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Graphs with bounded tree-width and large odd-girth are almost bipartite

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We prove that for every k and every $\varepsilon > 0$, there exists g such that every graph with tree-width at most k and odd-girth at least g has circular chromatic number at most $2 + \varepsilon$.

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1. Introduction

It has been a challenging problem to prove the existence of graphs of arbitrarily high girth and chromatic number [2]. On the other hand, graphs with large girth that avoid a fixed minor are known to have low chromatic number (in particular, this applies to graphs embedded on a fixed surface). More precisely, as Thomassen observed [8], a graph that avoids a fixed minor and has large girth is 2-degenerate, and hence 3-colorable. Further, Galluccio, Goddyn and Hell [3] proved the following theorem, which essentially states that graphs with large girth that avoid a fixed minor are almost bipartite.

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Theorem 1. (See Galluccio, Goddyn and Hell, 2001.) For every graph H and every $\varepsilon > 0$, there exists an integer g such that the circular chromatic number of every H -minor free graph of girth at least g is at most $2 + \varepsilon$.

A natural way to weaken the girth-condition is to require the graphs to have high odd-girth (the *odd-girth* is the length of a shortest odd cycle). However, Youngs [9] constructed 4-chromatic projective graphs with arbitrarily high odd-girth. Thus, the high odd-girth requirement is not sufficient to ensure 3-colorability, even for graphs embedded on a fixed surface. Klostermeyer and Zhang [4], though, proved that the circular chromatic number of every planar graph of sufficiently high odd-girth is arbitrarily close to 2. In particular, the same is true for K_4 -minor free graphs, i.e. graphs with tree-width at most 2. We prove that the conclusion is still true for any class of graphs with bounded tree-width, which answers a question of Pan and Zhu [6, Question 6.2] also appearing as Question 8.12 in the survey by Zhu [10].

Theorem 2. For every k and every $\varepsilon > 0$, there exists g such that every graph with tree-width at most k and odd-girth at least g has circular chromatic number at most $2 + \varepsilon$.

Motivated by tree-width duality, Nešetřil and Zhu [5] proved the following theorem.

Theorem 3. (See Nešetřil and Zhu, 1996.) For every k and every $\varepsilon > 0$, there exists g such that every graph G with tree-width at most k and homomorphic to a graph H with girth at least g has circular chromatic number at most $2 + \varepsilon$.

To see that Theorem 2 implies Theorem 3, observe that if G has an odd cycle of length g , then H has an odd cycle of length at most g .

2. Notation

A (p, q) -coloring of a graph is a coloring c of the vertices with colors from the set $\{0, \dots, p-1\}$ such that the colors of any two adjacent vertices u and v satisfy $q \leq |c(u) - c(v)| \leq p - q$. The circular chromatic number $\chi_c(G)$ of a graph G is the infimum (and it can be shown to be the minimum) of the ratios p/q such that G has a (p, q) -coloring. For every finite graph G , it holds that $\chi(G) = \lceil \chi_c(G) \rceil$ and there is a (p, q) -coloring of G for every p and q with $p/q \geq \chi_c(G)$. In particular, the circular chromatic number of G is at most $2 + 1/k$ if and only if G is homomorphic to a cycle of length $2k + 1$. The reader is referred to the surveys by Zhu [10,11] for more information about circular colorings.

A p -precoloring is a coloring φ of a subset A of vertices of a graph G with colors from $\{0, \dots, p-1\}$, and its extension is a coloring of the whole graph G that coincides with φ on A . The following lemma can be seen as a corollary of a theorem of Albertson and West [1, Theorem 1], and it is the only tool we use from this area.

Lemma 4. For every p and q with $2 < p/q$, there exists d such that any p -precoloring of vertices with mutual distances at least d of a bipartite graph H extends to a (p, q) -coloring of H .

A k -tree is a graph obtained from a complete graph of order $k + 1$ by successively adding vertices joined to exactly k pairwise adjacent vertices. The tree-width of a graph G is the smallest k such that G is a subgraph of a k -tree. Graphs with tree-width at most k are also called *partial k -trees*.

