



Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb


Some constructive bounds on Ramsey numbers

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ARTICLE INFO

Article history:

Received 4 April 2009

Available online 2 February 2010

Keywords:

Ramsey numbers

Lower bounds

Explicit constructions

ABSTRACT

We present explicit constructions of three families of graphs that yield the following lower bounds on Ramsey numbers: $R(4, m) \geq \Omega(m^{8/5})$, $R(5, m) \geq \Omega(m^{5/3})$, $R(6, m) \geq \Omega(m^2)$.

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1. Introduction

For two positive integers s and m , the Ramsey number $R(s, m)$ is the least integer R such that every graph on R vertices contains either a clique of size s or an independent set of size m . The best lower bounds on Ramsey numbers are proven by probabilistic methods which do not give explicit constructions of graphs without a clique of size s and an independent set of size m . All known constructions give weaker lower bounds.

In this note we show constructions that give nontrivial lower bounds on the Ramsey numbers $R(s, m)$ for $s = 4, 5, 6$. We describe three families of explicitly defined graphs that give the following bounds:

$$R(4, m) \geq \Omega(m^{8/5}), \quad R(5, m) \geq \Omega(m^{5/3}), \quad R(6, m) \geq \Omega(m^2).$$

These graphs are defined similarly to the graphs introduced in [6], which give the constructive lower bound

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¹ Research of this author was supported by NSF grant DMS-0650784 and grant 08-01-00673 of the Russian Foundation for Fundamental Research.

² Partially supported by grant MSM 0021620838.

³ Partially supported by NSF grant DMS 0800070.

$$R(3, m) \geq \Omega(m^{3/2}). \tag{1}$$

In [6] the lower bound was proved using an algebraic argument. For the sake of completeness, we repeat this construction and give a combinatorial proof of the lower bound. The first constructive lower bound of the form (1) was found by Alon [1].

A different generalization of the construction of [6] was presented in [2]. That construction gives polynomial lower bounds for $R(s, m)$ with the exponent of the polynomials increasing with s . However, for $s = 3, 4, 5, 6$, the construction does not give anything interesting.

To the best of our knowledge, the above bounds are the best known constructive bounds. However, they are far from nonconstructive lower bounds [7,10]:

$$R(3, m) \geq \Omega(m^2 / \log m), \quad R(4, m) \geq \Omega((m / \log m)^{2.5}),$$

$$R(5, m) \geq \Omega((m / \log m)^3), \quad R(6, m) \geq \Omega((m / \log m)^{3.5}).$$

2. Preliminaries

Let G be a graph. The *superline graph*, H_G , of G is constructed as follows. The vertices of H_G are the edges of G , and ef is an edge in H_G if e and f are disjoint edges of G and there exists an edge g of G that connects an end of e with an end of f (i.e., if the edges e, g and f form a path in G).

Let $\alpha(G)$ denote the *independence number* of G , i.e., the size of a largest independent set in G .

Lemma 2.1. *For every triangle-free graph G and its superline graph H_G ,*

$$\alpha(H_G) \leq \alpha(G).$$

Proof. Let A be an independent set in H_G . Let B be the set of vertices in G of the edges in A . Then the subgraph $G[B]$ of G induced by B has no triangles and does not contain paths of length 3. So, the components of $G[B]$ are stars, hence $G[B]$ has an independent set of size $|A|$. \square

We also need the following modification of the superline graph construction. Let G be a bipartite graph with bipartition of vertices (U, V) and let $<$ be a linear ordering of the edges of G . We denote by $H_G^<$ the graph whose vertices are the edges of G and a pair $\{uv, u'v'\}$, with $u \neq u' \in U$ and $v \neq v' \in V$ is an edge in $H_G^<$ if either $uv < u'v'$ and uv' is an edge in G or $u'v' < uv$ and $u'v$ is an edge in G . (In particular, $H_G^<$ is a subgraph of H_G .)

Lemma 2.2. *For every bipartite graph G on n vertices, every bipartition of G and every ordering $<$ of its edges,*

$$\alpha(H_G^<) < n.$$

Proof. Let A be a set of n edges of G . Then A contains a cycle. Let $(u_0, v_0, u_1, v_1, \dots, u_{k-1}, v_{k-1}, u_0)$ be a cycle formed by some edges in A . Then for some $0 \leq i < k$, $u_i v_i < u_{i+1} v_{i+1}$ (where we count $i + 1$ modulo k). Hence $\{u_i v_i, u_{i+1} v_{i+1}\}$ is an edge in $H_G^<$, which proves that A is not an independent set. \square

We also need the following concepts. A bipartite graph is called a *generalized k -gon*, if it is regular and has diameter k and girth $2k$.

