_1 - \cdots - C_k$) has an r-coloring f with $f(b_i) \neq f(b'_i)$ for all $i = 1, \dots, k$, a contradiction to Lemma 13. □

Suppose that $|B| = m$. Lemma 18 implies the following.

Lemma 19. *The number of K_r-components of $G(S)$ is less than $m/(r - 1)$.*

Let X be the set of vertices in S that are components of $G(S)$, and let R be the set of K_r-components of $G(S)$. Counting the edges between B and S, by Lemmas 14 and 15 we have

$$(r + 1)m \geqslant \big(|X| + |R|\big)r + (2r - 2)\big(m - \big(|X| + |R|\big)\big),$$

and hence $|X| + |R| \geqslant (r - 3)m/(r - 2)$. By Lemma 19, $|R| < m/(r - 1)$. It follows that

$$|X| > m\left(\frac{r - 3}{r - 2} - \frac{1}{r - 1}\right). \tag{4}$$

The next useful fact on list colorings is a direct consequence of a seminal theorem by Alon and Tarsi [2] (it also follows from known facts on kernel-perfect digraphs).

Lemma 20. *Let H be a bipartite graph. Suppose that each vertex v of H is given a list $L(v)$ of admissible colors. If H has an orientation D such that $d^+(v) < |L(v)|$ for every $v \in V(D)$, then H is L-colorable, i.e., has a proper coloring f such that $f(v) \in L(v)$ for every $v \in V(H)$.*

We derive the next lemma from Lemma 20.

Lemma 21. *Let $H = (X, Y; E)$ be a bipartite graph with partite sets X and Y and with $\delta(H) \geqslant 3$. Let L be a list assignment for H such that $|L(y)| \geqslant d(y) - 1$ for each $y \in Y$ and $|L(x)| \geqslant d(x)$ for each $x \in X$. Then H is L-colorable.*

Proof. Let H' be obtained from H by splitting each vertex $y \in Y$ into $\lceil d(y)/3 \rceil$ vertices of degree at most 3. Each split vertex has degree at most 3, and X only has vertices of degree at least 3. By Hall's Theorem, some matching M' in H' covers all vertices in X. Let M be obtained from M' by rejoining the split vertices. Then every vertex $y \in Y$ is incident with at most $\lceil d(y)/3 \rceil \leqslant d(y) - 2$ edges of M. Orient in H every edge of M towards X and the remaining edges towards Y. Since this orientation satisfies Lemma 20, the lemma is proved. □

If the subgraph G'' of G induced by $B \cup X$ contains a subgraph F with minimum degree at least 3, then we color properly $G - V(F)$ and by Lemma 21 can extend it to F. So suppose that there is no such F. Then G'' is 2-degenerate and hence $|E(G'')| \leqslant 2|V(G'')| - 3$. Since $|V(G'')| = m + |X|$, we have $r|X| < 2m + 2|X|$, i.e., $(r - 2)|X| < 2m$. Plugging (4) into this inequality and dividing by m, we get

$$(r-2)\left(\frac{r-3}{r-2} - \frac{1}{r-1}\right) < 2.$$

This means $r \leqslant 5$. So the theorem is proved.

3. Equitable coloring

It turns out that Theorem 6 easily extends as follows.

Theorem 22. *Let G be an r-colorable graph with $\theta(G) \leqslant 2r$ and $|G|$ be divisible by r. The following are equivalent:*

(A) *G is r-decomposable;*
(B) *G is r-equitable;*
(C) *G has an equitable r-coloring, but does not have a nearly equitable r-coloring.*

As with Theorem 6 itself, it is easy to see that (A) ⇒ (B) and (B) ⇒ (C). The next lemma reduces the last implication to the case $\Delta(G) \leqslant r$ which is proved in Theorem 6.

Lemma 23. *Let $r \geqslant 3$ and let n be divisible by r. Suppose that G is an n-vertex graph with $\theta(G) \leqslant 2r$ that has an equitable r-coloring. If $\Delta(G) > r$, then G has a nearly equitable r-coloring.*

Proof. Suppose that G has no nearly equitable r-colorings. Fix an equitable r-coloring f of G. If a vertex v has no neighbor in a color class distinct from $f(v)$, then by moving v there we obtain a nearly equitable r-coloring. Thus we assume that each v has neighbors in all color classes but its own. In particular, $\delta(G) \geqslant r - 1$, and hence $\Delta(G) \leqslant r + 1$.

Suppose that $v \in V(G)$ and $d(v) = r + 1$. Each of its neighbors has degree at most $r - 1$ and hence has exactly one neighbor in each color class but its own. If a color class W contains exactly two neighbors of v, then moving these two neighbors to the color class of v and v to W, we get a nearly equitable coloring. So, the only possible distribution of neighbors of v is that $r - 2$ color classes each contain exactly one neighbor of v and some class W contains three neighbors of v. Let u be a neighbor of v not in W. Since $d(u) \leqslant r - 1$, it is not adjacent to some other neighbor u' of v. Then moving u and u' into the color class of v and moving v to the former color class of u again yields a nearly equitable coloring of G. This proves the lemma and thus the whole Theorem 22. □

Proof of Theorem 10. Let G satisfy the conditions of the theorem. Suppose that Theorem 10 does not hold for G. Then G contains $K_{m,2r-m}$ for some odd m. Let $G_1, \ldots, G_l$ be all copies of graphs $K_{m,2r-m}$ for odd $m \leqslant r$ that are subgraphs of G. Let $G_0 = G - G_1 - \cdots - G_l$. By Conjecture 8, G_0 has an equitable

r-coloring; and if $l \geqslant 2$, then this coloring can be extended to an equitable r-coloring of G. Let $l = 1$. If $|V(G_0)|$ is not divisible by r, then we obtain an equitable r-coloring of G by combining an equitable r-coloring of G_0 with a nearly equitable r-coloring of G_1. So, suppose that $|V(G_0)| = rs$. If G_0 has no partition into the r-equitable graphs from the main list, then by Theorem 22 it has a nearly equitable r-coloring f. And f can be combined with a nearly equitable r-coloring of G_1 to get an equitable coloring of G. If G_0 has such a partition, then our theorem holds for G. □

Acknowledgment

We thank Michael Stiebitz for helpful comments.

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