A *rooted partial k -tree* is a partial k -tree G with $k + 1$ distinguished vertices v_1, \dots, v_{k+1} such that there exists a k -tree G' that is a supergraph of G and the vertices v_1, \dots, v_{k+1} form a clique in G' . We also say that the partial k -tree is *rooted* at v_1, \dots, v_{k+1} . If G is a partial k -tree rooted at v_1, \dots, v_{k+1} and G' is a partial k -tree rooted at v'_1, \dots, v'_{k+1} , then the graph $G \oplus G'$ obtained by identifying v_i and v'_i is again a rooted partial k -tree (identify the cliques in the corresponding k -trees).

Fix p and q . If G is a rooted partial k -tree, then $\mathcal{F}(G)$ is the set of all p -precolorings of the $k + 1$ distinguished vertices of G that can be extended to a (p, q) -coloring of G .

The next lemma is a standard application of results in the area of graphs of bounded tree-width [7].

Lemma 5. *Let k and N be positive integers such that $N \geq k + 1$. If G is a partial k -tree with at least $3N$ vertices, then there exist partial rooted k -trees G_1 and G_2 such that G is isomorphic to $G_1 \oplus G_2$ and G_1 has at least $N + 1$ and at most $2N$ vertices.*

If G is a partial k -tree rooted at v_1, \dots, v_{k+1} , then its type is a $(k + 1) \times (k + 1)$ matrix M such that M_{ij} is the length of the shortest path between the vertices v_i and v_j . If there is no such path, M_{ij} is equal to ∞ . Any matrix M that is a type of a partial rooted k -tree satisfies the triangle inequality (setting $\infty + x = \infty$ for any x). A symmetric matrix M whose entries are non-negative integers and ∞ (and zeroes only on the main diagonal) that satisfies the triangle inequality is a type. A type is bipartite if $M_{ij} + M_{jk} + M_{ik} \equiv 0 \pmod 2$ for any three finite entries M_{ij} , M_{jk} and M_{ik} . Two bipartite types M and M' are compatible if M_{ij} and M'_{ij} have the same parity whenever both of them are finite. We define a binary relation on bipartite types as follows: $M \preceq M'$ if and only if M and M' are compatible and $M_{ij} \leq M'_{ij}$ for every i and j . Note that the relation \preceq is a partial order.

We finish this section with the following lemma. Its straightforward proof is included to help us in familiarizing with the just introduced notation.

Lemma 6. *Let G^1 and G^2 be two bipartite rooted partial k -trees with types M^1 and M^2 such that there exists a bipartite type M^0 with $M^0 \preceq M^1$ and $M^0 \preceq M^2$. Then the types M^1 and M^2 are compatible, $G^1 \oplus G^2$ is a bipartite rooted partial k -tree and its type M satisfies $M^0 \preceq M$.*

Proof. The types M^1 and M^2 are compatible: if both M^1_{ij} and M^2_{ij} are finite, then M^0_{ij} is finite and has the same parity as M^1_{ij} and M^2_{ij} . Hence, the entries M^1_{ij} and M^2_{ij} have the same parity.

Let M be the type of $G^1 \oplus G^2$. Note that it does not hold in general that $M_{ij} = \min\{M^1_{ij}, M^2_{ij}\}$. We show that $M^0 \preceq M$ which will also imply that $G^1 \oplus G^2$ is bipartite since M^0 is a bipartite type. Consider a shortest path P between two distinguished vertices v_t and $v_{t'}$ and split P into paths P_1, \dots, P_ℓ delimited by distinguished vertices on P . Note that $\ell \leq k$ since P is a path. Let $j_0 = t$ and let j_i be the index of the end-vertex of P_i for $i \in \{1, \dots, \ell\}$. In particular, $j_\ell = t'$. Each of the paths P_1, \dots, P_ℓ is fully contained in G^1 or in G^2 (possibly in both if it is a single edge). Since $M^0 \preceq M^1$ and $M^0 \preceq M^2$, the length of P_i is at least $M^0_{j_{i-1}j_i}$, and it has the same parity as $M^0_{j_{i-1}j_i}$. Since M^0 is a bipartite type (among others, it satisfies the triangle inequality), the length of P , which is $M_{tt'}$, has the same parity as $M^0_{j_0j_\ell} = M^0_{tt'}$ and is at least $M^0_{tt'}$. This implies that $M^0 \preceq M$. \square

3. The main lemma

In this section, we prove a lemma which forms the core of our argument. To this end, we first prove another lemma that asserts that for every k , p and q , the set of types of all bipartite rooted partial k -trees forbidding a fixed set of p -precolorings from extending (and maybe some other precolorings, too) has always a maximal element. We state the lemma slightly differently to facilitate its application.