Let G be a bipartite graph with bipartition (U, V) . A mapping $\pi : U \rightarrow V$ is called a *polarity*, if it is a bijection such that for every $u_1, u_2 \in U$

$$u_1 \pi(u_2) \in E(G) \quad \text{iff} \quad u_2 \pi(u_1) \in E(G).$$

If a polarity exists, then the *polarity graph* G^π is the graph with vertex set U , and $u_1 u_2 \in E(G^\pi)$ iff $u_1 \neq u_2$ and $u_1 \pi(u_2) \in E(G)$.

We also define the *polarity graph with loops* $G^{\pi,0}$. It is obtained from the polarity graph G^π by adding loops uu for every u such that $u\pi(u)$ is an edge in G . The reason for using these graphs is that this construction preserves regularity; namely, if G is d -regular, then so is $G^{\pi,0}$. Our convention is that by adding a loop to a vertex v its degree increases by 1 (note that some authors postulate that the increase of the degree is 2). We use graphs $G^{\pi,0}$ to estimate the independence numbers of graphs G^π . Otherwise all graphs in this paper will be loop-free.

Recall the well-known inequality

$$\alpha(G) \leq n \frac{\lambda}{d}, \tag{2}$$

where G is d -regular graph without loops of order n , λ is the second largest absolute value of an eigenvalue and $\alpha(G)$ denotes the cardinality of the largest independent set in G . For graphs with loops the corresponding inequality is

$$\alpha(G) \leq n \frac{\lambda + 1}{d}, \tag{3}$$

where we allow loops to be present in independent sets. Both inequalities are immediate consequences of the well-known inequality

$$\left| e(A, B) - \frac{d}{n} |A| \cdot |B| \right| \leq \lambda \sqrt{|A| \cdot |B|}, \tag{4}$$

where $e(A, B)$ denotes the number of edges (u, v) such that $u \in A$ and $v \in B$ (including loops when A and B are not disjoint); this inequality is Corollary 2.5, Chapter 9 of [3].

Let G be a bipartite graph with bipartition (U, V) . Let M be the matrix of G such that the rows of M are indexed by vertices in U , the columns of M are indexed by vertices in V , M_{uv} equals 1 if (u, v) is an edge, and equals 0 otherwise. We use the eigenvalues of MM^T for generalized k -gons as they are presented in Tanner [11]. Observe that G has a polarity if and only if one can choose an ordering of vertices of G so that the matrix M is symmetric. In such a case, $MM^T = M^2$ and M is also the adjacency matrix of the polarity graph with loops $G^{\pi,0}$. Since the eigenvalues of M^2 are squares of the eigenvalues of M , one can determine the absolute values of the eigenvalues of $G^{\pi,0}$ from the eigenvalues of MM^T .

3. Constructive bound $R(3, m) \geq \Omega(m^{3/2})$

Let P_q be the incidence graph of the classical projective plane $PG(2, q)$ with $q^2 + q + 1$ points and $q^2 + q + 1$ lines, where q is a prime power. P_q is a regular graph of degree $q + 1$. Let $<$ be an arbitrary ordering of the edges of P_q . We use the graphs $H_{P_q}^<$. The following properties can be easily verified:

1. $H_{P_q}^<$ has $(q + 1)(q^2 + q + 1)$ vertices.
2. $H_{P_q}^<$ is triangle-free.

Proof. If $p_1l_1 < p_2l_2 < p_3l_3$ would form a triangle, then (p_1, l_3, p_2, l_2) would be a 4-cycle in P_q . \square

3. The largest independent set in $H_{P_q}^<$ has size $\leq 2(q^2 + q + 1)$, by Lemma 2.2.

4. Constructive bound $R(5, m) \geq \Omega(m^{5/3})$

The construction and the proof is the same as above, except that instead of graphs without $K_{2,2}$ we use denser graphs that do not contain $K_{3,3}$. Such graphs were constructed in [5,4].

We use the bipartite version of Brown's construction. Let U and V be two copies of \mathbb{F}_p^3 , where p is a prime and \mathbb{F}_p is the finite field with p elements. Connect two vertices $(u_1, u_2, u_3) \in U$ and $(v_1, v_2, v_3) \in V$ by an edge if they are different and $(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 = 0 \pmod{p}$. The resulting graph B_p

1. has $2p^3$ vertices,
2. is regular of degree $p^2 - p$, hence it has $p^5 - p^4$ edges,
3. does not contain $K_{3,3}$.