Lemma 7. *For every k , p and q , there exists a finite number of (bipartite) types M^1, \dots, M^m such that for any bipartite rooted partial k -tree G with type M , there exists a bipartite rooted partial k -tree G' with type M^i for some $i \in \{1, \dots, m\}$ such that $\mathcal{F}(G') \subseteq \mathcal{F}(G)$ and $M \preceq M^i$.*

Proof. Let $d \geq 2$ be the constant from Lemma 4 applied for p and q . Let M^1, \dots, M^m be all bipartite types with entries from the set $\{1, \dots, D^{(k+1)^2}\} \cup \{\infty\}$ where $D = 4d$. Thus, m is finite and does not exceed $(D^{(k+1)^2} + 1)^{k(k+1)/2}$.

Let G be a bipartite rooted partial k -tree with type M . If M is one of the types M^1, \dots, M^m , then there is nothing to prove (just choose i such that $M = M^i$). Otherwise, one of its entries is finite and exceeds $D^{(k+1)^2}$.

For $i \in \{1, \dots, (k+1)^2\}$, let J^i be the set of all positive integers between D^{i-1} and $D^i - 1$ (inclusively). Let i_0 be the smallest integer such that no entry of M is contained in J^{i_0} . Since M has at most $k(k+1)/2$ different entries, such an index i_0 exists. Note that if $i_0 = 1$, then Lemma 4 implies that $\mathcal{F}(G)$ contains all possible p -precolorings, and the sought graph G' is the bipartite rooted partial k -tree composed of $k+1$ isolated vertices, with the all- ∞ type.

Two vertices v_i and v_j at which G is rooted are close if M_{ij} is less than D^{i_0-1} . The relation \approx of being close is an equivalence relation on v_1, \dots, v_{k+1} . Indeed, it is reflexive and symmetric by the definition, and we now show that it is transitive. Suppose that M_{ij} and M_{jt} are both less than D^{i_0-1} . Then, the distance between v_i and v_t is at most $M_{ij} + M_{jt} \leq 2D^{i_0-1} - 2 \leq D^{i_0} - 1$ since $D \geq 2$. Consequently, by the choice of i_0 , the distance between v_i and v_t is at most $D^{i_0-1} - 1$ and thus $v_i \approx v_t$.

Let C_1, \dots, C_ℓ be the equivalence classes of the relation \approx . Note that C_1, \dots, C_ℓ is a finer partition than that given by the equivalence relation of being connected.

Since G is bipartite, we can partition its vertices into two color classes, say red and blue. For every $i \in \{1, \dots, \ell\}$, contract the closed neighborhood of a vertex v if v is a blue vertex and its distance from any vertex of C_i is at least D^{i_0-1} and keep doing so as long as such a vertex exists. Observe that the resulting graph is uniquely defined. After discarding the components that do not contain the vertices of C_i , we obtain a bipartite partial k -tree G_i rooted at the vertices of C_i : it is bipartite as we have always contracted closed neighborhoods of vertices of the same color (blue) to a single (red) vertex, and its tree-width is at most k since the tree-width is preserved by contractions. Moreover, the distance between any two vertices of C_i has not decreased since any path between them through any of the newly arising vertices has length at least $2D^{i_0-1} - 2 \geq D^{i_0-1}$.

Now, let G' be the bipartite rooted partial k -tree obtained by taking the disjoint union of G_1, \dots, G_ℓ . The type M' of G' can be obtained from the type of G : set M'_{ij} to be M_{ij} if the vertices v_i and v_j are close, and ∞ otherwise. Thus, M' is one of the types M^1, \dots, M^m and $M \preccurlyeq M'$. It remains to show that $\mathcal{F}(G') \subseteq \mathcal{F}(G)$.