Consider the graph $H_{B_p}^<$, where $<$ is an arbitrary linear order. Let $n = p^5 - p^4$ be the number of vertices of $H_{B_p}^<$. The maximal independent set has size $m \leq 2p^3$, according to Lemma 2.2, hence $n = \Omega(m^{5/3})$. The proof that $H_{B_p}^<$ does not contain K_5 is also similar. By way of contradiction, suppose that $u_1 v_1 < \dots < u_5 v_5$ are vertices of K_5 in $H_{B_p}^<$. Then vertices $u_1, u_2, u_3, v_3, v_4, v_5$ induce a $K_{3,3}$ in B_p .

5. Constructive bound $R(4, m) \geq \Omega(m^{8/5})$

Constructions of generalized 4-gons are known for every prime power q , but those with polarities exist if and only if q is an odd power of 2. For an explicit presentation of such a polarity, see [8]. Let Q_q^π and $Q_q^{\pi,0}$ denote the polarity graph and the polarity graph with loops of a generalized 4-gon for q an odd power of 2. We need the following properties:

1. $Q_q^{\pi,0}$ is a $(q + 1)$ -regular graph of order $q^3 + q^2 + q + 1$.
2. Q_q^π has at least $(q^4 + q^3 + q^2 + q)/2$ edges.
3. Q_q^π does not contain cycles C_3, C_4, C_6 (but it does contain C_5).
4. The second largest absolute value of an eigenvalue of $Q_q^{\pi,0}$ is $\sqrt{2q}$.
5. The size of any independent set in Q_q^π is at most $\frac{(q^3+q^2+q+1)(\sqrt{2q+1})}{q+1}$.

Properties 1 and 3 were proved in [8]. Property 2 immediately follows from Property 1. According to [11], the eigenvalues of the square of the adjacency matrix of $Q_q^{\pi,0}$ are $(q + 1)^2, 2q, 0$, which implies Property 4. Property 5 follows from Property 4 and inequality (3).

Consider the superline graph $H_{Q_q^\pi}$. Let m be its independence number and n be its order. Then $m = O(q^{5/2})$ and $n = \Theta(q^4)$, whence $n = \Omega(m^{8/5})$. It remains to prove that it does not contain K_4 . To this end we use Property 3 of Q_q .

Lemma 5.1. *If G is a graph such that K_4 is a subgraph of H_G , then G contains a cycle C_3 , or C_4 , or C_6 .*

Proof. Suppose that the edges $11', \dots, 44'$ of G are vertices of a K_4 in H_G . We consider two cases depending on how the first edge is connected to the three others, see Figs. 1–3.

Case 1. The sets $\{2, 2'\}, \{3, 3'\}$, and $\{4, 4'\}$ have to be adjacent to each other. Since Q_q is C_3 - and C_4 -free, they can be connected only by edges $i'j'$, $2 \leq i < j \leq 4$. But then we would have triangle $(2', 3', 4')$.

Case 2. We consider the possibilities how sets $\{2, 2'\}, \{3, 3'\}, \{4, 4'\}$ are connected to each other without creating cycles of lengths 3, 4 or 6. For $22'$ and $33'$ there is the unique possibility $2'3'$.

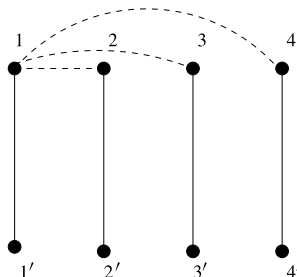


Fig. 1. Case 1.

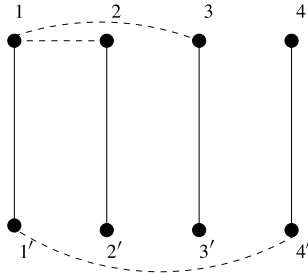


Fig. 2. Case 2.

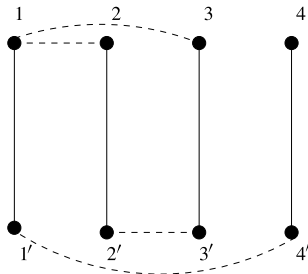


Fig. 3. Case 2, continued.

One can check that each of the edges $2'4, 3'4'$ would produce a C_6 and each of the edges $24', 34'$ would produce a C_4 . Hence we must have $(24 \text{ or } 2'4')$ and $(34 \text{ or } 3'4)$. Thus we have 4 possibilities; each of them produces C_4 or C_6 :

- $24, 34 \mapsto (1, 2, 4, 3),$
- $24, 3'4 \mapsto (2, 2', 3'4),$
- $2'4', 34 \mapsto (1, 2, 2', 4', 4, 3),$
- $2'4', 3'4 \mapsto (2', 4', 4, 3'). \quad \square$

6. Constructive bound $R(6, m) \geq \Omega(m^2)$

Let P_q^π be the polarity graph of the classical projective plane $PG(2, q)$, where q is a prime power. This graph has a simple explicit definition. The vertices of P_q^π are the *points* of the projective plane P_q . Given two distinct points with homogeneous coordinates (a, b, c) and (a', b', c') , $a, b, c, a', b', c' \in GF_q$, they form an edge in P_q^π , if $aa' + bb' + cc' = 0$. P_q^π has $n = q^2 + q + 1$ vertices; $q + 1$ vertices have degree q , and q^2 vertices have degree $q + 1$, hence the number of edges is $\frac{1}{2}q(q + 1)^2$. This graph is also known as the *Erdős-Rényi graph*; for more information on its independence number, see [9].

The second largest absolute value of an eigenvalue of the polarity graph with loops $P_q^{\pi,0}$ is $\lambda = \sqrt{q}$. This fact also follows from the results of [11], since P_q is a generalized 3-gon.

Whence by (3),

$$\alpha(P_q^\pi) = \alpha(P_q^{\pi,0}) \leq n \frac{\sqrt{q} + 1}{q + 1} = O(n^{3/4}).$$

Our explicit lower bound on $R(6, m)$ is based on $H_{P_q^\pi}$. The number of vertices of this graph is $\Omega(n^{3/2})$ and $\alpha(H_{P_q^\pi}) \leq \alpha(P_q^\pi) = O(n^{3/4}) = O(\sqrt{n^{3/2}})$. Hence it remains to prove that $H_{P_q^\pi}$ does not contain K_6 . Again, the only property we use is that P_q^π does not contain C_4 .

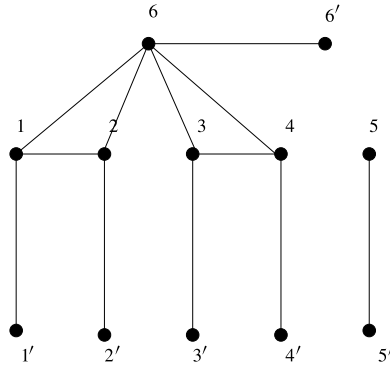


Fig. 4. Case 1.

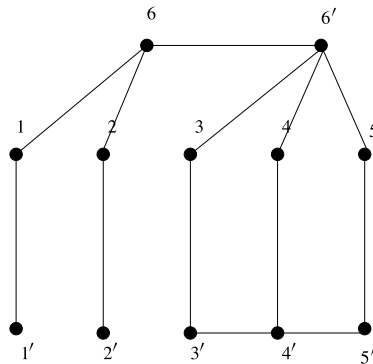


Fig. 5. Case 2.

Lemma 6.1. *If G is a graph such that K_6 is a subgraph of H_G , then C_4 is a subgraph of G .*

Proof. Suppose that $11', \dots, 66'$ are disjoint edges of a graph G that form a K_6 in H_G .

Case 1. One end of $66'$ is adjacent to ends of at least four other edges, say, 6 is adjacent to 1, 2, 3, 4. Assuming there is no C_4 in G , we cannot have edges ij' , for $1 \leq i \neq j \leq 4$. Furthermore we cannot have two non-disjoint edges on $\{1, 2, 3, 4\}$, or five edges on $\{1', 2', 3', 4'\}$. Thus we must have two disjoint edges on $\{1, 2, 3, 4\}$, say, 12 and 34, see Fig. 4. This forces the remaining four edges connecting $11', \dots, 44'$ to be $1'3', 1'4', 2'3', 2'4'$. Due to symmetry, it suffices to consider how $11'$ is connected with $33'$. The edge 13 would produce the 4-cycle $(1, 3, 4, 6)$; $13'$ would produce the cycle $(1, 3', 3, 6)$; $1'3$ would produce the cycle $(1', 3, 6, 1)$. But if we have edges $1'3', 1'4', 2'3', 2'4'$, then we get the 4-cycle $(1', 3', 2', 4')$.

Hence Case 1 leads to a contradiction.