Let $c \in \mathcal{F}(G')$ be a p -precoloring that extends to G' , and recall that $D \geq 4$. For $i \in \{1, \dots, \ell\}$, let A_i be the set of all red vertices at distance at most D^{i_0-1} and all blue vertices at distance at most $D^{i_0-1} - 1$ from C_i , and let R_i be the set of all red vertices at distance $D^{i_0-1} - 1$ or D^{i_0-1} from C_i . Set $B_i = A_i \setminus R_i$ (B_i is the “interior” of A_i and R_i its “boundary”). The extension of c to G_i naturally defines a coloring of all vertices of A_i : G_i is the subgraph of G induced by A_i with some red vertices of R_i identified (two vertices of R_i are identified if and only if they are in the same component of the graph $G - B_i$).

Let H be the following auxiliary graph obtained from G : remove the vertices of $B = B_1 \cup \dots \cup B_\ell$ and, for $i \in \{1, \dots, \ell\}$, identify every pair of vertices of R_i that are in the same component of $G - B$. Let R be the set of vertices of H corresponding to some vertices of $R_1 \cup \dots \cup R_\ell$. Precolor the vertices of R with the colors given by the colorings of the graphs G_i (note that two vertices of R_i in the same component of $G - B_i$ are also in the same component of $G - B$, so this is well defined). The graph H is bipartite as only red vertices have been identified. The distance between any two precolored vertices is at least d : consider two precolored vertices r and r' at distance at most $d - 1$. Let i and i' be such that $r \in R_i$ and $r' \in R_{i'}$. If $i = i'$, then r and r' are in the same component of $G - B$ and thus $r = r'$. If $i \neq i'$ then by the definition of R_i and $R_{i'}$, the vertex r is in G at distance at most D^{i_0-1} from some vertex v of C_i and r' is at distance at most D^{i_0-1} from some vertex v' of $C_{i'}$. So, the distance between v and v' is at most $2D^{i_0-1} + d - 1 \leq D^{i_0} - 1$. Since M has no entry from J^{i_0} , the vertices v and v' must be close and thus $i = i'$, a contradiction.

Since the distance between any two precolored vertices is at least d , the precoloring extends to H by Lemma 4 and in a natural way it defines a coloring of G . We conclude that every p -precoloring that extends to G' also extends to G and thus $\mathcal{F}(G') \subseteq \mathcal{F}(G)$. \square

We now prove our main lemma, which basically states that there is only a finite number of bipartite rooted partial k -trees that can appear in a minimal non- (p, q) -colorable graph with tree-width k and a given odd-girth.

Lemma 8. *For every k, p and q , there exist a finite number m and bipartite rooted partial k -trees G^1, \dots, G^m with types M^1, \dots, M^m such that for any bipartite rooted partial k -tree G with type M there exists i such that $\mathcal{F}(G^i) \subseteq \mathcal{F}(G)$ and $M \preceq M^i$.*

Proof. Let M^1, \dots, M^m be the types from Lemma 7. We define the graph G^i as follows: for every p -precoloring c that does not extend to a bipartite partial rooted k -tree with type M^i , fix a partial rooted k -tree G_c^i with type M^i such that c does not extend to G_c^i . Set $G^i = \bigoplus_c G_c^i$, where c runs over all such p -precolorings. If the above sum of partial k -trees is non-empty, then the type M of G^i is M^i . Indeed, $M \preceq M^i$ by the definition of G^i , and Lemma 6 implies that $M^i \preceq M$. If all the p -precolorings of the $k+1$ vertices in the root extend to each partial k -tree of type M^i , then let G^i be the graph consisting of $k+1$ isolated vertices. This happens in particular for the all- ∞ type.

Let us verify the statement of the lemma. Let G be a bipartite rooted partial k -tree and let M be the type of G . If $\mathcal{F}(G)$ is composed of all p -precolorings, the sought graph G^i is the one composed of $k+1$ isolated vertices. Hence, we assume that $\mathcal{F}(G)$ does not contain all p -precolorings, i.e., there are p -precolorings that do not extend to G . By Lemma 7, there exists a bipartite rooted partial k -tree G' with type M' such that $M \preceq M' = M^i$ for some i and $\mathcal{F}(G') \subseteq \mathcal{F}(G)$. For every p -precoloring c that does not extend to G' (and there exists at least one such p -precoloring c), some graph G_c^i has been glued into G^i . Hence, $\mathcal{F}(G^i) \subseteq \mathcal{F}(G') \subseteq \mathcal{F}(G)$. Since the type of G^i is M^i , the conclusion of the lemma follows. \square

4. Proof of Theorem 2

We are now ready to prove Theorem 2, which is recalled below.