Case 2. One end of $66'$ is adjacent to ends of two edges and the other is adjacent to ends of three edges, say, as shown in Fig. 5.

Assuming G does not contain C_4 , there are no edges of the form ij' , $i \neq j$ either for $1 \leq i, j \leq 2$ or for $3 \leq i, j \leq 5$. Furthermore, there can be only one edge of the form ij for $3 \leq i, j, \leq 5$. Thus we have at least two edges on $\{3', 4', 5'\}$, say, $3'4', 4'5'$.

In the following two claims we assume that C_4 is not a subgraph of G .

Claim 1. *There is no edge connecting the sets $\{1, 2\}$ and $\{3, 4, 5\}$.*

Proof. Such an edge would form C_4 with $6, 6'$. \square

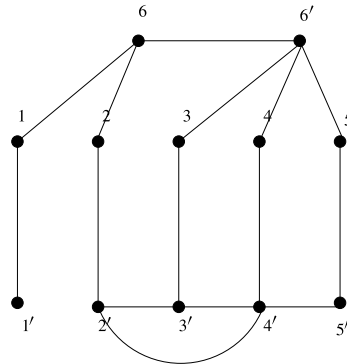


Fig. 6. Case 2, continued.

Claim 2. There is no edge connecting the sets $\{1', 2'\}$ and $\{3', 4', 5'\}$.

Proof. Suppose $i'j'$ is such an edge $i < j$. W.l.o.g. assume $i = 2$ and $j \in \{3, 4\}$. Let k be the other element of $\{3, 4\}$, thus jk is an edge. Then the only way $\{i, i'\}$ can be connected with $\{k, k'\}$ is by the edge $i'k'$. Indeed, $2k'$ would form the 4-cycle $(2, k', j', 2')$ and $2'k$ would form the cycle $(2', k, k', j')$. Thus we have the situation shown in Fig. 6.

Now consider how $\{2, 2'\}$ is connected to $\{5, 5'\}$. The edge $2'5'$ would produce 4-cycle $(2', 3', 4', 5')$; $2'5$ would produce 4-cycle $(2', 5, 5', 4')$; $25'$ would produce 4-cycle $(2, 5', 5, 4')$. This contradiction finishes the proof of Claim 2. \square

According to Claims 1 and 2, there are only cross edges connecting pairs $(1, 1'), (2, 2')$ with pairs $(3, 3'), (4, 4'), (5, 5')$. Since we need 6 edges to connect these sets, one of the vertices $1', 2', 3', 4', 5'$ must be incident with two such edges. Suppose for instance that it is $1'$ and it is adjacent to 3 and 4. Then $(1', 3, 6', 4)$ is a 4-cycle. The other cases are similar. \square

Acknowledgments

We thank Noga Alon for many fruitful conversations and both referees for their helpful comments.

References

[1] N. Alon, Explicit Ramsey graphs and orthonormal labelings, *Electron. J. Combin.* 1 (1994) R12, 8 pp.
 [2] N. Alon, P. Pudlák, Constructive lower bounds for off-diagonal Ramsey numbers, *Israel J. Math.* 122 (2001) 243–251.
 [3] N. Alon, J. Spencer, *The Probabilistic Method*, Wiley, 1991.
 [4] N. Alon, L. Rónyai, T. Szabó, Norm-graphs: variations and applications, *J. Combin. Theory Ser. B* 76 (1999) 280–290.
 [5] W.G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* 9 (1966) 281–289.
 [6] B. Codenotti, P. Pudlák, G. Resta, Some structural properties of low rank matrices related to computational complexity, *Theoret. Comput. Sci.* 235 (2000) 89–107.
 [7] J.H. Kim, The Ramsey number $R(3, t)$ has order of magnitude $t^2 / \log t$, *Random Structures Algorithms* 7 (1995) 173–207.
 [8] F. Lazebnik, V.A. Ustimenko, A.J. Woldar, Polarities and $2k$ -cycle-free graphs, *Discrete Math.* 197/198 (1999) 503–513.
 [9] D. Mubayi, J. Williford, On the independence number of the Erdős–Rényi and projective norm graph and related hypergraphs, *J. Graph Theory* 56 (2007) 113–127.
 [10] J. Spencer, Asymptotic lower bounds for Ramsey functions, *Discrete Math.* 20 (1977/1978) 69–76.
 [11] R.M. Tanner, Explicit concentrators from generalized N -gons, *SIAM J. Alg. Disc. Meth.* 5 (3) (1984) 287–293.