Theorem 2. *For every k and every $\varepsilon > 0$, there exists g such that every graph with tree-width at most k and odd-girth at least g has circular chromatic number at most $2 + \varepsilon$.*

Proof. Fix p and q such that $2 < p/q \leq 2 + \varepsilon$. Let G^1, \dots, G^m be the bipartite partial k -trees from Lemma 8 applied for k, p and q . Set N to be the largest order of the graphs G^i and set g to be $3N$. We assert that each partial k -tree with odd-girth g has circular chromatic number at most p/q . Assume that this is not the case and let G be a counterexample with the fewest vertices.

The graph G has at least $3N$ vertices (otherwise, it has no odd cycles and thus is bipartite). By Lemma 5, G is isomorphic to $G_1 \oplus G_2$, where G_1 and G_2 are rooted partial k -trees and the number of vertices of G_1 is between $N+1$ and $2N$. By the choice of g , the graph G_1 has no odd cycle and thus it is a bipartite rooted partial k -tree. By Lemma 8, there exists i such that $\mathcal{F}(G^i) \subseteq \mathcal{F}(G_1)$ and $M_1 \preceq M^i$ where M_1 is the type of G_1 and M^i is the type of G^i . Let G' be the partial k -tree $G^i \oplus G_2$.

First, G' has fewer vertices than G since the number of vertices of G^i is at most N and the number of vertices of G_1 is at least $N+1$. Second, G' has no (p, q) -coloring: if it had a (p, q) -coloring, then the corresponding p -precoloring of the $k+1$ vertices shared by G^i and G_2 would extend to G_1 since $\mathcal{F}(G^i) \subseteq \mathcal{F}(G_1)$ and thus G would have a (p, q) -coloring, too. Finally, G' has no odd cycle of length less than g : if it had such a cycle, replace any path between vertices v_j and $v_{j'}$ of the root of G^i with a path of at most the same length between them in G_1 (recall that $M_1 \preceq M^i$). If such paths for different pairs of v_j and $v_{j'}$ on the considered odd cycle intersect, take their symmetric difference. In this way, we obtain an Eulerian subgraph of $G = G_1 \oplus G_2$ with an odd number of edges such that the number of its edges is less than g . Consequently, this Eulerian subgraph has an odd cycle of length less than g , which violates the assumption on the odd-girth of G . We conclude that G' is a counterexample with fewer vertices than G , a contradiction. \square

We end by pointing out that the approach used yields an upper bound of $3(k+1) \cdot 2^{2^{p^{k+1}}((4d)^{(k+1)^2} + 1)^{k^2}}$ for the smallest g such that all graphs with tree-width at most k and odd-girth at least g have circular chromatic number at most p/q , whenever $p/q > 2$. More precisely, the value of N cannot exceed $(k+1) \cdot 2^{2^{p^{k+1}}((4d)^{(k+1)^2} + 1)^{k^2}}$. To see this, we consider all pairs $P = (C, M)$ where C is a set of p -precolorings of the root and M is a type such that there is a bipartite rooted partial k -tree of type M to which no coloring of C extends. Let n_P be the size of a smallest such partial k -tree. We obtain a sequence of at most $2^{2^{p^{k+1}}((4d)^{(k+1)^2} + 1)^{k^2}}$ integers. The announced bound follows from the following fact: if the sequence is sorted in increasing order, then each term is at most twice the previous one.

Indeed, consider the tree-decomposition of the partial k -tree G_P chosen for the pair P . If the bag containing the root has a single child, then we delete a vertex of the root, and set a vertex in the single child to be part of the root. We obtain a partial k -tree to which some p -precolorings of C do not extend. Thus, $n_P \leq 1 + n_{P'}$ for some pair P' and $n_{P'} < n_P$. If the bag containing the root has more than one child, then G_P can be obtained by identifying the roots of two smaller partial k -trees G and G' . By the minimality of G_P , the orders of G and G' are n_{P_1} and n_{P_2} for two pairs P_1 and P_2 such that $n_{P_i} < n_P$ for $i \in \{1, 2\}$. This yields the stated fact, which in turn implies the given bound, since the smallest element of the sequence is $k+1$.

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